Vietnam Journal of Mechanics, Vol. 44, No. 3 (2022), pp. 280 – 290
DOI: https://doi.org/10.15625/0866-7136/17546

A NOVEL DAMAGE INDEX EXTRACTED FROM FREQUENCY RESPONSE OF CRACKED TIMOSHENKO BEAM SUBJECTED TO MOVING HARMONIC LOAD

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Received: 15 August 2022 / Published online: 30 September 2022

Abstract. In this paper, there is proposed a novel damage index extracted from frequency response of cracked Timoshenko beam under moving harmonic load using the so-called Modal Assurance Criterion (MAC) concept. First, frequency response of a cracked Timoshenko beam subjected to harmonic force moving on the beam with constant speed is obtained in an analytical expression. Then, a scalar characteristic like the coherence between the frequency responses of intact and cracked beams is determined and called herein Spectral Assurance Criterion (SAC). Such designed criterion is dependent upon crack parameters (location and depth), the load frequency and speed as well as position on beam where the responses have been measured. Numerical analysis shows that SAC is much more sensitive to crack than natural frequencies and can be used as a novel damage index for crack detection in beam using moving load. The effect of moving load frequency and speed has been also examined with the aim to have got an indicator most adequate for the crack detection problem.

Keywords: cracked beams, moving load, frequency response, damage index.

1. INTRODUCTION

Damage detection is a crucial problem in Structural Health Monitoring (SHM) and it is usually accomplished by processing the data gathered from the measured vibration of a structure under consideration [1–4]. The key to the Damage Detection Problem (DDP) is the so-called damage index chosen as an indicator for identifying deterioration that happened in the structure. For a long time until nowadays, the dynamical characteristics such as natural frequencies and mode shapes acknowledged as modal parameters of a structure have been selected as the most popular indicator for damage detection in engineering structures. It is because any deterioration occurred in a structure should lead to a change in the structure's dynamic characteristics. However, the modal parameter-based damage detection has faced so far with the following constraints. First, the modal parameters are not directly measured but they are usually extracted from the measured response of a structure under a controlled load. So that integrated measurement and extraction errors may hide the damage signature in the modal parameter indicator. Second, while the more easily measured quantitative indicators such as natural frequencies are weakly sensitive to local damage, the functional features such as mode shapes more sensitive to damage are difficult to accurately measure. Therefore, the damage indexes that are more sensitive to damage and could be easily measured need to be developed. Various techniques have been proposed for damage detection using response of structures subjected to moving load [5–8] and they demonstrated an advantage of moving load used for the damage detection. Most of works in the widespread literature on the moving load-based damage detection used time history response to the moving load in combination with the signal processing methods such as wavelet or Hilbert-Huang transforms. Very little works on the moving load-based damage detection employed frequency response of structures subjected to moving load [9, 10].

In the present study, a novel damage index extracted from frequency response of cracked Timoshenko beam under moving harmonic load is proposed using the so-called Modal Assurance Criterion (MAC) concept [11–21]. First, frequency response of a cracked Timoshenko beam subjected to harmonic force moving on the beam with constant speed is obtained in an analytical expression [22]. Then, a scalar characteristic like the coherence between the frequency responses of intact and cracked beams is determined and called herein Spectral Assurance Criterion (SAC). Such designed criterion is dependent upon crack parameters (location and depth), the load frequency and speed as well as position on beam where the responses have been measured. Numerical analysis shows that SAC is much more sensitive to crack than natural frequencies and can be used as a novel damage index for crack detection in beam using moving load. The effect of moving load frequency and speed has been also examined with the aim to have got an indicator most adequate for the crack detection problem.

2. FREQUENCY RESPONSE OF CRACKED TIMOSHENKO BEAM TO MOVING LOAD

Consider a uniform beam of length ℓ; material density (ρ); elasticity (E) and shear (G) modulus; area $A = b \times h$ and moment of inertia $I = bh^3/12$ of cross section, subjected to a transverse load of density $p(x,t)$. Using the Timoshenko theory of beam

$$u(x,z,t) = u_0(x,t) - z\theta(x,t), \quad w(x,z,t) = w_0(x,t), \tag{1}$$

and constituting equations

$$\varepsilon_x = \partial u_0/\partial x - z\partial\theta/\partial x, \quad \gamma_{xz} = \partial w_0/\partial x - \theta, \quad \sigma_x = E\varepsilon_x, \quad \tau_{xz} = \kappa G\gamma_{xz}, \tag{2}$$

and Hamilton principle, the equations for free vibration of the beam can be established as

$$\rho A\ddot{w}_0(x,t) - \kappa GA(w_0'' - \theta') = p(x,t), \quad \rho I\ddot{\theta}(x,t) - EI\theta'' - \kappa GA(w' - \theta) = 0, \tag{3}$$

Furthermore, it is assumed that the beam has been cracked at positions e and the crack is modeled by rotational springs of stiffness K calculated from crack depth [23].

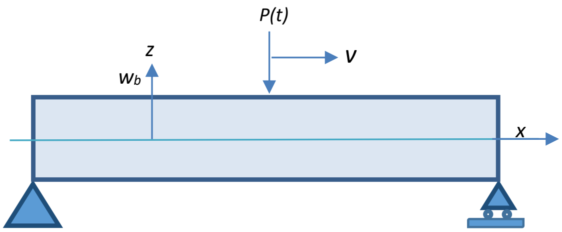


Fig. 1. Simply supported beam subjected to moving force

Therefore, conditions that must be satisfied at the crack section are

$$\begin{gathered} w_0(e+0,t) = w_0(e-0,t) = w_0(e,t), \quad \theta'_x(e_j+0,t) = \theta'_x(e-0,t) = \theta'_x(e,t), \\ \theta(e+0,t) = \theta(e-0,t) + \gamma\theta'_x(e,t), \quad w'_{0x}(e_j+0,t) = w'_{0x}(e-0,t) + \gamma\theta'_x(e), \\ \gamma = EI/K. \end{gathered} \tag{4}$$

Under the Fourier transformation equations (3) become

$$\rho A\omega^2 W(x,\omega) + \kappa GA(W'' - \Theta') = P(x,\omega), \ \rho I\omega^2\Theta(x,\omega) + EI\Theta'' + \kappa GA(W' - \Theta) = 0, \tag{5}$$

where

$$W(x,\omega) = \int_{-\infty}^{\infty} w_0(x,t)e^{-i\omega t}dt, \ \Theta(x,\omega) = \int_{-\infty}^{\infty} \theta(x,t)e^{-i\omega t}dt, \ P(x,\omega) = \int_{-\infty}^{\infty} p(x,t)e^{-i\omega t}dt,$$

and conditions at crack positions (4) get the form

$$\begin{gathered} W_0(e+0,\omega) = W_0(e-0,\omega) = W_0(e,\omega), \quad \Theta'_x(e_j+0,\omega) = \Theta'_x(e_j-0,\omega) = \Theta'(e_j,\omega), \\ \Theta(e+0,\omega) = \Theta(e-0,\omega) + \gamma\Theta'_x(e,\omega), \quad W'_{0x}(e+0,\omega) = W'_{0x}(e-0,\omega) + \gamma\Theta'_x(e,\omega). \end{gathered} \tag{6}$$

First, let's consider homogeneous equations

$$\begin{gathered} \rho A\omega^2 W_0(x,\omega) + \kappa GA(W''_0 - \Theta') = 0, \\ \rho I\omega^2\Theta(x,\omega) + EI\Theta'' + \kappa GA(W'_0 - \Theta) = 0, \end{gathered} \tag{7}$$

general solution of which, as shown in [22], can be represented in the form

$$\begin{aligned} & W_0(x) = C_1\cosh k_1x + C_2\sinh k_1x + C_3\cos k_2x + C_4\sin k_2x, \\ & \Theta_0(x) = r_1C_1\sinh k_1x + r_1C_2\cosh k_1x + r_2C_3\sin k_2x - r_2C_4\cos k_2x, \\ & k_1 = \sqrt{(\sqrt{\alpha^2+4\beta} - \alpha)/2}, \quad k_2 = \sqrt{(\sqrt{\alpha^2+4\beta} + \alpha)/2}, \\ & r_1 = (\rho\omega^2/\kappa Gk_1 + k_1), \quad r_2 = (\rho\omega^2/\kappa Gk_2 - k_2), \\ & \alpha = (\rho\omega^2/E)(1 + E/\kappa G), \quad \beta = (\rho\omega^2/E)\left(\rho\omega^2/\kappa G - A/I\right). \end{aligned} \tag{8}$$

Particularly, solution (8) satisfying the conditions $W_0(0) = 0, W'_0(0) = 1, \Theta_0(0) = 1, \Theta'_0(0) = 0$ is

$$S_w(x) = S_1 \sinh k_1 x + S_2 \sin k_2 x, \quad S_\theta(x) = r_1 S_1 \cosh k_1 x - r_2 S_2 \cos k_2 x, \tag{9}$$

$$S_1 = (r_2 + k_2)/(r_1 k_2 + r_2 k_1), \quad S_2 = (r_1 - k_1)/(r_1 k_2 + r_2 k_1) \tag{10}$$

Using obtained above solution, general solution of homogeneous Eq. (7) satisfying conditions (6) at cracks is represented by

$$\begin{aligned} W_0^c(x,\omega) &= C_1 W_1(k_1,x) + C_2 W_2(k_1,x) + C_3 W_3(k_2,x) + C_4 W_4(k_2,x), \\ \Theta_0^c(x,\omega) &= C_1 \Theta_1(k_1,x) + C_2 \Theta_2(k_1,x) + C_3 \Theta_3(k_2,x) + C_4 \Theta_4(k_2,x), \end{aligned} \tag{11}$$

where

$$\begin{aligned} &W_1(x) = \cosh k_1 x + \gamma L_1(e) K_w(x-e), \quad W_2(x) = \sinh k_1 x + \gamma L_2(e) K_w(x-e), \\ &W_3(x) = \cos k_2 x + \gamma L_3(e) K_w(x-e), \quad W_4(x) = \sin k_2 x + \gamma L_4(e) K_w(x-e), \\ &\Theta_1(x) = r_1 \sinh k_1 x + \gamma L_1(e) K_\theta(x-e), \quad \Theta_2(x) = r_1 \cosh k_1 x + \gamma L_2(e) K_\theta(x-e), \\ &\Theta_3(x) = r_2 \sin k_2 x + \gamma L_3(e) K_\theta(x-e), \quad \Theta_4(x) = -r_2 \cos k_2 x + \gamma L_4(e) K_\theta(x-e), \\ &K_w(x) = \begin{cases} 0, & \text{if } x < 0 \\ S_w(x), & \text{if } x \geq 0 \end{cases} \quad K'_w(x) = \begin{cases} 0, & \text{if } x < 0 \\ S'_w(x), & \text{if } x \geq 0 \end{cases} \\ &K_\theta(x) = \begin{cases} 0, & \text{if } x < 0 \\ S_\theta(x), & \text{if } x \geq 0 \end{cases} \quad K'_\theta(x) = \begin{cases} 0, & \text{if } x < 0 \\ S'_\theta(x), & \text{if } x \geq 0 \end{cases} \end{aligned} \tag{12}$$

$$\begin{aligned} L_1(x) &= k_1 r_1 \cosh k_1 x, \quad L_2(x) = k_1 r_1 \sinh k_1 x, \\ L_3(x) &= k_2 r_2 \cos k_2 x, \quad L_4(x) = k_2 r_2 \sin k_2 x. \end{aligned} \tag{13}$$

Now, suppose that the transverse load is a harmonic force $P_0 e^{i\Omega t}$ moving on the beam with constant speed v, that means

$$p(x,t) = P_0 e^{i\Omega t} \delta(x - vt). \tag{14}$$

So, the right-hand side of Eq. (5) can be calculated as

$$P(x,\omega) = \bar{P}_0 \exp\{i\bar{\Omega} x\}, \quad \bar{P}_0 = P_0/v, \quad \bar{\Omega} = (\Omega - \omega)/v, \tag{15}$$

that allows one to find a particular solution of Eq. (5) in the form

$$W_p(x,\omega) = D_w \exp\{i\bar{\Omega} x\}, \quad \Theta_p(x,\omega) = D_\theta \exp\{i\bar{\Omega} x\}, \tag{16}$$

with

$$\begin{aligned} D_w &= \bar{P}_0 (EI\bar{\Omega}^2 - \rho I \omega^2 + \kappa GA)/D, \quad D_\theta = \bar{P}_0 \kappa GA i\bar{\Omega}/D, \\ D &= [(EI\bar{\Omega}^2 - \rho I \omega^2)(\kappa GA\bar{\Omega}^2 - \rho A \omega^2) - \kappa GA^2 \rho \omega^2]. \end{aligned} \tag{17}$$

Therefore, general solution of Eq. (5) satisfying conditions (6) is

$$\begin{aligned} W(x,\omega) &= C_1 W_1(k_1,x) + C_2 W_2(k_1,x) + C_3 W_3(k_2,x) + C_4 W_4(k_2,x) + W_p(x,\omega), \\ \Theta(x,\omega) &= C_1 \Theta_1(k_1,x) + C_2 \Theta_2(k_1,x) + C_3 \Theta_3(k_2,x) + C_4 \Theta_4(k_2,x) + \Theta_p(x,\omega). \end{aligned} \tag{18}$$

Substituting expressions (18) to given boundary conditions, for example,

$$W(0,\omega) = 0, \quad \Theta'(0,\omega) = 0, \quad W(L,\omega) = 0, \quad \Theta'(L,\omega) = 0,$$

in case of simply supported beams, one can find the constants C_1, C_2, C_3, C_4 as

$$\begin{aligned}
&C_1 = (r_2k_2D_w - i\bar{\Omega}D_\theta)/(r_1k_1 - r_2k_2), \quad C_3 = (r_1k_1D_w - i\bar{\Omega}D_\theta)/(r_2k_2 - r_1k_1)\\
&C_2 = [W_4(k_2, L)Q_\theta - \Theta_4'(k_2, L)Q_w)]/\Delta, \quad C_4 = [\Theta_2'(k_1, L)Q_w - W_2(k_1, L)Q_\theta)]/\Delta,\\
&Q_w = [W_p(L, \omega) + C_1W_1(k_1, L) + C_3W_3(k_2, L)],\\
&Q_\theta = [\Theta_p'(L, \omega) + C_1\Theta_1'(k_1, L) + C_3\Theta_3'(k_2, L)],\\
&\Delta = W_2(k_1, L)\Theta_4'(k_2, L) - W_4(k_2, L)\Theta_2'(k_1, L).
\end{aligned} \tag{19}$$

It is not difficult to verify that natural frequencies of the beam can be found from equation

$$\Delta = W_2(k_1, L)\Theta_4'(k_2, L) - W_4(k_2, L)\Theta_2'(k_1, L) = 0. \tag{20}$$

Thus, frequency response of the simply supported beam subjected to moving harmonic load has been conducted in the form (18) with constants C_1, C_2, C_3, C_4 determined in (19). Modules of the response components are acknowledged as deflection and slope spectrums

$$S_w(x, \omega) = |W(x, \omega)|, \quad S_\theta(x, \omega) = |\Theta(x, \omega)|, \tag{21}$$

that would be examined below in dependence upon crack parameters.

3. MODAL ASSURANCE CRITERION AND ITS APPLICATION

The modal assurance criterion (MAC) was proposed first to check orthogonality and consistency of predicted and measured mode shape vectors [11, 12] and then employed for model updating and structural damage monitoring [13, 14]. Since the criterion is insensitive to small changes in the compared mode shapes, it was extended for detecting structural damages based on the changes in natural frequencies and termed by Multiple Damage Location Assurance Criterion (MDLAC) [15, 16]. Nevertheless, either MAC or MDLAC provide only particular and limited information on the damaged structure condition, therefore, more fruitful assurance criterions were developed for flexibility [17] or frequency response function [18] matrices. While the latter criterion is formulated by comparison of the frequency response functions at the same frequency, because of shifted natural frequencies and vibration phase due to damage the authors of works [19–21] proposed the so-called frequency domain assurance criterions that compare the responses at different frequencies. To the authors' better knowledge, no similar assurance criterion conducted from frequency responses of beams under moving load has been reported in the literature.

As well-known, so-called assurance criterion between two signals (vectors) $\{S_j, j = 1, \ldots, N\}$ and $\{Q_j, j = 1, \ldots, N\}$ is determined as

$$H(S, Q) = \left[\left(\sum_{k=1}^{N} S_kQ_k \right)^2 / \left(\sum_{k=1}^{N} S_k^2 \times \sum_{k=1}^{N} Q_k^2 \right) \right]^{1/2}. \tag{22}$$

Accordingly, two signals are considered as similar or strongly correlated if the coefficient between them is about unique and its deviation from unique provides a measure of the signal's dissimilarity. In this case the coherence coefficient is termed hereby similarity index of two signals.

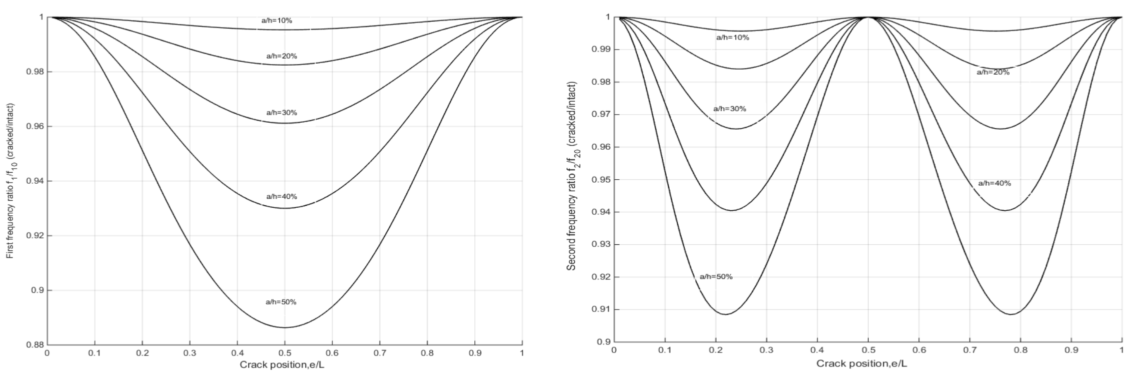
If the compared signals are frequency response spectrums of cracked and intact beams, both measured at position x, the coefficient would describe the change of the responses due to crack. Therefore, the coherence coefficient calculated for cracked and intact frequency responses as

$$\mathrm{SAC}(e,a,x) = \left[\frac{\left(\sum_{k=1}^{N} S_0\left(\omega_j, x\right) S_c\left(\omega_j, x, e, a\right) \right)^2}{\left(\sum_{k=1}^{N} S_0^2\left(\omega_j, x\right) \times \sum_{k=1}^{N} S_c^2\left(\omega_j, x, e, a\right) \right)} \right]^{1/2}, \tag{23}$$

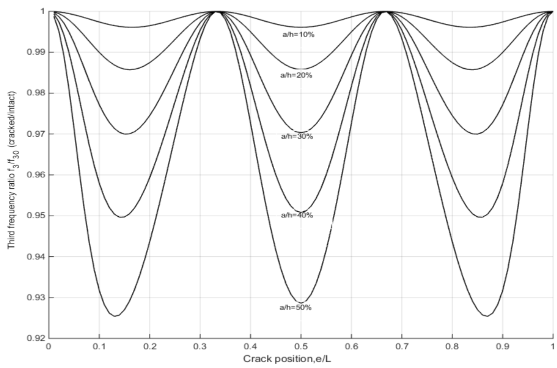
can be acknowledged as spectral assurance criterion (SAC) like the frequency domain assurance criterion [21]. The SAC is examined below in dependence upon not only the crack parameters but also the moving load ones and it would be shown that the criterion provides a novel useful indicator for crack detection using distributed sensor in combination with moving load.

4. NUMERICAL ANALYSIS

First, for validation of the above theoretical development, the changes in the three lowest natural frequencies of a simply supported beam due to crack are computed and the obtained results are provided in Fig. 2, where the ratios of cracked frequencies to intact ones are presented. Obviously, the crack-induced changes in natural frequencies are the same as given in Ref. [22]. Namely, crack at the beam middle creates maximal change in the odd frequencies and makes no effect on the even ones. Moreover, the variations are symmetrical about the middle of the beam with symmetric boundary conditions.



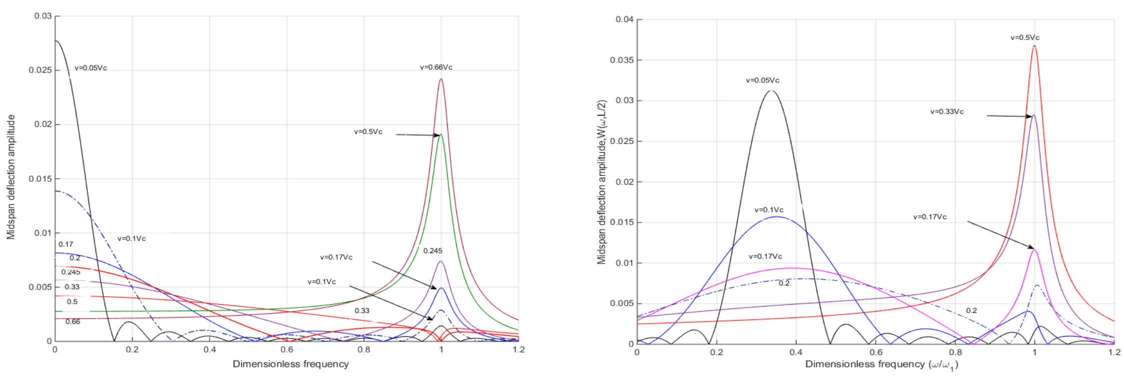
(a) First frequency (b) Second frequency



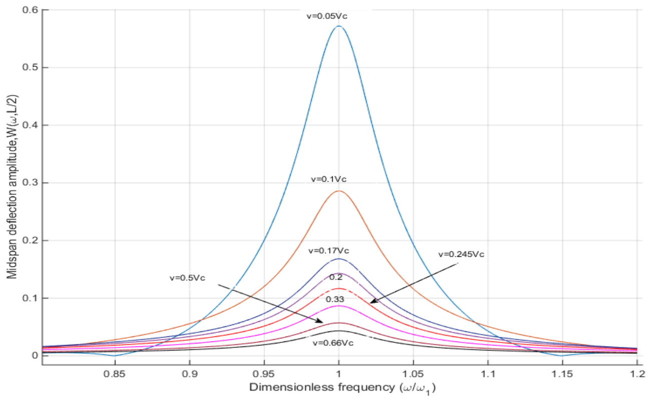
(c) Third frequency

Fig. 2. Variation of natural frequencies due to crack location and depth

Next, for spectral analysis of moving load-induced response let's introduce the following parameters: ω_{01} – fundamental frequency of undamaged beam; $V_c = \omega_{01}L/\pi$ – critical speed of moving load; dimensionless frequency and speed $\bar{\omega} = \omega/\omega_{01}$ and $v = V/V_c$. The moving harmonic load is called resonant if load frequency equals to natural frequency, $\Omega = \omega_{01}$. So, the midspan deflection spectrum, $S_w(\omega) = |W(L/2, \omega)|$, computed for undamaged beam in various load speeds and frequencies are shown in Fig. 3, where three cases of load frequency: (a) constant force, $\Omega = 0$; (b) one third resonant load, $\Omega = \omega_{01}/3$ and (c) resonant load, $\Omega = \omega_{01}$ are provided. Recalling the results obtained by the authors in [8, 9] we can see also in Fig. 3 that under moving harmonic load dominant vibration components of the beam response are vibrations with the load frequency and eigenfrequency called herein forced mode and eigenmode respectively. Amplitudes of the vibration components are strongly dependent upon speed of the moving load. Namely, the forced mode of vibration is dominant for the load moving at low speed, while the eigenmode gets to be prevalent at the high speed of the load. Evidently, the amplitude of resonant vibration mode is highest, and it is rapidly reduced for increasing load speed.



(a) Constant load ($\Omega = 0$)   (b) One third resonant load ($\Omega = \omega_{01}/3$)



(c) Resonant load $(\Omega = \omega_{01})$

Fig. 3. Midspan deflection spectrums for uncracked simply supported beam in various frequency and speed of moving load

Finally, the spectral assurance criterion determined by Eq. (23) is numerically examined in dependence upon crack and load parameters, and results of the computation are displayed in Figs. 4–6, where given graphs demonstrate SAC as function of crack location for various crack depth, moving load speed and frequency. The effect of the position on beam where the responses have been measured is also investigated.

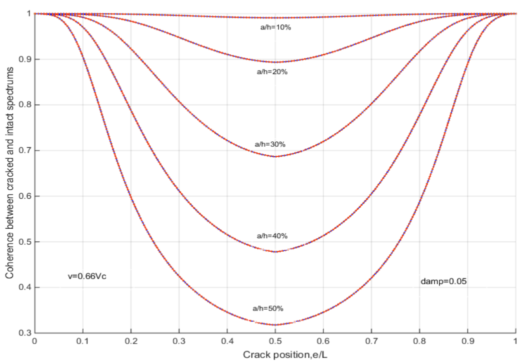


Fig. 4. Spectral assurance criterion versus crack position in different crack depth and load frequency (Dot lines represent the case of resonant load; solid and dash-dot lines – constant and non-resonant load) $v = 0.245V_c$, damp $= 0.05$

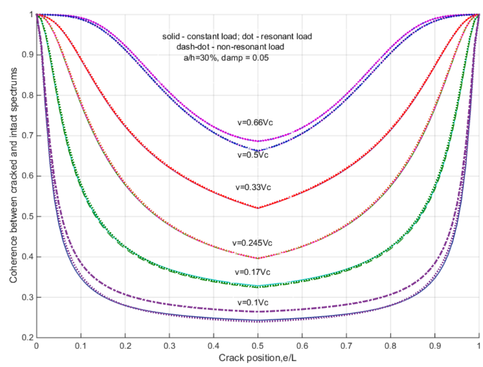


Fig. 5. Spectral assurance criterion versus crack position in different load frequency and speed $a/h = 0.3$, damp $= 0.05$

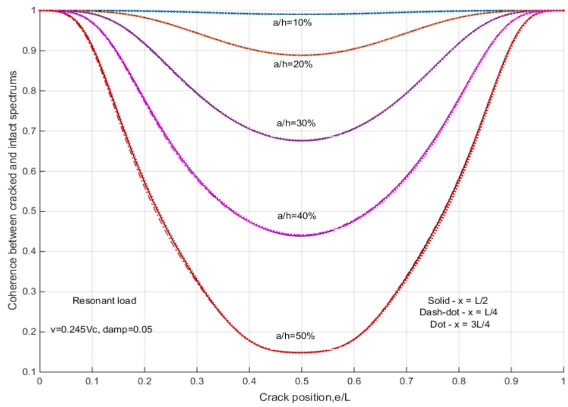


Fig. 6. Spectral assurance criterion versus crack position in different crack depth and measuring locations $\Omega = \omega_{01}, v = 0.245V_c$, damp $= 0.05$

5. CONCLUSIONS

Thus, a novel damage index, called spectral assurance criterion (SAC), has been introduced and examined in the present paper. It is extracted from frequency response of cracked Timoshenko beam subjected to moving harmonic load using the well-known concept of modal assurance criterion.

The analytical expression obtained for the frequency response allows thoroughly examining the damage index in dependence upon not only crack location and depth but also the moving load parameters such as frequency and speed as well as position on beam where the response is measured.

As a quantitative feature, like natural frequencies of a structure under consideration but much more sensitive to crack than the natural frequencies, SAC shows to be a promising novel indicator for structural crack identification.

It has been shown a significant effect of moving load speed on SAC's sensitivity to crack, while the moving load frequency and the position on beam where the frequency response is measured make no effect on the crack-induced change in the SAC.

Though the novel damage index, SAC, has been proposed for beam structures under moving load, it can be developed further for more complicate structures such as frames which allow consistent measurements of frequency responses in both intact and damaged structure conditions.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

FUNDING

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

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