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UNIQUENESS OF ELASTIC CONTINUATION IN A SEMILINEAR ELASTIC BODY

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ABSTRACT. The authors prove a theorem on uniqueness of elastic continuation in a nonhomogeneous elastic solid with a displacement-dependent tension modulus, generalizing an earlier result by Ang, Ikehata, Trong and Yamamoto for a nonhomogeneous linear elastic solid.

Let Ω be a bounded domain in $\mathbf{R}^3$ representing an elastic body. We consider the problem of uniqueness for the determination of the stress field in Ω from the displacements and surface stresses given on an open portion Γ of the boundary $\partial\Omega$ of Ω, a problem referred to as one of elastic continuation. In [AITY], uniqueness of elastic continuation is proved for a nonhomogeneous linear elastic solid. In the present paper, we address the problem of uniqueness of elastic continuation in the case of a nonhomogeneous elastic solid with a tension modulus depending not only on $x = (x_1, x_2, x_3)$ but also on the displacement $u = (u_1, u_2, u_3)$. More precisely, we shall assume

$$\lambda = \lambda(x, u), \tag{1}$$

where λ is a multiple of the tension modulus (cf. [TG]).

Let $x = (x_1, x_2, x_3)$ be in $\Omega \subset \mathbf{R}^3$. For $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, we denote by σ_i, τ_{jk} the components of the normal stress and of the shear stress corresponding to the x_i-direction. We shall consider the following system (cf. [TG])

$$\frac{\partial \sigma_i}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial \tau_{ik}}{\partial x_k} = -X_i \tag{2}$$

subject to the boundary conditions

$$u\big|_{\Gamma} = u^0 \tag{3}$$

and

$$n_i \sigma_i + n_j \tau_{ij} + n_k \tau_{ik} = \overline{X}_i \quad \text{on } \Gamma, \tag{4}$$

where $u^0 = (u_1^0, u_2^0, u_3^0)$ and $n = (n_1, n_2, n_3)$ is the outer unit normal vector to $\partial\Omega$.

The displacement $u = (u_1, u_2, u_3)$ and the stresses σ_i, τ_{jk} satisfy the following relations (cf. [TG])

$$\sigma_i = \lambda e + 2G\varepsilon_i, \tag{5}$$

$$\tau_{ij} = G\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \tag{6}$$

where

$$\varepsilon_i = \frac{\partial u_i}{\partial x_i}, \tag{7}$$

$$e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3. \tag{8}$$

From now on, we shall assume that

$$\lambda \in C^2(\mathbf{R}^3 \times \mathbf{R}^3), \quad G \in C^2(\mathbf{R}^3)$$

and that

$$\lambda(x, u) > 0 \quad \text{for all } x \in \overline{\Omega},\ u \in \mathbf{R}^3.$$

Following is the main result of this paper.

Theorem 1. *Let Γ be C^1-smooth, let λ, G be respectively in $C^2(\mathbf{R}^3 \times \mathbf{R}^3)$, $C^2(\mathbf{R}^3)$. Then the system (2), (5)-(8) subjected to conditions (1), (3), (4) has at most one solution u in $(C^3(\Omega \cup \Gamma))^3$.*

We note that, with an almost identical proof, we obtain a result similar to that of Theorem 1 in the case of a two dimensional elastic body.

The proof of Theorem 1 relies on the following

Lemma 1. *Let Γ, λ, G be as in Theorem 1 and let $u^1 = (u_1^1, u_2^1, u_3^1)$, $u^2 = (u_1^2, u_2^2, u_3^2)$ be in $(C^3(\Omega \cup \Gamma))^3$ and satisfy (2)-(4). Then there exists an open subset Γ_0 of Γ such that*

$$\nabla u_i^1 = \nabla u_i^2, \quad \nabla \varepsilon_i^1 = \nabla \varepsilon_i^2, \quad i = 1, 2, 3 \quad \text{on } \Gamma_0,$$

where $\nabla = \Big(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}\Big)$.

Proof. By (3), one has,

$$u^1(x) = u^0(x) = u^2(x) \quad x \in \Gamma.$$

Hence, we have

$$\lambda(x, u^1(x)) = \lambda(x, u^2(x)) \quad \text{for all } x \in \Gamma. \tag{9}$$

For $i, j = 1, 2, 3$, we put

$$\begin{aligned} v_i &= u_i^1 - u_i^2, & \varepsilon_i &= \varepsilon_i^1 - \varepsilon_i^2, \\ s_i &= \sigma_i^1 - \sigma_i^2, & t_{ij} &= \tau_{ij}^1 - \tau_{ij}^2, \end{aligned} \tag{10}$$

where ε_i^k, τ_{ij}^k, $k = 1, 2$, $i, j = 1, 2, 3$ are defined from (6), (7) corresponding to u^k.

From (3), one has

$$v_i = 0 \quad \text{on } \Gamma, \quad i = 1, 2, 3. \tag{11}$$

We have to consider the following three cases

(i) There is a z_0 in Γ such that $n(z_0) = (n_1(z_0), n_2(z_0), n_3(z_0))$ satisfies $n_1(z_0)n_2(z_0)n_3(z_0) \neq 0$. In this case, we can find a neigborhood U of z_0 such that $n_1(z)n_2(z)n_3(z) \neq 0$ for all z in $U \cap \Gamma$ and we put $\Gamma_0 = U \cap \Gamma$.

(ii) Case (i) does not hold and there are z_1 and an $i \in \{1, 2, 3\}$ such that $n_i(z_1) = 0$ and $n_j(z_1) \neq 0$ for $j \in \{1, 2, 3\} \setminus \{i\}$. In this case, we can find a neighbohood U of z_1 such that $n_i(z) = 0$ and $n_j(z) \neq 0$ for $j \in \{1, 2, 3\} \setminus \{i\}$ and $z \in U \cap \Gamma \equiv \Gamma_0$.

(iii) Cases (i), (ii) do not hold and there are $z_2 \in \Gamma$ and $i, j \in \{1, 2, 3\}$, $i \neq j$ such that $n_i(z_2) = n_j(z_2) = 0$ and that $n_k(z_2) \neq 0$ for $\{k\} = \{1, 2, 3\} \setminus \{i, j\}$. By the above argument, we can choose $\Gamma_0 \subset \Gamma$ such that $n_i(z) = n_j(z) = 0$ and $n_k(z) \neq 0$ for $\{k\} = \{1, 2, 3\} \setminus \{i, j\}$ and $z \in \Gamma_0$.

We shall give the proof for Case (i). The proof of the lemma in Cases (ii), (iii) are similar to (in fact, easier than) that for Case (i), and hence, will be omitted.

In Case (i), the vectors $(0, -n_3(z), n_2(z))$, $(-n_3(z), 0, n_1(z))$ are vectors tangential to Γ_0 for all z in Γ_0. Hence, taking the tangential derivatives of (11) respectively in the directions of the above vectors, we get, for $i, j = 1, 2, 3$,

$$\frac{\partial v_i}{\partial x_j} = \frac{n_j}{n_i}\varepsilon_i \quad \text{on } \Gamma_0. \tag{12}$$

Hence

$$t_{ij} = G\Big(\frac{n_j}{n_i}\varepsilon_i + \frac{n_i}{n_j}\varepsilon_j\Big) \qquad \text{on } \Gamma_0 \tag{13}$$

for $i \neq j$, $i, j = 1, 2, 3$.

Substituting (12), (13) into (4) gives after some rearrangements

$$\Big(1 + \frac{1 - 2\nu}{n_i^2}\Big)\varepsilon_i + \varepsilon_j + \varepsilon_k = 0 \qquad \text{on } \Gamma_0, \tag{14}$$

where $i \neq j$, $j \neq k$, $k \neq i$, $i, j, k = 1, 2, 3$ and $\nu = \lambda/2(G + \lambda)$.

The system (14) implies

$$\varepsilon_i = 0 \qquad \text{on } \Gamma_0, \quad i = 1, 2, 3. \tag{15}$$

In view of (12), (15) we get

$$\nabla v_i = 0 \qquad \text{on } \Gamma_0, \quad i = 1, 2, 3. \tag{16}$$

Now, from (16), we can take the tangential derivatives of (15) to get

$$\frac{\partial \varepsilon_i}{\partial x_j} = \frac{n_j}{n_i}\frac{\partial \varepsilon_i}{\partial x_i} \qquad \text{on } \Gamma_0, \quad i \neq j. \tag{17}$$

Using (9), (15), (16), (17) we get, on Γ_0,

$$\frac{\partial t_{ij}}{\partial x_i} = G\Big(\frac{n_j}{n_i}\frac{\partial \varepsilon_i}{\partial x_i} + \frac{n_i^2}{n_j^2}\frac{\partial \varepsilon_j}{\partial x_j}\Big), \tag{18}$$

$$\frac{\partial s_i}{\partial x_i} = \lambda(x, u^1)\nu^{-1}\Big((1-\nu)\frac{\partial \varepsilon_i}{\partial x_i} + \frac{\nu n_i}{n_j}\frac{\partial \varepsilon_j}{\partial x_j} + \frac{\nu n_i}{n_k}\frac{\partial \varepsilon_k}{\partial x_k}\Big). \tag{19}$$

Substituting (18), (19) into (2) one gets on Γ_0

$$\Big(1 + \frac{1-2\nu}{n_i^2}\Big)\frac{\partial \varepsilon_i}{\partial x_i} + \frac{n_i}{n_j}\frac{\partial \varepsilon_j}{\partial x_j} + \frac{n_i}{n_k}\frac{\partial \varepsilon_k}{\partial x_k} = 0 \tag{20}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ successively.

From the system (20) we get

$$\frac{\partial \varepsilon_i}{\partial x_i} = 0 \qquad \text{on } \Gamma_0, \quad i = 1, 2, 3. \tag{21}$$

In view of (17), (21) one has

$$\nabla \varepsilon_i = 0 \qquad \text{on } \Gamma_0, \quad i = 1, 2, 3.$$

This completes the proof of Lemma 1.

We now turn to the

Proof of Theorem 1. For convenience, we put

$$\lambda^\ell(x) = \lambda(x, u^\ell(x)) \qquad \ell = 1, 2.$$

Substituting (5) into (2) gives after some computations

$$G\Delta u_i^\ell + F_i^\ell = -X_i, \qquad i = 1, 2, 3, \ \ell = 1, 2 \tag{22}$$

with

$$F_i^l = (\lambda^\ell + G)\frac{\partial e^\ell}{\partial x_i} + \frac{\partial \lambda^\ell}{\partial x_i}e^\ell + 2\frac{\partial G}{\partial x_i}\frac{\partial u_i^\ell}{\partial x_i} + \frac{1}{G}\frac{\partial G}{\partial x_j}\tau_{ij}^\ell + \frac{1}{G}\frac{\partial G}{\partial x_k}\tau_{ik}^\ell$$

for $\ell = 1, 2$, $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

Differentiating (22) with respect to x_i, one gets

$$-\frac{\partial X_i}{\partial x_i} = G\Delta \frac{\partial u_i^\ell}{\partial x_i} + \frac{\partial G}{\partial x_i}\Delta u_i^\ell + (\lambda^\ell + G)\frac{\partial^2 e^\ell}{\partial x_i^2} + 2\frac{\partial G}{\partial x_i}\frac{\partial^2 u_i^\ell}{\partial x_i^2}$$
$$+ \frac{\partial G}{\partial x_j}\Big(\frac{\partial^2 u_i^\ell}{\partial x_i \partial x_j} + \frac{\partial^2 u_j^\ell}{\partial x_i^2}\Big) + \frac{\partial G}{\partial x_k}\Big(\frac{\partial^2 u_i^\ell}{\partial x_i \partial x_k} + \frac{\partial^2 u_k^\ell}{\partial x_i^2}\Big)$$
$$+ \sum_{s=1}^{3} \frac{\partial \lambda}{\partial u_s}(x, u^\ell)\frac{\partial^2 u_s^\ell}{\partial x_i^2} e^l + F_{1i}^\ell, \tag{23}$$

where ℓ, (i, j, k) are as in the formulas for F_i^ℓ and

$$F_{1i}^l = 2\Big(\frac{\partial \lambda^\ell}{\partial x_i} + \frac{\partial G}{\partial x_i}\Big)\frac{\partial e^\ell}{\partial x_i} + \frac{\partial^2 \lambda}{\partial x_i^2}(x, u^\ell)e^\ell + \sum_{r,s=1}^{3} \frac{\partial^2 \lambda}{\partial u_r \partial u_s}(x, u^\ell)\frac{\partial u_r^\ell}{\partial x_i}\frac{\partial u_s^\ell}{\partial x_i} e^\ell$$
$$+ \frac{\partial^2 G}{\partial x_i \partial x_j}\Big(\frac{\partial u_i^\ell}{\partial x_j} + \frac{\partial u_j^\ell}{\partial x_i}\Big) + \frac{\partial^2 G}{\partial x_i \partial x_k}\Big(\frac{\partial u_i^\ell}{\partial x_k} + \frac{\partial u_k^\ell}{\partial x_i}\Big) + 2\frac{\partial^2 G}{\partial x_i^2}\frac{\partial u_i^l}{\partial x_i}.$$

Letting $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ successively in (23) and adding together the results thus obtained one has

$$(\lambda^\ell + 2G)\Delta e^\ell + 2\sum_{i=1}^{3}\Big(\frac{\partial G}{\partial x_i} + \frac{\partial \lambda}{\partial u_i}(x, u^\ell)\Big)\Delta u_i^\ell + \sum_{i=1}^{3} F_{1i}^\ell = -\sum_{i=1}^{3}\frac{\partial X_i}{\partial x_i}\cdot \tag{24}$$

From (22), it follows that

$$(\lambda^\ell + 2G)\Delta e^\ell - 2\sum_{i=1}^{3} G^{-1}\Big(\frac{\partial G}{\partial x_i} + \frac{\partial \lambda}{\partial u_i}(x, u^\ell)\Big)(F_i^\ell + X_i) + \sum_{i=1}^{3} F_{1i}^\ell = -\sum_{i=1}^{3}\frac{\partial X_i}{\partial x_i}\cdot \tag{25}$$

From (22), (25) we get

$$\Delta u_i^\ell + G^{-1}F_i^\ell = -X_i, \tag{26}$$
$$\Delta e^\ell + F_e^\ell = -\sum_{i=1}^{3}\frac{\partial X_i}{\partial x_i}, \tag{27}$$

where $i = 1, 2, 3$, $\ell = 1, 2$ and

$$F_e^\ell = (\lambda^\ell + 2G)^{-1}\left(-2\sum_{i=1}^{3} G^{-1}\Big(\frac{\partial G}{\partial x_i} + \frac{\partial \lambda}{\partial u_i}(x, u^\ell)\Big)(F_i^\ell + X_i) + \sum_{i=1}^{3} F_{1i}^\ell\right).$$

Put $v_i = u_i^1 - u_i^2$, $i = 1, 2, 3$, $v_4 = e^1 - e^2$. By (26), (27) and the mean value theorem of Lagrange, we can find functions a_{ijk}, b_{ip} in $C(\Omega \cup \Gamma)$, $j, k = 1, 2, 3$, $i, p = 1, 2, 3, 4$, such that

$$\Delta v_i + \sum_{j,k=1}^{3} a_{ijk} \frac{\partial v_j}{\partial x_k} + \sum_{p=1}^{4} b_{ip} v_p = 0, \qquad i = 1, 2, 3, 4. \tag{28}$$

In view of Lemma 1, one has, for an open portion Γ_0 of Γ,

$$v_i\big|_{\Gamma_0} = \nabla v_i\big|_{\Gamma_0} = 0. \tag{29}$$

We claim that $v_i \equiv 0$ on Ω $(i = 1, 2, 3, 4)$. In fact, let ω be an open subset of $\mathbf{R}^3 \setminus \overline{\Omega}$ such that $\Omega_0 = \omega \cup \Gamma_0 \cup \Omega$ is connected. Using the reflexive method (see, e.g., [F], page 10), we can extend the functions a_{ijk}, b_{ip} $(i, p = 1, 2, 3, 4,\ j, k = 1, 2, 3)$ to functions $\tilde{a}_{ijk}, \tilde{b}_{ip}$ continuous on Ω_0 such that

$$\tilde{a}_{ijk}|_{\Omega} = a_{ijk}, \tag{30}$$

$$\tilde{b}_{ip}|_{\Omega} = b_{ip}. \tag{31}$$

From (29), we can also extend v_i to $\tilde{v}_i$ $(i = 1, 2, 3, 4)$ in $H^2_{loc}(\Omega_0)$ by putting

$$\tilde{v}_i = \begin{cases} v_i & x \in \Omega \\ 0 & x \in \Omega_0 = \omega \cup \Gamma. \end{cases} \tag{32}$$

We shall prove that $\tilde{v}_i = 0$ (i=1,2,3,4) on Γ_0. In view of (30)-(32), the system (28) gives

$$\Delta \tilde{v}_i + \sum_{j,k=1}^{3} \tilde{a}_{ijk} \frac{\partial \tilde{v}_j}{\partial x_k} + \sum_{p=1}^{4} \tilde{b}_{ip} \tilde{v}_p = 0, \qquad i = 1, 2, 3, 4. \tag{33}$$

Put

$$U = \{x \in \Omega_0 : \text{there is a neighbohood } N_x \text{ of } x \text{ in } \Omega_0 \text{ such that } \tilde{v}_i(z) = 0 \text{ for } z \in N_x,\ i = 1, 2, 3, 4\}.$$

It is clear that $U \neq \emptyset$ (since $\omega \subset U$) and that U is relatively open in Ω_0. It is sufficient to prove that

$$U \text{ is relatively closed in } \Omega_0. \tag{34}$$

Indeed, if (34) holds, then by the connectedness of Ω_0 one has $U = \Omega_0$, i.e., $\tilde{v}_i = 0$ for all x in Ω_0 which will complete the proof of the theorem. Thus, we prove (34).

Let (z_n) be a sequence in U and let $z_0 \in \Omega_0$ such that

$$z_n \longrightarrow z_0 \quad \text{as} \quad n \to \infty. \tag{35}$$

From (35), we can find an n_0 and an $r > 0$ such that the ball B_r of radius r centered at z_{n_0} satisfies

$$z_0 \in B_{r/4}, \quad B_r \subset \Omega_0. \tag{36}$$

We shall prove that $\tilde{v}_i(z) = 0$ for all $z \in B_{r/4}$.

Using the estimates in [P], one has for $\alpha, m > 0$ large enough

$$\int\limits_{B_r} \exp(\alpha\rho^{-m})|\Delta(\phi\tilde{v}_i)|^2 dV \geq \frac{m^2\alpha}{4} \int\limits_{B_r} \exp(\alpha\rho^{-m})\Big(\frac{|\nabla(\phi\tilde{v}_i)|^2}{\rho^{m+2}} + \frac{|\phi\tilde{v}_i|^2}{\rho^{3m+4}}\Big) dV, \tag{37}$$

where $\rho = |x - z_{n_0}|$, $dV = dx_1 dx_2 dx_3$, $i = 1, 2, 3, 4$, and $\phi = \phi(\rho)$ is a C^1-smooth function satisfying

$$\phi(\rho) = \begin{cases} 0 & \text{for } \rho \geq r, \\ 1 & \text{for } \rho < r/2. \end{cases}$$

Using (33), (37), we shall get, after some rearrangements (see [P] for the details), that

$$\exp(-\alpha(4^m - 2^m)r^{-m}) \int\limits_{B_r \setminus B_{r/2}} \sum_{i=1}^{4} |\Delta\phi\tilde{v}_i|^2 dV \geq \int\limits_{B_{r/4}} \sum_{i=1}^{4} \Big(\frac{|\nabla\tilde{v}_i|^2}{\rho^{m+2}} + \frac{|\tilde{v}_i|^2}{\rho^{3m+4}}\Big) dV.$$

By letting $\alpha \to +\infty$ we see that $\tilde{v}_i = 0$ on $B_{r/4}$. Since $z_0 \in B_{r/4}$, we get that $z_0 \in U$, i.e., U is closed in Ω_0. This completes the proof of our theorem. ∎

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VỀ TÍNH THÁC TRIỂN DUY NHẤT
CỦA VẬT THỂ ĐÀN HỒI KHÔNG THUẦN NHẤT

Các tác giả chứng minh một định lý về tính thác triển duy nhất của hệ vật thể đàn hồi không thuần nhất với mô đun đàn hồi phụ thuộc vào dịch chuyển. Kết quả của bài báo này tổng quát hóa các kết quả của các tác giả Áng, Ikehata, Trọng, Yamamoto về hệ vật thể đàn hồi tuyến tính.

TÁC GIẢ GỬI BÀI ĐĂNG CHÚ Ý

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