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FORMULAS FOR THE VELOCITY OF STONELEY WAVES AT THE UNBONDED INTERFACE BETWEEN TWO MICROPOLAR ELASTIC HALF-SPACES

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Abstract. In this paper, we investigate the propagation of Stoneley waves along the unbonded interface between two micropolar elastic half-spaces. Using the complex function method, we derive explicit formulas for the wave velocity. These formulas have applications in various scientific fields, particularly in non-destructive evaluation. Two numerical examples illustrate how the obtained wave velocity formulas can be used to evaluate material parameters non-destructively.

Keywords: Stoneley waves, micropolar half-spaces, the complex function method, formulas for the Stoneley wave velocity, unbonded interface.

1. INTRODUCTION

The propagation of Stoneley waves along the welded interface between two dissimilar isotropic elastic half-spaces, with wave amplitudes decaying away from the interface, was first investigated by Stoneley [1] in 1924. Initially, applications of Stoneley waves were mainly found in seismology; however, they have increasingly attracted the attention of researchers due to their significant potential in various scientific and engineering fields. Phan et al. [2] employed the reciprocity theorem to derive closed-form expressions for the Stoneley wave amplitudes and the scattering caused by interfacial delamination. Gu et al. [3] investigated the influence of nematic elastic material parameters on the Stoneley wave velocity, contributing to the design of smart interfaces.

Nowadays, new materials are frequently created in laboratories, and estimating their material parameters is extremely important. Surface waves in general and Stoneley waves in particular are useful tools for this task because their signals are easily detected and their velocities are easily measured. The inverse problem in nondestructive evaluation is formulated based on measured wave velocity values and the corresponding secular equation. Solving the inverse problem is fitting the experimental secular curve to the theoretical one in order to determine the material parameters. However, if explicit formulas for the wave velocity and slowness are available, the problem reduces to solving algebraic equations. This approach is significantly simpler than performing curve fitting between experimental and theoretical secular curves.

So far, a lot of formulas for the velocity of Rayleigh waves in different elastic materials have been derived (see [4–11]). However, only a few formulas for the velocity of Stoneley waves have been found due to the complexity of their secular equation. In particular, the velocity formulas for Stoneley waves propagating along the loosely bonded interface of two elastic half-spaces were presented by Vinh and Giang [12], and those for propagation along the bonded interface of two elastic half-spaces with identical bulk wave velocities were provided by Vinh et al. [13].

In recent years, internal structures or microstructured materials have become increasingly common in practical applications, such as porous materials [14], reinforced soils [15], and granular materials [16]. Bones (of both human and animals) are also considered natural structural materials and are modeled as micropolar elastic media in [17]. The classical elasticity theory is insufficient to describe the micro-behavior of these materials. To address this, Eringen [18, 19] proposed the theory of micropolar elasticity and microstretch elasticity. Using the microstretch theory, Tomar and Dilbag Singh [20] investigated Stoneley waves and derived the secular equation for the wave. Using micropolar theory, Eringen [19] studied the propagation of Rayleigh waves in an isotropic micropolar elastic medium and derived the secular equation in explicit form. Tajuddin [21] derived the secular equation for Stoneley waves propagating along the unbonded interface between two isotropic micropolar elastic half-spaces and proved the existence of Stoneley waves in two special cases: when the half-spaces are either incompressible or Poisson materials with closely related properties. Since the corresponding material parameters of the two half-spaces are very close, Tajuddin supposed the Stoneley wave velocity as a perturbation of the Rayleigh wave velocity and applied an asymptotic expansion method to derive the Stoneley wave velocity formula. As a result, the velocity formulas presented in that work are approximate. In contrast, the study by Giang and Vinh [22] investigates Stoneley wave propagation along the interface between two generally isotropic micropolar elastic half-spaces, without imposing any specific assumptions on the material parameter values. In their work, the authors provided the necessary and sufficient conditions for the existence of Stoneley waves, along with explicit formulas for Stoneley wave slowness.

In the formulation and solution of inverse problems, the more independent equations that can be established, the more material parameters can be determined. Explicit velocity formulas are particularly valuable in inverse problems, as they provide independent equations.

In this paper, the velocity formulas for Stoneley waves propagating along the unbonded interface between two isotropic micropolar elastic half-spaces are derived using the complex function method [23, 24]. It should be noted that the velocity formulas derived in this paper are completely independent of the slowness formulas derived by Giang and Vinh [22]. Two numerical examples (Subsections 4.1 and 4.2) are presented to illustrate the use of the wave velocity formula in solving inverse problems. In these examples, it is assumed that only one material parameter is unknown, while the others have been determined by other methods. An algebraic equation is then formulated based on the wave velocity formula and the measured wave velocity. Solving this equation yields the value of the unknown material parameter. However, when the velocity and slowness formulas are combined, two independent algebraic equations can be established from a single velocity measurement, allowing the simultaneous determination of two material parameters (see the example presented in Subsection 4.3). Therefore, the velocity formulas derived in this paper, together with the slowness formulas in [22], are useful tools for solving inverse problems.

2. SECULAR EQUATION

Consider a Stoneley wave propagating along the interface between two micropolar elastic half-spaces, denoted by Ω ($x_3 \leq 0$) and Ω^* ($x_3 \geq 0$). Assume that the interface at $x_3 = 0$ is unbonded, meaning that the normal displacement, the normal stress, the couple stress, and the microrotation are continuous, while the shear stress components vanish across the interface [21]. The wave propagates in the x_1 direction with velocity c. The secular equation for the Stoneley wave was first introduced by Tajuddin [21], although the derivation was not provided in detail. In a more recent study, Giang and Vinh [22] presented the rigorous and detailed derivation of the secular equation, which is given as follows

$$\frac{\rho^* {c_T^*}^4}{\sqrt{1 - c^2({c_L^*}^2 + \varepsilon^* {c_T^*}^2)^{-1}}} R^*(c) + \frac{\rho c_T^4}{\sqrt{1 - c^2(c_L^2 + \varepsilon c_T^2)^{-1}}} R(c) = 0, \tag{1}$$

in which

$$R(c) = (2 + \varepsilon - \frac{c^2}{{c_T}^2})^2 - (2 + \varepsilon)^2 \sqrt{1 - \frac{c^2}{{c_L}^2 + \varepsilon {c_T}^2}} \sqrt{(1 - \frac{c^2}{(1+\varepsilon){c_T}^2)}}, \tag{2}$$

$$R^*(c) = (2+\varepsilon^* - \frac{c^2}{c_T^{*2}})^2 - (2+\varepsilon^*)^2\sqrt{1 - \frac{c^2}{c_L^{*2} + \varepsilon^* c_T^{*2}}}\sqrt{(1 - \frac{c^2}{(1+\varepsilon^*)c_T^{*2}}}. \quad (3)$$

In the above equation, $c_T = \sqrt{\frac{\mu}{\rho}}, c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$ are the speeds of transverse and longitudinal waves, respectively; ρ is the mass density; and λ, μ are Lamé constants. $\varepsilon = \frac{\kappa}{\mu}$ where κ is the micropolar constant of the half-space Ω. The same quantities related to Ω^* have the same symbol but are systematically distinguished by an asterisk.

Multiplying both sides of (3) by $\sqrt{1 - c^2(c_L^2 + \varepsilon c_T^2)^{-1}}$, we have

$$\begin{aligned} &\rho^* c_T^{*4} \frac{\sqrt{1 - c^2(c_L^2 + \varepsilon c_T^2)^{-1}}}{\sqrt{1 - c^2(c_L^{*2} + \varepsilon^* c_T^{*2})^{-1}}} \Big\{ (2+\varepsilon^* - c^2 c_T^{*-2})^2 \\ &- (2+\varepsilon^*)^2 \sqrt{1 - c^2(c_L^{*2} + \varepsilon^* c_T^{*2})^{-1}} \sqrt{(1 - c^2(1+\varepsilon^*)^{-1} c_T^{*-2})} \Big\} \\ &+ \rho c_T^4 \Big\{ (2+\varepsilon - c^2 {c_T}^{-2})^2 - (2+\varepsilon)^2 \sqrt{1 - c^2({c_L}^2 + \varepsilon {c_T}^2)^{-1}} \sqrt{1 - c^2(1+\varepsilon)^{-1}{c_T}^{-2}} \Big\} = 0. \end{aligned} \quad (4)$$

We introduce the following notations

$$x = \frac{c^2}{(1+\varepsilon)c_T^2}, \quad a = \frac{c_T^2}{c_T^{*2}}, \quad b = \frac{(1+\varepsilon)c_T^2}{c_L^{*2} + \varepsilon^* c_T^{*2}}, \quad (5)$$

$$d = \frac{(1+\varepsilon)c_T^2}{(1+\varepsilon^*)c_T^{*2}}, \quad e = \frac{(1+\varepsilon)c_T^2}{{c_L}^2 + \varepsilon {c_T}^2}, \quad f = \frac{\rho}{\rho^*}. \quad (6)$$

In terms of these notations, the secular equation is rewritten as

$$\begin{aligned} f(x) := \frac{\sqrt{1-ex}}{\sqrt{1-bx}} \Big\{ \big[2+\varepsilon^* - x(1+\varepsilon)a\big]^2 - (2+\varepsilon^*)^2\sqrt{1-bx}\sqrt{1-dx} \Big\} \\ + fa^2 \Big\{ \big[2+\varepsilon - x(1+\varepsilon)\big]^2 - (2+\varepsilon)^2 \sqrt{1-ex}\sqrt{1-x} \Big\} = 0. \end{aligned} \quad (7)$$

Without loss of generality, we can suppose that $(1+\varepsilon)c_T^2 \leq (1+\varepsilon^*)c_T^{*2}$. With this assumption, we have (see [22]):

Remark 1. *The necessary and sufficient condition for a Stoneley wave to exist is that Eq. (7) has a root in the interval* $(0, 1)$.

In addition, from the fact $c_T < c_L, c_T^* < c_L^*$, *it is easy to see that* $b, e < 1, b < d, d \leq 1$. *Therefore, there are three basic possibilities (see [22]):*

Case 1: $b < d < e < 1$.

Case 2: $b < e < d < 1$.

Case 3: $e < b < d < 1$.

and some special cases

Case 1.1: $b < d = e < 1$.

Case 2.1: $b = e < d < 1$.

Case 2.2: $b < e < d = 1$.

Case 2.3: $b = e < d = 1$.

Case 3.1: $e < b < d = 1$.

In the complex plane $\mathbf{C}$, *we consider the equation:*

$$F(z) = \frac{\sqrt{e}\sqrt{z-1/e}}{\sqrt{b}\sqrt{z-1/b}}\Big((2+\varepsilon^* - z(1+\varepsilon^*)a)^2 + (2+\varepsilon^*)^2\sqrt{bd}\sqrt{z-1/b}\sqrt{z-1/d}\Big) + fa^2\Big((2+\varepsilon - z(1+\varepsilon))^2 + (2+\varepsilon)^2\sqrt{e}\sqrt{z-1/e}\sqrt{z-1}\Big) = 0, \tag{8}$$

where $\sqrt{z-1/e}$, $\sqrt{z-1/b}$, $\sqrt{z-1/d}$, $\sqrt{z-1}$ *are chosen as the principal branches of the corresponding square roots. Eq. (8) coincides with Eq. (7) for real values of z belonging to* $(0,1)$.

Remark 2. $z_0 = 0$ *is a root of Eq. (8).*

3. FORMULAS FOR THE VELOCITY

In this section, we will find a root of Eq. (7) corresponding to the Stoneley wave velocity by employing the complex function method [10–13]. The complex function method is a powerful mathematical tool that was first used by Nkemzi [4] to derive the Rayleigh wave velocity formula in an isotropic elastic half-space. Later, this method was successfully employed by Vinh, Giang, and their collaborators to solve the secular equations of Rayleigh and Stoneley waves to derive the wave velocity formulas for these waves [12, 13]. When the elastic medium is anisotropic, the secular equations become more complicated, harder to solve, and, in many cases, remain open. So far, the complex function method can be used not only to derive explicit formulas for surface wave velocities, but also to establish necessary and sufficient conditions for the existence of such waves [10, 11, 22, 25].

3.1. Case 1: $0 < b < d < e < 1$

Denote $L = L_1 \cup L_2 \cup L_3$ with $L_1 = [1, 1/e]$, $L_2 = [1/e, 1/d]$, $L_3 = [1/d, 1/b]$, $S = \{z \in \mathbf{C}, z \notin L\}$, $N(z_0) = \{z \in S : 0 < |z - z_0| < \delta\}$, where δ is a sufficiently small positive number and z_0 is some point in the complex plane $\mathbf{C}$. If a function $\phi(z)$ is holomorphic in

$\Omega \subset \mathbf{C}$, we write $\phi(z) \in H(\Omega)$. From Eq. (8), it is not difficult to show that the function $F(z)$ has the following properties:

- (f_1) $F(z) \in H(S)$.

- (f_2) $F(z)$ is bounded in $N(1)$.

- (f_3) $F(z) = O\left(z - \dfrac{1}{b}\right)^{1/2}$ as $|z| \to \dfrac{1}{b}$.

- (f_4) $F(z) = O(z^2)$ as $|z| \to \infty$.

- (f_5) $F(z)$ is continuous on L from the left and from the right (see [23]), with the boundary values $F^+(t)$ (the right boundary value of $F(z)$) and $F^-(t)$ (the left boundary value of $F(z)$) defined as follows

$$F_k^{\pm}(t) = R_k(t) \pm i I_k(t), \quad t \in L_k, \quad k = \overline{1:3}, \tag{9}$$

where

$$\begin{aligned} R_1(t) = &\frac{\sqrt{e}\sqrt{1/e - t}}{\sqrt{b}\sqrt{1/b - t}}\Big((2 + \varepsilon^* - t(1+\varepsilon)a)^2 - (2+\varepsilon^*)^2\sqrt{bd}\sqrt{1/b - t}\sqrt{1/d - t}\Big) \\ &+ fa^2(2 + \varepsilon - t(1+\varepsilon)^2, \end{aligned} \tag{10}$$

$$I_1(t) = fa^2(2+\varepsilon)^2\sqrt{e}\sqrt{1/e - t}\sqrt{t-1}), \tag{11}$$

$$R_2(t) = fa^2\Big((2 + \varepsilon - t(1+\varepsilon))^2 + (2+\varepsilon)^2\sqrt{e}\sqrt{t - 1/e}\sqrt{t-1}\Big), \tag{12}$$

$$I_2(t) = \sqrt{e}\sqrt{t - 1/e}(2+\varepsilon^*)^2\sqrt{d}\sqrt{1/d - t} - \frac{\sqrt{e}\sqrt{t - 1/e}}{\sqrt{b}\sqrt{1/b - t}}(2 + \varepsilon^* - t(1+\varepsilon)a)^2, \tag{13}$$

$$\begin{aligned} R_3(t) = &fa^2((2 + \varepsilon - t(1+\varepsilon))^2 + (2+\varepsilon)^2\sqrt{e}\sqrt{t - 1/e}\sqrt{t-1}) \\ &+ (2+\varepsilon^*)^2\sqrt{ed}\sqrt{t - 1/e}\sqrt{t - 1/d}, \end{aligned} \tag{14}$$

$$I_3(t) = -\frac{\sqrt{e}\sqrt{t - 1/e}}{\sqrt{b}\sqrt{1/b - t}}(2 + \varepsilon^* - t(1+\varepsilon)a)^2. \tag{15}$$

Now, we introduce the function

$$g(t) = \frac{F^+(t)}{F^-(t)}, \quad t \in L. \tag{16}$$

Since $F^+(t)$ and $F^-(t)$ are conjugate to each other on L, we can easily deduce that

$$\log g(t) = i\phi(t), \tag{17}$$

in which

$$\phi(t) = \operatorname{Arg} g(t) = 2\operatorname{Arg} F^+(t), \quad t \in L. \tag{18}$$

Note that if $F^+(t)$ vanishes or goes to ∞ at some $t \in L$, $g(t)$ is understood as

$$g(t) = \lim_{\tau \to t} \frac{F_k^+(\tau)}{F_k^-(\tau)}. \tag{19}$$

Using Eqs. (10)–(15), it is not difficult to show that

$$g(1) = \begin{cases} 1, & \text{if } R_1(1) > 0 \\ \mathbf{e}^{i\pi}, & \text{if } R_1(1) = 0 \\ \mathbf{e}^{i2\pi}, & \text{if } R_1(1) < 0 \end{cases} \qquad g(1/b) = \mathbf{e}^{-i\pi}. \tag{20}$$

From (16), it is clear that

$$F^+(t) = g(t)F^-(t), \quad t \in L. \tag{21}$$

Consider the function $\Gamma(z)$ defined by

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\log g(t)}{t-z}\mathrm{d}t. \tag{22}$$

It is not difficult to verify that

$(\gamma_1)\,\Gamma(z) \in H(S)$,

$(\gamma_2)\,\Gamma(\infty) = 0$,

$$(\gamma_3)\,\Gamma(z) = \begin{cases} \Omega_0(z), & \text{if } R_1(1) > 0 \\ \dfrac{1}{2}\log\dfrac{1}{z-1} + \Omega_1(z), & \text{if } R_1(1) = 0 \\ \log\dfrac{1}{z-1} + \Omega_2(z), & \text{if } R_1(1) < 0 \end{cases} \qquad z \in N(1)$$

where $\Omega_0(z), \Omega_1(z), \Omega_2(z)$ are bounded in $N(1)$ and take a defined value at $z = 1$. (γ_4) $\Gamma(z) = \dfrac{1}{2}\log\dfrac{1}{z-1/b} + \Omega_3(z)$ for $z \in N(1/b)$, where $\Omega_3(z)$ is bounded in $N(1/b)$ and takes a defined value at $z = 1/b$.

Introduce the function $\Phi(z)$ given by

$$\Phi(z) = \exp\Gamma(z). \tag{23}$$

It is implied from $(\gamma_1) - (\gamma_4)$ that

(ϕ_1) $\Phi(z) \in H(S)$,

(ϕ_2) $\Phi(z) \neq 0, \quad \forall z \in S$,

$$(\phi_3)\ \Phi(z) = \begin{cases} \mathbf{e}^{\Omega_0(z)}, & \text{if } R_1(1) > 0 \\ (z-1)^{-1/2}\mathbf{e}^{\Omega_1(z)}, & \text{if } R_1(1) = 0 \\ (z-1)\mathbf{e}^{\Omega_2(z)}, & \text{if } R_1(1) < 0 \end{cases} \qquad z \in N(1)$$

$$(\phi_4)\ \Phi(z) = \left(z - \frac{1}{b}\right)^{-1/2} \mathbf{e}^{\Omega_3(z)} \text{ for } z \in N\left(\frac{1}{b}\right).$$

From the Plemelj formula [23], the function $\Phi(z)$ is seen directly to satisfy the boundary condition

$$\Phi^+(t) = g(t)\Phi^-(t), \quad t \in L. \tag{24}$$

We now consider the function $Y(z)$ defined by

$$Y(z) = F(z)/\Phi(z). \tag{25}$$

From $(f_1) - (f_5)$, (16), $(\phi_1) - (\phi_4)$ and (24), it follows that

(y_1) $Y(z) \in H(S)$,

(y_2) $Y(z) = O(z^2)$ as $|z| \to \infty$,

$$(y_3)\ Y(z) = \begin{cases} \mathbf{e}^{-\Omega_0(z)}F(z), & \text{in the case } R_1(1) > 0 \\ (z-1)^{1/2}\mathbf{e}^{-\Omega_1(z)}F(z), & \text{in the case } R_1(1) = 0 \\ (z-1)\mathbf{e}^{-\Omega_2(z)}F(z), & \text{in the case } R_1(1) < 0 \end{cases} \qquad z \in N(1)$$

(y_4) $Y(z) = \mathbf{e}^{-\Omega_3(z)}(z - 1/b)^{1/2}F(z) = O(1)$ for $z \in N(1/b)$,

(y_5) $Y^+(t) = Y^-(t), \quad t \in L$.

Properties (y_1) and (y_5) of the function $Y(z)$ show that $Y(z)$ is holomorphic in the entire complex plane **C** with the possible exception of the points $z = 1$ and $z = 1/b$. By (y_3) and (y_4), these points are removable singular points, and it may be assumed that the function $Y(z)$ is holomorphic in the entire complex plane **C** (see [24]). Thus, by the generalised Liouville theorem [24] and taking into account (y_2), $Y(z)$ is a second-order polynomial

$$Y(z) := P(z) = A_2z^2 + A_1z + A_0, \tag{26}$$

where A_2, A_1, A_0 are constants.

From (25) and (26), we have

$$F(z) = \Phi(z)P(z). \tag{27}$$

Since $\Phi(z) \neq 0, \forall z \in S$. Therefore, we have the following proposition:

Proposition 1.

$$F(z) = 0 \leftrightarrow P(z) = 0, \quad \forall z \in S. \tag{28}$$

From Remark 2 and this proposition, we deduce that $P(z)$ has a root $z_0 = 0$. The remaining root, denoted by x_s, corresponds to the Stoneley wave velocity, if the Stoneley wave exists. Readers may refer to Ref. [22] for more details on the existence conditions and the uniqueness of the Stoneley wave.

To find x_s, we first need to determine the coefficients of $P(z)$. From (23) and (27) we have

$$P(z) = F(z)e^{-\Gamma(z)}. \tag{29}$$

From (23) and (18), it follows (see also [4])

$$-\Gamma(z) = \sum_{n=0}^{\infty} \frac{I_n}{z^{n+1}}, \tag{30}$$

in which

$$I_n = \frac{1}{2\pi} \int_L t^n \phi(t) \mathrm{d}t, \quad n = 0, 1, 2, 3, \dots \tag{31}$$

Using (30), we can express $e^{-\Gamma(z)}$ as follows

$$e^{-\Gamma(z)} = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + O(z^{-3}), \tag{32}$$

where a_1, a_2, a_3 are constants to be determined. Employing the identity

$$\left(e^{-\Gamma(z)}\right)' = \left(-\Gamma(z)\right)' e^{-\Gamma(z)}, \tag{33}$$

and substituting (30) and (32) into (33) yields

$$a_1 = I_0, \quad a_2 = \frac{I_0^2}{2} + I_1. \tag{34}$$

By expanding $F(z)$ into a Laurent series at infinity, it is not difficult to verify that

$$F(z) = B_2 z^2 + B_1 z + B_0 + O(z^{-1}), \tag{35}$$

where

$$B_2 = \sqrt{e/b}(a^2 + 2a^2\varepsilon^* + a^2\varepsilon^{*2}) + a^2(1 + 2\varepsilon + \varepsilon^2)f, \tag{36}$$

$$\begin{aligned} B_1 = \sqrt{e/b}\big(& -4a - 6a\varepsilon^* - 2a\varepsilon^2 + \sqrt{bd}(2+\varepsilon^*)^2 \\ & + (1/2b - 1/2e)(a^2 + 2a^2\varepsilon^* + a^2\varepsilon^{*2})\big) + a^2 f(-4 - 6\varepsilon - 2\varepsilon^2 + \sqrt{e}(2+\varepsilon)^2). \end{aligned} \tag{37}$$

Substituting (32) and (35) into (29) yields

$$P(z) = B_2 z^2 + (B_2 a_1 + B_1)z + (B_2 a_2 + B_1 a_1 + B_0). \tag{38}$$

From Remark 2 and (28), we have

$$P(0) = 0. \tag{39}$$

Using (34), (37), (38), and Vieta's theorem, it is easy to see that the second root of the equation $P(z) = 0$, denoted by x_s is

$$x_s = -B_1/B_2 - I_0. \tag{40}$$

Theorem 3.1. *In the case $b < d < e < 1$, if the Stoneley wave exists, the formula for the velocity is*

$$x_s = -B_1/B_2 - I_0, \tag{41}$$

where B_2, B_1 are represented by (36) and (37) and I_0 is calculated as follows

$$I_0 = \frac{1}{\pi}\sum_{k=1}^{3}\int_{L_k} Arg\ F_k^+(t)\mathrm{d}t, \tag{42}$$

in which $F_k^+(t) = R_k + iI_k$ and $R_k, I_k, k = \overline{1,3}$ are given by: (9)–(15).

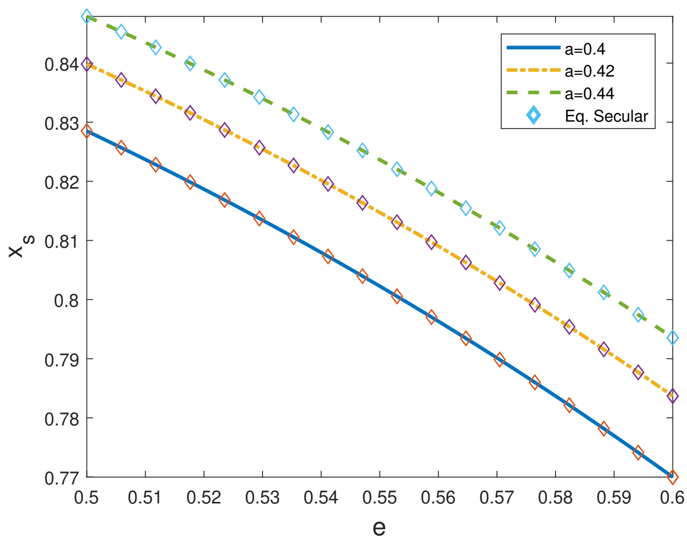

Fig. 1. The dependence of the Stoneley wave velocity on the parameter e for case 1 with $b = 0.3, f = 3, \varepsilon = 0.003, \varepsilon^* = 0.005$

In Fig. 1, the solid, dash-dot, and dashed curves represent the dependence of the dimensionless velocity x_s on e for the cases $a = 0.4, a = 0.42, a = 0.44$, respectively, computed using formula (41). The diamond curve shows the velocity values obtained by numerically solving the secular equation (7). It can be found from the figure that the results obtained from formula (41) coincide with those from the numerical solution of the secular equation, thereby confirming the validity of the derived formula.

3.2. Case 2: $0 < b < e < d < 1$

In this section, $L = L_1 \cup L_2 \cup L_3$ with $L_1 = [1, 1/d], L_2 = [1/d, 1/e], L_3 = [1/e, 1/b]$, $S = \{z \in \mathbf{C}, z \notin L\}$. With the same procedure in the previous section, we have the following theorem:

Theorem 3.2. *In the case $0 < b < e < d < 1$, if the Stoneley wave exists, the formula for the velocity is:*

$$x_s = -B_1/B_2 - I_0, \tag{43}$$

where B_1, B_2 are represented by (36) and (37), and I_0 is calculated as follows

$$I_0 = \frac{1}{\pi} \sum_{k=1}^{3} \int_{L_k} Arg\ F_k^+(t) \mathrm{d}t, \tag{44}$$

in which $F_k^+(t) = R_k + iI_k$ and $R_k, I_k, k = 1, 3$ are given by (10), (11), (14), (15) and $R_2(t), I_2(t)$ are given by

$$R_2 = \frac{\sqrt{e}\sqrt{1/e - t}}{\sqrt{b}\sqrt{1/b - t}} \left(2 + \varepsilon^* - t(1 + \varepsilon^*)a\right)^2 + fa^2 \left(2 + \varepsilon - t(1 + \varepsilon)\right)^2, \tag{45}$$

$$I_2 = \sqrt{e}(2 + \varepsilon^*)^2 \sqrt{d} \sqrt{\frac{1}{e} - t} \sqrt{t - \frac{1}{d}} + fa^2((2 + \varepsilon)^2 \sqrt{e} \sqrt{\frac{1}{e} - t} \sqrt{t - 1}. \tag{46}$$

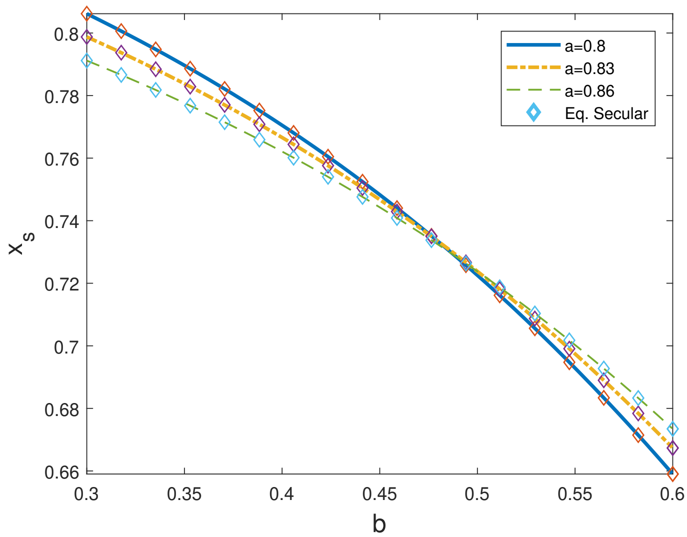

Fig. 2. The dependence of the Stoneley wave velocity on the parameter b for case 2 with $e = 0, f = 2, \varepsilon = 0.005, \varepsilon^* = 0.004$

In Fig. 2, the solid, dash-dot, and dashed curves represent the dependence of the dimensionless velocity x_s on b for the cases $a = 0.8, a = 0.83, a = 0.86$, respectively, computed using formula (43). The diamond curve shows the velocity values obtained by numerically solving the secular equation (7). It can be found from the figure that the results obtained from formula (43) coincide with those from the numerical solution of the secular equation, thereby confirming the validity of the derived formula.

3.3. Case 3: $0 < e < b < d < 1$

In this case, since $1/e$ (instead of $1/b$) is the right endpoint of L, we use the equation

$$\begin{aligned}\hat{F}(z) = \sqrt{\frac{e}{b}}\Big((2+\varepsilon^* - z(1+\varepsilon^*)a)^2 + (2+\varepsilon^*)^2\sqrt{bd}\sqrt{z-1/b}\sqrt{z-1/d}\Big)\\ + fa^2\frac{\sqrt{z-1/b}}{\sqrt{z-1/e}}\Big((2+\varepsilon - z(1+\varepsilon))^2 + (2+\varepsilon)^2\sqrt{e}\sqrt{z-1/e}\sqrt{z-1}\Big) = 0.\end{aligned} \tag{47}$$

Denote $L = L_1 \cup L_2 \cup L_3$ with $L_1 = [1, 1/d], L_2 = [1/d, 1/b], L_3 = [1/b, 1/e]$, $S = \{z \in \mathbf{C}, z \notin L\}$. It is not difficult to show that the function $\hat{F}(z)$ has following the properties:

- $(\hat{f}_1)$ $\hat{F}(z) \in H(S)$.
- $(\hat{f}_2)$ $\hat{F}(z)$ is bounded in $N(1)$.
- $(\hat{f}_3)$ $\hat{F}(z) = O\left(z - \dfrac{1}{e}\right)^{1/2}$ as $|z| \to \dfrac{1}{e}$.
- $(\hat{f}_4)$ $\hat{F}(z) = O(z^2)$ as $|z| \to \infty$.
- $(\hat{f}_5)$ $\hat{F}(z)$ is continuous on L from the left and from the right (see [23]) with the boundary values $\hat{F}^+(t)$ (the right boundary value of $\hat{F}(z)$) and $\hat{F}^-(t)$ (the left boundary value of $\hat{F}(z)$) defined as follows

$$\hat{F}_k^{\pm}(t) = \hat{R}_k(t) \pm i\hat{I}_k(t), \quad t \in L_k, \quad k = \overline{1:3}, \tag{48}$$

where

$$\begin{aligned}\hat{R}_1(t) = \sqrt{\frac{e}{b}}\big((2+\varepsilon^* - t(1+\varepsilon^*)a)^2 - (2+\varepsilon^*)^2\sqrt{bd}\sqrt{1/b-t}\sqrt{1/d-t}\big)\\ + fa^2\frac{\sqrt{1/b-t}}{\sqrt{1/e-t}}\Big(2+\varepsilon - t(1+\varepsilon)\Big)^2,\end{aligned} \tag{49}$$

$$\hat{I}_1(t) = fa^2(2+\varepsilon)^2\sqrt{e}\sqrt{1/b-t}\sqrt{t-1}, \tag{50}$$

$$\hat{R}_2(t) = \sqrt{\frac{e}{b}}(2+\varepsilon^* - t(1+\varepsilon^*)a)^2 + fa^2\frac{\sqrt{1/b-t}}{\sqrt{1/e-t}}\Big((2+\varepsilon - t(1+\varepsilon)\Big)^2, \tag{51}$$

$$\hat{I}_2(t) = \sqrt{ed}(2+\varepsilon^*)^2\sqrt{1/b-t}\sqrt{t-1/d} + fa^2(2+\varepsilon)^2\sqrt{e}\sqrt{1/b-t}\sqrt{t-1}, \tag{52}$$

$$\hat{R}_3(t) = \sqrt{\frac{e}{b}}(2+\varepsilon^* - t(1+\varepsilon^*)a)^2 + fa^2(2+\varepsilon)^2\sqrt{e}\sqrt{t-1/b}\sqrt{t-1} + \sqrt{ed}\sqrt{t-1/b}\sqrt{t-1/d}(2+\varepsilon^*)^2, \tag{53}$$

$$\hat{I}_3(t) = -fa^2\frac{\sqrt{t-1/b}}{\sqrt{1/e-t}}\Big(2+\varepsilon - t(1+\varepsilon)\Big)^2, \tag{54}$$

$$\hat{g}(t) = \frac{\hat{F}^+(t)}{\hat{F}^-(t)}, \quad t \in L. \tag{55}$$

Using Eqs. (49)–(54), it is not difficult to show that

$$\hat{g}(1) = \begin{cases} 1, & \text{if } \hat{R}_1(1) > 0 \\ \mathbf{e}^{i\pi}, & \text{if } \hat{R}_1(1) = 0 \\ \mathbf{e}^{i2\pi}, & \text{if } \hat{R}_1(1) < 0 \end{cases} \qquad \hat{g}(1/e) = \mathbf{e}^{-i\pi}. \tag{56}$$

Therefore, following the same procedure as in Case 1, we have the following theorem:

Theorem 3.3. *In the case $0 < e < b < d < 1$, if the Stoneley wave exists, the formula for the velocity is*

$$x_s = -\hat{B}_1/\hat{B}_2 - I_0, \tag{57}$$

where $\hat{B}_1, \hat{B}_2$ are represented as follow

$$\hat{B}_2 = \sqrt{e/b}(a^2 + 2a^2\varepsilon^* + a^2\varepsilon^{*2}) + fa^2(1+2\varepsilon+\varepsilon^2), \tag{58}$$

$$\hat{B}_1 = \sqrt{e/b}\big(-4a - 6a\varepsilon^* - 2a\varepsilon^{*2} + \sqrt{bd}(2+\varepsilon^*)^2\big) + fa^2\big(-4 - 6\varepsilon - 2\varepsilon^{*2} + \sqrt{e}(2+\varepsilon)^2 + (-1/2b + 1/2e)(1+2\varepsilon^* + \varepsilon^{*2})\big), \tag{59}$$

and $\hat{I}_0$ is calculated as follows

$$\hat{I}_0 = \frac{1}{\pi}\sum_{k=1}^{3}\int_{L_k} Arg\ \hat{F}_k^+(t)\mathrm{d}t. \tag{60}$$

In Fig. 3, the solid, dash-dot, and dashed curves represent the dependence of the dimensionless velocity x_s on a for the cases $e = 0.4, e = 0.42, e = 0.44$, respectively, computed using formula (57). The diamond curve shows the velocity values obtained by numerically solving the secular equation (7). It can be found from the figure that the results obtained from formula (57) coincide with those from the numerical solution of the secular equation, thereby confirming the validity of the derived formula.

Remark 3. *The velocity formulas for Stoneley waves in special cases can be easily obtained from those for the three main cases.*

Remark 4. *The results in this paper recover those in the paper [12] for the case when two half-spaces are purely elastic by taking $\varepsilon = \varepsilon^* = 0$.*

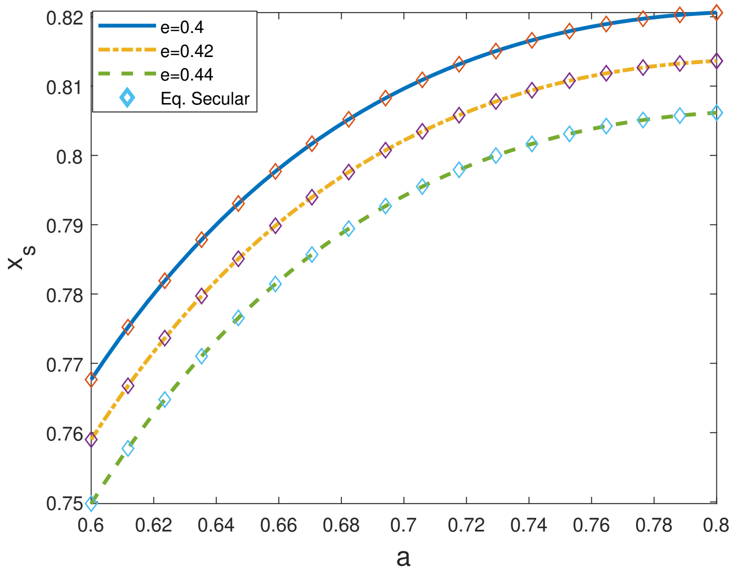

Fig. 3. The dependence of the Stoneley wave velocity on the parameter a for case 3 with $b = 0.5, f = 2, \varepsilon = 0.005, \varepsilon^* = 0.004$

4. EVALUATION OF MECHANICAL PROPERTIES OF TWO HALF-SPACES BY USING FORMULAS OF STONELEY WAVE VELOCITY

As emphasized in Giang's thesis [26], the velocity formulas for surface waves, including Stoneley waves, are extremely convenient tools for non-destructively evaluating the mechanical characteristics of structures before and during loading. In this section, two examples are presented to illustrate the application of the Stoneley wave velocity formula in evaluating material parameters.

4.1. Evaluation of c_T^*

In this subsection, we assume that the material parameters, except for c_T^*, have been measured. The input data consists of $c_T = \sqrt{2}\,\text{m/s}, c_L = \sqrt{6}\,\text{m/s}, c_L^* = \sqrt{8}\,\text{m/s}, f = 2, \varepsilon = 0.04, \varepsilon^* = 0.06$, along with the measured dimensionless Stoneley wave velocity x_m. The output is the transverse wave velocity c_T^* of the half-space Ω^*. Due to the lack of actual measured results, the input velocity data is synthetic, created from the correct wave velocity values (41) for the case $c_T^* = \sqrt{7}\,\text{m/s}$ with a noise level of 1%. To derive the solution to the problem, the author developed a program to solve the equation

$$x_s(c_T^*) = x_m, \tag{61}$$

using the bisection method in Matlab R2017b.

In Eq. (61), $x_s(c_T^*)$ is a function of c_T^* constructed in Matlab using the obtained formula for the Stoneley wave velocity (41) and the input data $c_T = \sqrt{2}$ m/s, $c_L = \sqrt{6}$ m/s, $c_L^* = \sqrt{8}$ m/s, $f = 2, \varepsilon = 0.04$ and $\varepsilon^* = 0.06$. The obtained result is $c_T^* = 2.673$ m/s with a relative error of 1%.

4.2. Evaluation of c_L^*

In this subsection, we assume that the material parameters, except for c_L^*, have been measured. The input data consists of $c_T = \sqrt{2}$ m/s, $c_L = \sqrt{5}$ m/s, $c_T^* = 2$ m/s, $f = 2, \varepsilon = 0.03$, and $\varepsilon^* = 0.05$ along with x_m, which is generated from the correct wave velocity value calculated using formula (43) for the case $c_L^* = \sqrt{6.5}$ m/s with a noise level of 0.5%.

The function $x_s(c_L^*)$ is constructed in Matlab as a function of c_L^* using the formula for the Stoneley wave velocity (43) with the input data $c_T = \sqrt{2}$ m/s, $c_L = \sqrt{5}$ m/s, $c_T^* = 2$ m/s, $f = 2, \varepsilon = 0.03$ and $\varepsilon^* = 0.05$. Solving the equation

$$x_s(c_L^*) = x_m, \tag{62}$$

yields the result $c_L^* = 2.518$ m/s with a relative error of 1.2%.

4.3. Simultaneous evaluation of c_L^* and c_T^*

In this subsection, we assume that the material parameters, except for c_L^* and c_T^*, have been measured. The input data consists of $c_T = \sqrt{3}$ m/s, $c_L = \sqrt{6}$ m/s, $f = 2, \varepsilon = 0.002$, and $\varepsilon^* = 0.004$ along with x_m, which is generated from the correct wave velocity value calculated using the formula (43) for the case $c_L^* = \sqrt{6.2}$ m/s, $c_T^* = \sqrt{4.5}$ m/s, with a noise level of 0.5%.

The functions $x_s(c_L^*, c_T^*)$ and $y_s(c_L^*, c_T^*)$ are constructed in Matlab using the formulas for the Stoneley wave velocity (43) and slowness (Eq. (76) in [22]), respectively, with the input data $c_T = \sqrt{3}$ m/s, $c_L = \sqrt{6}$ m/s, $f = 2, \varepsilon = 0.002$, and $\varepsilon^* = 0.004$. Solving the equations

$$x_s(c_L^*, c_T^*) = x_m, \tag{63}$$

$$y_s(c_L^*, c_T^*) = \frac{1}{x_m}, \tag{64}$$

yields the results $c_T^* = 2.1395$ m/s, $c_L^* = 2.48998$ m/s with a relative errors of 0.855% and 1.4×10^{-4}%, respectively.

5. CONCLUSION

In this paper, formulas for the velocity of Stoneley waves propagating along the unbonded interface between two micropolar isotropic elastic half-spaces have been derived using the complex function method. The two numerical examples presented demonstrate that the derived wave velocity formulas are convenient tools for non-destructive evaluation.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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REFERENCES

[1] R. Stoneley. Elastic waves at the surface of separation of two solids. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, **106**, (1924), pp. 416–428. https://doi.org/10.1098/rspa.1924.0079.

[2] H. Phan, T. Q. Bui, H. T.-L. Nguyen, and C. V. Pham. Computation of interface wave motions by reciprocity considerations. *Wave Motion*, **79**, (2018), pp. 10–22. https://doi.org/10.1016/j.wavemoti.2018.02.008.

[3] Q. Gu, Y. Liu, and T. Liang. Stoneley wave at the interface of elastic-nematic elastomer half spaces. *Physica B: Condensed Matter*, **652**, (2023). https://doi.org/10.1016/j.physb.2022.414629.

[4] D. Nkemzi. A new formula for the velocity of Rayleigh waves. *Wave Motion*, **26**, (1997), pp. 199–205. https://doi.org/10.1016/s0165-2125(97)00004-8.

[5] P. G. Malischewsky. Comment to "A new formula for the velocity of Rayleigh waves" by D. Nkemzi [Wave Motion 26 (1997) 199–205]. *Wave Motion*, **31**, (2000), pp. 93–96. https://doi.org/10.1016/s0165-2125(99)00025-6.

[6] P. C. Vinh and R. W. Ogden. On formulas for the Rayleigh wave speed. *Wave Motion*, **39**, (2004), pp. 191–197. https://doi.org/10.1016/j.wavemoti.2003.08.004.

[7] P. C. Vinh and R. W. Ogden. Formulas for the Rayleigh wave speed in orthotropic elastic solids. *Archive of Mechanics*, **56**, (2004), pp. 247–265.

[8] P. C. Vinh. On formulas for the velocity of Rayleigh waves in prestrained incompressible elastic solids. *Journal of Applied Mechanics*, **77**, (2010), pp. 1–9. https://doi.org/10.1115/1.3197139.

[9] P. C. Vinh. On formulas for the Rayleigh wave velocity in pre-stressed compressible solids. *Wave Motion*, **48**, (2011), pp. 614–625. https://doi.org/10.1016/j.wavemoti.2011.04.015.

[10] P. C. Vinh and N. Q. Xuan. Rayleigh waves with impedance boundary condition: Formula for the velocity, existence and uniqueness. *European Journal of Mechanics - A/Solids*, **61**, (2017), pp. 180–185. https://doi.org/10.1016/j.euromechsol.2016.09.011.

[11] P. T. H. Giang and P. C. Vinh. On the existence of Rayleigh waves with full impedance boundary condition. *Mathematics and Mechanics of Solids*, **30**, (2024), pp. 927–941. https://doi.org/10.1177/10812865241266809.

[12] P. C. Vinh and P. T. H. Giang. On formulas for the velocity of Stoneley waves propagating along the loosely bonded interface of two elastic half-spaces. *Wave Motion*, **48**, (2011), pp. 647–657. https://doi.org/10.1016/j.wavemoti.2011.05.002.

[13] P. C. Vinh, P. G. Malischewsky, and P. T. H. Giang. Formulas for the speed and slowness of Stoneley waves in bonded isotropic elastic half-spaces with the same bulk wave velocities. *International Journal of Engineering Science*, **60**, (2012), pp. 53–58. https://doi.org/10.1016/j.ijengsci.2012.05.002.

[14] R. S. Lakes. Experimental microelasticity of two porous solids. *International Journal of Solids and Structures*, **22**, (1), (1986), pp. 55–63. https://doi.org/10.1016/0020-7683(86)90103-4.

[15] G. Hassen and P. de Buhan. A two-phase model and related numerical tool for the design of soil structures reinforced by stiff linear inclusions. *European Journal of Mechanics A/Solids*, **24**, (2005), pp. 987–1001. https://doi.org/10.1016/j.euromechsol.2005.06.009.

[16] N. P. Kruyt. On weak and strong contact force networks in granular materials. *International Journal of Solids and Structures*, **92–93**, (2016), pp. 135–140. https://doi.org/10.1016/j.ijsolstr.2016.02.039.

[17] J. F. C. Yang and R. S. Lakes. Experimental study of micropolar and couple stress elasticity in compact bone in bending. *Journal of Biomechanics*, **15**, (1982), pp. 91–98. https://doi.org/10.1016/0021-9290(82)90040-9.

[18] A. C. Eringen. Linear theory of micropolar elasticity. *Journal of Mathematics and Mechanics*, **15**, (6), (1966), pp. 909–923.

[19] C. Eringen. *Theory of microcontinuum field theories*. Springer, (1999).

[20] S. K. Tomar and D. Singh. Propagation of Stoneley waves at an interface between two microstretch elastic half-spaces. *Journal of Vibration and Control*, **12**, (2006), pp. 995–1009. https://doi.org/10.1177/1077546306068689.

[21] M. Tajuddin. Existence of Stoneley waves at an unbonded interface between two micropolar elastic half-spaces. *Journal of Applied Mechanics*, **62**, (1995), pp. 255–257. https://doi.org/10.1115/1.2895919.

[22] P. T. H. Giang and P. C. Vinh. Existence and uniqueness of micropolar elastic Stoneley waves with sliding contact and formulas for the wave slowness. *Journal of Applied Mechanics*, **92**, (2025). https://doi.org/10.1115/1.4066956.

[23] N. I. Muskhelishvili. *Singular integral equations*. Noordhoff-Groningen, (1953). https://doi.org/10.1007/978-94-009-9994-7.

[24] N. I. Muskhelishvili. *Some basic problems of mathematical theory of elasticity*. Noordhoff, Netherlands, (1963). https://doi.org/10.1007/978-94-017-3034-1.

[25] P. T. H. Giang and P. C. Vinh. Existence and uniqueness of Rayleigh waves with normal impedance boundary conditions and formula for the wave velocity. *Journal of Engineering Mathematics*, **130**, (2021). https://doi.org/10.1007/s10665-021-10170-y.

[26] P. T. H. Giang. *The velocity formulas of Rayleigh waves and Stoneley waves*. Phd thesis, VNU University of Science, (2017).