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INCREMENTALLY UPDATING APPROXIMATION IN INCOMPLETE INFORMATION SYSTEMS
UNDER THE VARIATION OF OBJECTS

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Abstract. In covering approximation space, the rough membership functions give numerical characterizations of covering-based rough set approximations. It is considered as a tool for establishing the relationship between covering-based rough sets and fuzzy covering-based rough sets. In this paper, we introduce a new method to update the approximation sets with rough membership functions in covering approximation space. Firstly, we present the third types of rough membership functions and study their properties. And then, we consider the change of them while simultaneously adding and removing objects in the information system. Based on that change, we propose a method for updating the approximation sets when the objects vary over time. We proved that the method facilitates knowledge maintenance without retrain from scratch.

Keywords. Rough set; Incomplete information systems; Covering-based rough set; The third types of rough membership functions; Incremental learning.

1. INTRODUCTION

Rough set theory was originally proposed by Pawlak in 1982 [27] and now it is used as a useful mathematical tool to solve problems containing uncertain data in information systems and data analysis. However, it can only be used in the complete information systems while real data is often imperfect. Therefore, many extensions have been made in recent years to deal with this problem. Some scholars have extended rough sets by replacing the equivalent relation with other binary relations. Those approaches are based on two cases. One is Lost value [13] in which unknown values of attributes are already lost and the other is Do not care [6–8, 15], which may be potentially replaced by any value in the domain.

In addition, the researchers also extended the rough set based on coverings of the universe of discourse [14, 20, 25]. First, Zakowski built the first type of covering-based rough sets by covering instead of a partition of the universe [20]. Bonikowski et al. used the concepts of extension and intension to propose the second type of covering-based rough sets [20]. And Pomykala included interior and closure operators from topology in the second type of covering-based rough sets [14]. Wang et al. established relationships between four matroidal structures of coverings and the second type of covering-based rough sets [5]. Tsang et al. introduced the third type of covering-based rough sets [9], and Zhu discussed difference

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between this model and Pawlaks rough sets [28]. Zhu and Wang studied the fourth type of covering-based rough sets and established axiomatic systems for the lower and upper approximation operators [21].

The dynamic information system can be divided into three aspects: variation of objects, variation of attributes, and variation of attributes values. As the information changes, the approximation sets also change. Thence, incremental learning techniques are used for mining dynamic databases. The main idea of those methods is using the results obtained previously in order to facilitate knowledge maintenance in the changing database without exploiting the total database from scratch. Based on rough set theory, studies on incremental data analysis have been developed. A method for incremental updating rough approximations in information system under the characteristic relation-based rough sets is proposed by Li et al. [18]. Chen et al. discussed a method for incremental approach for updating approximations of variable precision rough-set model [12]. They updated the properties of information granulation and approximations with the refining and coarsening of attribute values. Luo et al. proposed incrementally updating approximations in the set-valued information systems [3]. Then, Luo et al. introduced an incremental method for updating probabilistic approximations when adding and removing objects based on characteristic relation [4]. It is easy to see that these incremental methods are all used to the ratio of overlap in the equivalence class without considering the degree of overlap in a basic set. The third type of rough membership function is defined the highest ratio of overlap in a decision set. If conditional probability pays close attention to the classification of an equivalence class then the third type of rough membership function pays close attention to the decision class. In this paper, we propose a method for updating the approximation sets based on the third type of rough membership function in the incomplete system when the objects vary.

This paper is organized as follows: Section 2 briefly reviews some basic concepts of Pawlaks rough sets and covering-based rough sets. Section 3 gives the concept and some properties of the third type of rough membership function by neighborhood operator in covering approximation space. Section 4 introduces an incremental updating method with approximation sets in covering approximation space and Section 5 presents the conclusions.

2. PRELIMINARIES

In this section, we briefly review some existing definitions and results of Pawlaks rough sets and covering-based rough sets.

The main idea of a rough set is based on the partition or indiscernibility relation to define subsets called the lower and upper approximation sets to approximate description of arbitrary subset in the universe. This partition or equivalence relation is still restrictive for various applications. Therefore, it is not applicable in information systems containing imperfect data. To deal with this problem, Kryszkiewicz introduced indiscernibility based on tolerance relations [15]. Here, a missing value was considered as a special value that may take any possible value.

Definition 1. [15] An information system usually is defined as $IS = (U, A, V, f)$ where U is a non-empty finite set of objects, A is a non-empty finite set of attributes, $V = \{V_a \,|a \in A\}$ is a domain of attribute a, f is a function from $U \times A$ into V.

If U contains at least an unknown value object, then IS is called an incomplete information system, denoted as IIS, otherwise complete. In incomplete information systems, unknown values are denoted by special symbol "$*$" and are supposed to be contained in the set V_a.

In practice, if we have $A = C \cup \{d\}$, where C denotes a nonempty finite set of conditional attributes and $d \notin C$ is a distinguished attribute called decision, then $IS = (U, C \cup \{d\}, V, f)$ is called a decision table.

Definition 2. [15] Let IIS $= (U, C \cup \{d\}, V, f)$ be an incomplete information system, and $P \subseteq C$. Then a Tolerance relation TOR_P denotes a binary relation between objects that are possibly equivalent in terms of values of attributes and defined as

$$\mathrm{TOR}_P = \{(x, y) \in U \times U \,|\forall a \in P, f_a(x) = f_a(y) \vee f_a(x) = * \vee f_a(y) = *\}, \tag{1}$$

where $f_a(x)$, $f_a(y)$ denote the values of objects x and y on a, $\vee$ denotes disjunction.

This relation is reexive and symmetric but does not need to be transitive.

Let $T_P(x) = \{y \in U \,|\mathrm{TOR}_P\,(y, x)\}$ be the set of objects which are in relation with x in terms of P in the sense of the above tolerance relation.

Definition 3. [15] Let IIS $= (U, C \cup \{d\}, V, f)$ be an incomplete information system, and $P \subseteq C, X \subseteq U$. The lower and upper approximations of X in terms of P are defined as follows

$$\underline{\mathrm{appr}}_P\,(X) = \{x \in U \,|T_P\,(x) \subseteq X\}, \tag{2}$$

$$\overline{\mathrm{appr}}_P\,(X) = \{x \in U \,|T_P\,(x) \cap X \neq \emptyset\}. \tag{3}$$

Definition 4. [25] Let U be a universe of discourse and $\mathcal{C}$ be a family of subsets of U. Then $\mathcal{C}$ is called a covering of U if none of elements of $\mathcal{C}$ is empty and $\cup\{C \,|C \in \mathcal{C}\} = U$. If K is an element of $\mathcal{C}$, K is called a covering block. Furthermore, $(U, \mathcal{C})$ is called a covering approximation space and denoted it by CAS.

In the incomplete information system IIS $= (U, C \cup \{d\}, V, f)$, with $P \subseteq C$, let $\mathcal{C} = \{T_{Pi}(x)\}$ then $\mathcal{C}$ is called a special characteristic covering of U [9].

Next, we recall some definitions of covering, which shall be needed in the sequel.

Definition 5. [25] Let CAS $= (U, \mathcal{C})$ be a covering approximation space. For any $x \in U$, $N_{\mathcal{C}}(x) = \bigcap\{K \in \mathcal{C} : x \in K\}$ is called the neighborhood of x.

Definition 6. [5] Let CAS $= (U, \mathcal{C})$ be a covering approximation space. $\mathrm{Cov}_{\mathcal{C}}(X) = \{N_{\mathcal{C}}(x) : x \in U\}$ is called the covering of neighborhoods induced by $\mathcal{C}$.

Definition 7. [5] Given U be a discourse of universe. $A \subseteq U$ is called a fuzzy set, or rather a fuzzy subset of U, if exist a function assigning each element x of U a value $A(x) \in [0, 1]$. At that time, the family of all fuzzy subsets of U, i.e., the set of all functions from U to $[0, 1]$ is called the fuzzy power set of U and denoted as $\mathcal{P}(U)$.

3. ROUGH SET MODEL BASED ON THE THIRD TYPE OF ROUGH MEMBERSHIP FUNCTION

One of the fundamental notions of set theory is the rough membership function. It was used to measure the uncertainty of a set in an information system. In Pawlak rough set, the rough membership function was also used to present numerical characterizations of rough set approximations. Yao made a survey on existing studies, and gave some new results on the decision-theoretic rough set model based on rough membership function [23]. Greco et al. introduced a generalization of the original definition of rough sets and variable precision rough sets using the concept of absolute and relative rough membership functions [17]. Ge et al. constructed a kind of rough membership function based on covering rough set [22]. It is considered the fourth type of rough membership. In CAS $= (U, \mathcal{C})$ with $x \in U$, $X \in \mathcal{P}(U)$, they defined the rough membership function as follows

$$\varphi_{\mathcal{C}}^{X}(x) = \max\left\{\frac{|X \cap C|}{|C|}\, | x \in C, C \in \mathcal{C}\right\}.$$

Based on this definition, we realize that $\varphi_{\mathcal{C}}^{X}(x)$ is only related to the covering blocks containing x. Yang et al. defined the first type of rough membership function as follows [1]

$$\sigma_{\mathcal{C}}^{X} = \frac{|X \cap N_{\mathcal{C}}(X)|}{|N_{\mathcal{C}}(X)|},$$

where $N_{\mathcal{C}}(X) = \bigcap\{C \in \mathcal{C} | x \in C\}$. The above definition means, in the case that object x related both the covering $\mathcal{C}$ and X, then it is important to measure the rough membership of x to X with respect $\mathcal{C}$. After that, they defined the second and third types of rough membership functions by generalizing the first and fourth types of rough membership functions, respectively. Here the second type of rough membership shows the ratio of $|X \cap N_{\mathcal{C}}(x)|$ and $|N_{\mathcal{C}}(x)|$, and the third type of rough membership shows the highest ratio of $|X \cap C|$ and $|X|$. Since $N_{\mathcal{C}}(x) \subseteq C$, then the second type of rough membership function is always less than or equal to the third type of rough membership function.

In the following, we review the definition about the third type of rough membership function in a covering approximation space and its properties.

Definition 8. [1] Let CAS $= (U, \mathcal{C})$ be a covering approximation space. For any $x \in U$, $X \in \mathcal{P}(U)$, the third type of rough membership function is defined as follows

$$V_{\mathcal{C}}^{X}(x) = \begin{cases} 0, & X = \emptyset, \\ \max\left\{\frac{|X \cap C|}{|X|}\, | x \in C, C \in \mathcal{C}\right\}, & X \neq \emptyset. \end{cases} \quad (4)$$

Here, $V_{\mathcal{C}}^{X}(x)$ is considered maximum coverage measure. If given a rule $C \to X$ then $V_{\mathcal{C}}^{X}(x)$ means the elements C are the most general in the decision class X. With $x \in X$ and $X \neq \emptyset$ then $V_{\mathcal{C}}^{X}(x) > 0$.

Based on Definition 8, some properties of $V_{\mathcal{C}}^{X}(x)$ are presented as follows.

Proposition 9. [1] *Let* CAS $= (U, \mathcal{C})$ *be a covering approximation space. For any* $X \in \mathcal{P}(U)$, $\forall x \in X$, *the following statements hold*

(i) $0 \leq V_{\mathcal{C}}^{X}(x) \leq 1$,

(ii) *If* $\exists \mathcal{C} \in \mathcal{C}$ *such that* $\emptyset \neq \mathcal{C} \subseteq \mathcal{C}$, *then* $V_{\mathcal{C}}^{X}(y) = 1, \forall y \in \mathcal{C}$.

According to the proposition above, assume that $\mathcal{C} = \{C_1, C_2, .., C_m\}$, if $x \in C_i$, then $V_{\mathcal{C}}^{C_i}(x) = 1$, for $i = 1, 2, .., m$. Thus, the family $\mathcal{C}_V = \{V_{\mathcal{C}}^{C_i} | i = 1, 2, .., m\}$ is a fuzzy $\beta-$covering of U for $\beta \in [0, 1]$.

Definition 10. [1] Let CAS $= (U, \mathcal{C})$ be a covering approximation space. $0 \leq \rho \leqslant 1$ and $X \in \mathcal{P}(U)$. The graded lower and graded upper approximations based on covering of X with respect to $(U, \mathcal{C})$ based on the parameter ρ are defined, respectively, as follows

$$\underline{\mathcal{C}}_{\rho}(X) = \left\{x \in U \,\middle|\, V_{\mathcal{C}}^{(\sim X)}(x) \leq \rho\right\}, \tag{5}$$

$$\overline{\mathcal{C}}_{\rho}(X) = \left\{x \in U \,\middle|\, V_{\mathcal{C}}^{(X)}(x) > \rho\right\}, \tag{6}$$

where $\sim X$ denotes a complementary set of X.

Below are the definitions of the positive region, negative region, upper boundary region, lower boundary region and boundary region based on covering.

Definition 11. [1] Let CAS $= (U, \mathcal{C})$ be a covering approximation space. $0 \leq \rho \leqslant 1$ and $X \in \mathcal{P}(U)$. The positive region, negative region, upper boundary region, lowers boundary region and boundary region are defined as

$$\mathrm{POS}_{\rho}(X) = \underline{\mathcal{C}}_{\rho}(X) \cap \overline{\mathcal{C}}_{\rho}(X); \tag{7}$$

$$\mathrm{NEG}_{\rho}(X) = U - (\underline{\mathcal{C}}_{\rho}(X) \cup \overline{\mathcal{C}}_{\rho}(X)); \tag{8}$$

$$\mathrm{LBND}_{\rho}(X) = \underline{\mathcal{C}}_{\rho}(X) - \overline{\mathcal{C}}_{\rho}(X); \tag{9}$$

$$\mathrm{UBND}_{\rho}(X) = \overline{\mathcal{C}}_{\rho}(X) - \underline{\mathcal{C}}_{\rho}(X); \tag{10}$$

$$\mathrm{BND}_{\rho}(X) = LBND_{\rho}(X) \cup UBND_{\rho}(X). \tag{11}$$

The sets $\mathrm{POS}_{\rho}(X)$, $\mathrm{NEG}_{\rho}(X)$, $\mathrm{LBND}_{\rho}(X)$, $\mathrm{UBND}_{\rho}(X)$, $\mathrm{BND}_{\rho}(X)$ are also called the graded covering-based positive region, negative region, lower boundary region, upper boundary region and boundary region of X, respectively.

From the definition above we can get the following properties of approximation space directly as:

(i) $\underline{\mathcal{C}}_{\rho}(U) = U$;

(ii) $\overline{\mathcal{C}}_{\rho}(\emptyset) = \emptyset$;

(iii) $\underline{\mathcal{C}}_{\rho}(\sim X) = \sim \overline{\mathcal{C}}_{\rho}(X)$;

(iv) $\overline{\mathcal{C}}_{\rho}(\sim X) = \sim \underline{\mathcal{C}}_{\rho}()$;

(v) If $\exists \mathcal{C} \in \mathcal{C}$ such that $X \subseteq \mathcal{C}$, then $X \subseteq \overline{\mathcal{C}}_{\rho}(X)$;

(vi) If $\exists \mathcal{C} \in \mathcal{C}$ such that $\sim X \subseteq \mathcal{C}$, then $\underline{\mathcal{C}}_{\rho}(X) \subseteq X$;

(vii) $\underline{\mathcal{C}}_{\rho}(X) \subseteq \underline{\mathcal{C}}_{\rho}\left(\underline{\mathcal{C}}_{\rho}(X)\right)$;

(viii) $\overline{\mathcal{C}}_{\rho}\left(\overline{\mathcal{C}}_{\rho}(X)\right) \subseteq \overline{\mathcal{C}}_{\rho}(X)$;

(ix) $\underline{\mathcal{C}}_0(X) = \{x \in U | \mathcal{C} \subseteq X, x \in \mathcal{C} \in \mathfrak{C}\}$,
$\overline{\mathfrak{C}}_0(X) = \{x \in U || X \cap \mathcal{C}| \neq \emptyset, \exists \mathcal{C} \in \mathfrak{C}$, such that $x \in \mathcal{C}\}$;

(x) If $|A| = | \sim A|$, then
$$\underline{\mathfrak{C}}_\rho(X) = \{x \in U || C| - |X \cap C| \leq \rho |X|,\ x \in \mathcal{C} \in \mathfrak{C},$$
and $\overline{\mathfrak{C}}_\rho(X) = \{x \in U || X \cap \mathcal{C}| > \rho |X|,\ \exists \mathcal{C} \in \mathfrak{C}$ such that $x \in \mathcal{C}$;

(xi) If $0 \leq \rho_1 \leq \rho_2 < 1$, then $\underline{\mathfrak{C}}_{\rho_2}(X) \subseteq \underline{\mathfrak{C}}_{\rho_1}(X)$ and $\overline{\mathfrak{C}}_{\rho_2}(X) \subseteq \overline{\mathfrak{C}}_{\rho_1}(X)$.

4. UPDATE APPROXIMATION SETS IN DYNAMIC COVERING INFORMATION SYSTEMS

Yao et al. studied the minimum, maximum and average rough membership functions, and their properties [24]. Xu and Zhang proposed new lower and upper approximations and obtained some important properties in generalized rough set induced by a covering [19]. Shi et al. discussed the uncertainty of covering in the covering approximation space and presented an approach which measures these similarities using a triangular norm [26]. Lin et al. presented three types of covering based multi-granulation rough sets by using different covering approximation operators [11].

In the dynamic systems, researchers investigated knowledge reduction by using incrementally updating approaches. Lang et al. provided some methods to computing the type-1 and type-2 characteristic matrices of dynamic coverings when the objects vary [10]. Cai et al. studied knowledge reduction of dynamic covering decision information systems caused by altering attribute values [16]. Hu et al. proposed a method for updating approximations based on equivalence relation matrix, diagonal matrix and cut matrix in multigranulation rough set when a single granular structure evolves over time [2].

In such an approach, there is a problem as to whether there is a way to update the approximation sets without using matrices. To deal with this issue, we propose a method to update the approximation sets based on the third type of rough membership function.

Let $\text{IIS} = (U, C \cup \{d\}, V, f)$ be an incomplete decision table and $P \subseteq C$. We call $\mathfrak{C}_P = \{T_P(x) | x \in U\}$ a special characteristic covering.

We describe this information system at time step t, when the object has not changed, as $\text{IIS}^{(t)} = (U^{(t)}, C^{(t)} \cup \{d\}^{(t)}, V, f)$. At time step $t+1$, when adding object $\overline{x}$ and deleting object $\underline{x}$ occur simultaneously, the information is denoted as $\text{IIS}^{(t+1)} = (U^{(t+1)}, C^{(t+1)} \cup \{d\}^{(t+1)}, V, f)$.

According to Definition 10, to update the lower and upper approximation sets, we first need to consider their change when the third type of rough membership function changes.

For simplicity, we denote $V^{(t)}$ the third type of rough membership function at time t and $V^{(t+1)}$ at time $t+1$.

In the following, we consider the change of approximation sets when the third type of rough member functions increases, decreases or is constant. We first consider the change of approximation sets when the third type of rough membership function does not change.

Theorem 12. *Suppose that at time $t+1$, the third type of rough membership functions does not change, i.e.,$V^{(t+1)} = V^{(t)}$, then*
$$\overline{\mathfrak{C}}_\rho^{(t+1)}(X) = \overline{\mathfrak{C}}_\rho^{(t)}(X) - \Delta + \Delta', \tag{12}$$

where

$$\Delta = \{\underline{x}|\underline{x} \in \overline{\mathcal{C}}_\rho^{(t)}(X)\}, \text{ and } \Delta' = \{\overline{x}|V^{(X)}(\overline{x}) > \rho\}. \tag{13}$$

$$\underline{\mathcal{C}}_\rho^{(t+1)}(X) = \underline{\mathcal{C}}_\rho^{(t)}(X) - \Delta_1 + \Delta_2, \tag{14}$$

where

$$\Delta_1 = \{\underline{x}|\underline{x} \in \underline{\mathcal{C}}_\rho^{(t)}(X)\} \text{ and } \Delta_2 = \{\overline{x}|V^{(\sim X)}(\overline{x}) \leq \rho\}. \tag{15}$$

Proof. It can be directly deduced from Definition 10. ■

Next, we will update the approximation sets as the third type of rough membership functions increases over time.

Theorem 13. *Suppose that at time $t+1$, the third type of rough membership functions increases, i.e., $V^{(t+1)} > V^{(t)}$, then*
If $V^{(X)(t+1)} > V^{(X)(t)}$ then

$$\overline{\mathcal{C}}_\rho^{(t+1)}(X) = \overline{\mathcal{C}}_\rho^{(t)}(X) - \Delta_1 + \Delta_2, \tag{16}$$

where

$$\Delta_1 = \{\underline{x}|V^{(X)}(\underline{x}) > \rho\}, \text{and } \Delta_2 = \{\overline{x}|V^{(X)}(\overline{x}) > \rho\}. \tag{17}$$

If $V^{(\sim X)(t+1)} > V^{(\sim X)(t)}$ then

$$\underline{\mathcal{C}}_\rho^{(t+1)}(X) = \underline{\mathcal{C}}_\rho^{(t)}(X) - \Delta + \Delta', \tag{18}$$

where

$$\Delta = \{\underline{x}, x \in U|V^{(\sim X)}(\underline{x}) \leq \rho, V^{(\sim X)(t+1)}(x) > \rho\}, \text{ and } \Delta' = \{\overline{x}|V^{(\sim X)}(\overline{x}) \leq \rho\}. \tag{19}$$

Proof. If $V^{(X)(t+1)} > V^{(X)(t)} > \rho$ and $V^{(X)}(\underline{x}) \wedge V^{(X)}(\overline{x}) > \rho$ then (16) hold based on Definition 10.

If $V^{(\sim X)(t+1)} > V^{(\sim X)(t)}$, since $V^{(\sim X)(t)} \leq \rho$, we consider two cases:

+ Case 1: If $V^{(\sim X)(t+1)}(x) \leq \rho$ then

If $V^{(\sim X)}(\underline{x}) \leq \rho \Rightarrow \underline{x} \in \underline{\mathcal{C}}_\rho^{(t)}(X) \Rightarrow \underline{\mathcal{C}}_\rho^{(t+1)}(X) = \underline{\mathcal{C}}_\rho^{(t)}(X) - \{\underline{x}\}$.

If $V^{(\sim X)}(\overline{x}) \leq \rho \Rightarrow \overline{x} \in \underline{\mathcal{C}}_\rho^{(t+1)}(X) \Rightarrow \underline{\mathcal{C}}_\rho^{(t+1)}(X) = \underline{\mathcal{C}}_\rho^{(t)}(X) \cup \{\overline{x}\}$.

+ Case 2: If $V^{(\sim X)(t+1)}(x) > \rho$, based on Definition 10, x does not belong to $\underline{\mathcal{C}}_\rho(X)$ at time $t+1$, furthermore, if $V^{(\sim X)}(\underline{x}) \leq \rho$, $V^{(\sim X)}(\overline{x}) \leq \rho$, then (18) holds. ■

And finally, we consider the change of the approximation sets when the third type of rough membership function decreases.

Theorem 14. *Suppose that at time $t+1$, the third type of rough membership functions decreases, i.e., $V^{(t+1)} < V^{(t)}$, then*
If $V^{(X)(t+1)} < V^{(X)(t)}$ then

$$\overline{\mathcal{C}}_\rho^{(t+1)}(X) = \overline{\mathcal{C}}_\rho^{(t)}(X) - \Delta + \Delta', \tag{20}$$

where

$$\Delta = \{\underline{x}, x \in U | V^{(X)}(\underline{x}) > \rho, V^{(X)(t+1)}(x) \leq \rho\}, \ and \ \ \Delta' = \{\overline{x} | V^{(X)}(\overline{x}) > \rho\}. \tag{21}$$

If $V^{(\sim X)(t+1)} < V^{(\sim X)(t)}$ *then*

$$\underline{\mathcal{C}}_{\rho}^{(t+1)}(X) = \underline{\mathcal{C}}_{\rho}^{(t)}(X) - \Delta_1 + \Delta_2, \tag{22}$$

where

$$\Delta_1 = \{\underline{x} | V^{(\sim X)}(\underline{x}) \leq \rho\}, \ and \ \ \Delta_2 = \{\overline{x} | V^{(\sim X)}(\overline{x}) \leq \rho\}. \tag{23}$$

Proof. The proof of this theorem is to that of Theorem 13. ■

In the following, we study the changing trend of the third type of rough membership functions when adding and removing objects simultaneously.

Let $\text{IIS}^{(t)} = (U^{(t)}, C^{(t)} \cup \{d\}^{(t)}, V, f)$ be an information system at time t, with which the tolerance classes and decision classes, respectively, are $U^{(t)}/TOL_P^{(t)} = \left\{T_{P1}^{(t)}, T_{P2}^{(t)}, .., T_{Pm}^{(t)}\right\}$ and $U^{(t)}\big/\{d\}^{(t)} = \left\{D_1^{(t)}, D_2^{(t)}, .., D_n^{(t)}\right\}$.

And the information system at time $t+1$ is $\text{IIS}^{(t+1)} = (U^{(t+1)}, C^{(t+1)} \cup \{d\}^{(t+1)}, V, f)$ with which the tolerance classes and decision classes, respectively, are $U^{(t+1)}\Big/TOL_P^{(t+1)} = \left\{T_{P1}^{(t+1)}, T_{P2}^{(t+1)}, .., T_{Pm}^{(t+1)}\right\}$ and $U^{(t+1)}\big/\{d\}^{(t+1)} = \left\{D_1^{(t+1)}, D_2^{(t)}, .., D_n^{(t+1)}\right\}$.

In order to easily update the third type of rough membership functions, in the following, we show how to update the tolerance and decision classes. We assume that, at time $t+1$, object $\overline{x}$ is added and object $\underline{x}$ is deleted simultaneously. Then, the change of tolerance and decision classes at time $t+1$ can be obtained as follows

$$T_{Pi}^{(t+1)} = \begin{cases} T_{Pi}^{(t)} - \{\underline{x}\} \text{ if } \ \underline{x} \in T_{Pi}^{(t)} \wedge \overline{x} \notin T_{Pi}^{(t)}, \\ T_{Pi}^{(t)} \cup \{\overline{x}\} \text{ if } \ \underline{x} \notin T_{Pi}^{(t)} \wedge \overline{x} \in T_{Pi}^{(t)}, \\ T_{Pi}^{(t)} \cup \{\overline{x}\} - \{\underline{x}\} \text{ if } \ \underline{x} \in T_{Pi}^{(t)} \wedge \overline{x} \in T_{Pi}^{(t)}, \\ T_{Pi}^{(t)}, \qquad \text{otherwise.} \end{cases} \tag{24}$$

$$D_j^{(t+1)} = \begin{cases} D_j^{(t)} - \{\underline{x}\} \text{ if } \ \underline{x} \in D_j^{(t)} \wedge \overline{x} \notin D_j^{(t)}, \\ D_j^{(t)} \cup \{\overline{x}\} \text{ if } \ \underline{x} \notin D_j^{(t)} \wedge \overline{x} \in D_j^{(t)}, \\ D_j^{(t)} \cup \{\overline{x}\} - \{\underline{x}\} \text{ if } \ \underline{x} \in D_j^{(t)} \wedge \overline{x} \in D_j^{(t)}, \\ D_j^{(t)}, \qquad \text{otherwise.} \end{cases} \tag{25}$$

Here we assume that object $\overline{x}$ belongs to existing tolerance classes and decision classes. In the opposite case, $\overline{x}$ will form a new class, respectively.

Since $\{T_{Pi}\}$ is a family of subset of U with $T_{Pi} \neq \emptyset$ and $\bigcup T_{Pi} = U$, then we consider $\mathcal{C}_P = \{T_{P1}, T_{P2}, .., T_{Pn}\}$ a special characteristic covering and $(U, \mathcal{C}_P)$ a covering approximation space. When $\{T_{Pi}\}$ changes, the changing trend of the third type of rough membership functions is as follows.

Theorem 15. *Let* $\text{IIS} = (U, C \cup \{d\}, V, f)$ *be an information system, where* $U = \{u_1, u_2, .., u_n\}$, $P \subseteq C$, $D \subseteq U$, TOL_P *is a tolerance relation on* U. *Suppose, object* $\overline{x}$ *is added and object* $\underline{x}$ *is deleted simultaneously from time* t *to time* $t+1$. *And*

If

$$
\begin{aligned}
&1.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&2.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&3.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),\\
&4.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&5.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&6.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)
\end{aligned}
$$

then $V^{(D)(t+1)} = V^{(D)(t)}$.

If

$$
\begin{aligned}
&7.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),\\
&8.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),\\
&9.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),\\
&10.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&11.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&12.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),
\end{aligned}
$$

then $V^{(D)(t+1)} > V^{(D)(t)}$.

If

$$
\begin{aligned}
&13.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),\\
&14.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&15.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),\\
&16.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),
\end{aligned}
$$

then $V^{(D)(t+1)} < V^{(D)(t)}$.

Proof.

1. Since $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right)$

$\Rightarrow T_{Pi}^{(t+1)} = T_{Pi}^{(t)}$ and $D^{(t+1)} = D^{(t)}$

$\Rightarrow |T_{Pi}^{(t+1)} \cap D^{(t+1)}| = |T_{Pi}^{(t)} \cap D^{(t)}|$ and $|D^{(t+1)}| = |D^{(t)}|$

$\Rightarrow \max\left\{\frac{\left|T_{Pi}^{(t+1)} \cap D^{(t+1)}\right|}{\left|D^{(t+1)}\right|} \,\middle|\, x \in T_{Pi}^{(t+1)}\right\} = \max\left\{\frac{\left|T_{Pi}^{(t)} \cap D^{(t)}\right|}{\left|D^{(t)}\right|} \,\middle|\, x \in T_{Pi}^{(t)}\right\}.$

According to Definition 8, $V^{(D)(t+1)} = V^{(D)(t)}$.

The proof of $2, 3, 4, 5$, and 6 is similar to that of 1.

7. Since $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$

$\Rightarrow T_{Pi}^{(t+1)} = T_{Pi}^{(t)}$ and $D^{(t+1)} = D^{(t)} - \{\underline{x}\}$

$\Rightarrow |T_{Pi}^{(t+1)} \cap D^{(t+1)}| = |T_{Pi}^{(t)} \cap D^{(t)}|$ and $|D^{(t+1)}| = |D^{(t)}| - 1 < |D^{(t)}|$,

$\frac{\left|T_{Pi}^{(t+1)} \cap D^{(t+1)}\right|}{\left|D^{(t+1)}\right|} > \frac{\left|T_{Pi}^{(t)} \cap D^{(t)}\right|}{\left|D^{(t)}\right|}$

$\Rightarrow \max\left\{\frac{\left|T_{Pi}^{(t+1)} \cap D^{(t+1)}\right|}{\left|D^{(t+1)}\right|} \middle| x \in T_{Pi}^{(t+1)}\right\} > \max\left\{\frac{\left|T_{Pi}^{(t)} \cap D^{(t)}\right|}{\left|D^{(t)}\right|} \middle| x \in T_{Pi}^{(t)}\right\}$.

According to Definition 8, $V^{(D)(t+1)} > V^{(D)(t)}$.

The proof of $8, 9, 10, 11$, and 12 is similar to that of 7.

13. Since $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$

$\Rightarrow T_{Pi}^{(t+1)} = T_{Pi}^{(t)} - \{\underline{x}\}$ and $D^{(t+1)} = D^{(t)} - \{\underline{x}\}$

$\Rightarrow |T_{Pi}^{(t+1)} \cap D^{(t+1)}| = |T_{Pi}^{(t)} \cap D^{(t)}| - 1$ and $|D^{(t+1)}| = |D^{(t)}| - 1 < |D^{(t)}|$,

$\frac{\left|T_{Pi}^{(t+1)} \cap D^{(t+1)}\right|}{\left|D^{(t+1)}\right|} < \frac{\left|T_{Pi}^{(t)} \cap D^{(t)}\right|}{\left|D^{(t)}\right|}$

$\Rightarrow \max\left\{\frac{\left|T_{Pi}^{(t+1)} \cap D^{(t+1)}\right|}{\left|D^{(t+1)}\right|} \middle| x \in T_{Pi}^{(t+1)}\right\} < \max\left\{\frac{\left|T_{Pi}^{(t)} \cap D^{(t)}\right|}{\left|D^{(t)}\right|} \middle| x \in T_{Pi}^{(t)}\right\}$.

According to Definition 8, $V^{(D)(t+1)} < V^{(D)(t)}$.

The proof of $14, 15$, and 16 is similar to that of 13. ∎

Theorem 16. *Let* $\text{IIS} = (U, C \cup \{d\}, V, f)$ *be an information system, where* $U = \{u_1, u_2, .., u_n\}$, $P \subseteq C$, $D \subseteq U$, TOL_P *is a tolerance relation on* U. *Suppose, object* $\overline{x}$ *is added and object* $\underline{x}$ *is deleted simultaneously from time* t *to time* $t+1$. *And*

If

1. $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right)$,
2. $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$,
3. $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$,
4. $\left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$,
5. $\left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right)$,
6. $\left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$,
7. $\left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right)$,

then $V^{(\sim D)(t+1)} = V^{(\sim D)(t)}$.

If

$$8.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),$$
$$9.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),$$
$$10.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),$$
$$11.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),$$
$$12.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \in D^{(t)}\right),$$
$$13.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \notin T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),$$

then $V^{(\sim D)(t+1)} > V^{(\sim D)(t)}$.

If

$$14.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \notin D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),$$
$$15.\ \left(\overline{x} \notin T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),$$
$$16.\ \left(\overline{x} \in T_{Pi}^{(t+1)} \wedge \overline{x} \in D^{(t+1)}\right) \wedge \left(\underline{x} \in T_{Pi}^{(t)} \wedge \underline{x} \notin D^{(t)}\right),$$

then $V^{(\sim D)(t+1)} < V^{(\sim D)(t)}$.

Proof. The proof of Theorem 16 is similar to that of Theorem 15. ■

This section shows a method for updating approximation sets when object $\overline{x}$ is added and object $\underline{x}$ is deleted in the incomplete information system. Next, we will present an example to illustrate this method.

Example 17. Given an information system is in Table 1.

Table 1. An incomplete information system at time step t

Car	a_1	a_2	a_3	a_4	d
x_1	Low	High	Full	High	Excel.
x_2	Medium	Medium	Full	Low	Excel.
x_3	Low	Medium	Medium	*	Poor
x_4	Low	*	*	High	Poor
x_5	High	Low	Full	High	Good
x_6	High	*	Full	High	Good
x_7	High	Low	Full	High	Poor
x_8	High	Low	Full	High	Good

Let $C = \{a_1, a_2, a_3, a_4\}$.

Based on Definition 2, we have

$T_C(1) = \{1\},\ T_C(2) = \{2\},\ T_C(3) = T_C(4) = \{3, 4\},$

$T_C(5) = T_C(6) = T_C(7) = T_C(8) = \{5, 6, 7, 8\}.$

From there we get the partition and the third type of rough membership function

$$U^{(t)} \Big/ TOL_C^{(t)} = \left\{ T_{C1}^{(t)}, T_{C2}^{(t)}, T_{C3}^{(t)}, T_{C4}^{(t)} \right\},$$

where
$T_{C1}^{(t)} = \{1\},\ T_{C2}^{(t)} = \{2\},\ T_{C3}^{(t)} = \{3,4\},\ T_{C4}^{(t)} = \{5,6,7,8\}$.
With $D^{(t)} = \{3,4,7\}$, we can calculate the third type of rough membership functions as follows

$$V_{T_C}^{D^{(t)}} = \frac{0}{x_1} + \frac{0}{x_2} + \frac{\frac{2}{3}}{x_3} + \frac{\frac{2}{3}}{x_4} + \frac{\frac{1}{3}}{x_5} + \frac{\frac{1}{3}}{x_6} + \frac{\frac{1}{3}}{x_7} + \frac{\frac{1}{3}}{x_8},$$
$$V_{T_C}^{\sim D^{(t)}} = \frac{\frac{1}{5}}{x_1} + \frac{\frac{1}{5}}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{\frac{3}{5}}{x_5} + \frac{\frac{3}{5}}{x_6} + \frac{\frac{3}{5}}{x_7} + \frac{\frac{3}{5}}{x_8}.$$

Let $\rho = 0.3$. According to Definition 10, the graded covering-based lower and upper approximations of $D^{(t)}$ can be obtained as follows

$$\overline{C}_{0.3}(D^{(t)}) = \{x_1, x_2, x_3, x_4\},$$
$$\underline{C}_{0.3}(D^{(t)}) = \{x_3, x_4, x_5, x_6, x_7, x_8\}.$$

Next, suppose that at time $t+1$, object x_9 is added and object x_1 is deleted shown in Table 2.

Table 2. An incomplete information system at time step $t+1$

Car	a_1	a_2	a_3	a_4	d
x_2	Medium	Medium	Full	Low	Excel.
x_3	Low	Medium	Medium	*	Poor
x_4	Low	*	*	High	Poor
x_5	High	Low	Full	High	Good
x_6	High	*	Full	High	Good
x_7	High	Low	Full	High	Poor
x_8	High	Low	Full	High	Good
x_9	Low	*	Medium	High	Good

Then the tolerance classes can be updated as follows

$$T_{C1}^{(t+1)} = T_{C1}^{(t)} - \{x_1\} = \emptyset,$$
$$T_{C2}^{(t+1)} = T_{C2}^{(t)} = \{x_2\},$$
$$T_{C3}^{(t+1)} = T_{C3}^{(t)} \cup \{x_9\} = \{x_3, x_4, x_9\},$$
$$T_{C4}^{(t+1)} = T_{C4}^{(t)} = \{x_5, x_6, x_7, x_8\}.$$

Since $x_9 \notin D^{(t+1)} \wedge x_1 \notin D^{(t)}$ then $D^{(t+1)} = D^{(t)}$.

According to Theorem 15 and Theorem 16, the third type of rough membership functions is calculated as follows

$$V_{T_C}^{D^{(t+1)}} = \frac{0}{x_2} + \frac{\frac{2}{3}}{x_3} + \frac{\frac{2}{3}}{x_4} + \frac{\frac{1}{3}}{x_5} + \frac{\frac{1}{3}}{x_6} + \frac{\frac{1}{3}}{x_7} + \frac{\frac{1}{3}}{x_8} + \frac{\frac{2}{3}}{x_9},$$

$$V_{T_C}^{\sim D^{(t+1)}} = \frac{\frac{1}{5}}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{\frac{3}{5}}{x_5} + \frac{\frac{3}{5}}{x_6} + \frac{\frac{3}{5}}{x_7} + \frac{\frac{3}{5}}{x_8} + \frac{\frac{1}{5}}{x_9}.$$

Based on Theorem 13 and Theorem 14, the graded covering-based lower and upper approximations of $D^{(t+1)}$ can be updated as follows

$$\overline{C}_{0.3}(D^{(t+1)}) = \overline{C}_{0.3}(D^{(t)}) - \{x_1\} \cup \{x_9\} = \{x_2, x_3, x_4, x_9\},$$

$$\underline{C}_{0.3}(D^{(t+1)}) = \underline{C}_{0.3}(D^{(t)}) \cup \{x_9\} = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\}.$$

By the above example, we illustrate that by using the third type of rough membership functions we can update the approximation sets. When the objects are added and deleted simultaneously from time t to time $t+1$, the third type of rough membership functions will change. Based on this change, we can update the approximation sets by modifying the original sets. According to the above example, to fix the approximation set, we just need to calculate the $V_{T_C}^{D^{(t+1)}}(x_9)$ and $V_{T_C}^{(\sim D)^{(t+1)}}(x_9)$ without recalculating all the objects.

5. CONCLUSION

Approximation sets are important concepts of the rough set theory. When the objects change over time, the approximation set also change. Our contribution is to introduce a method for updating graded covering-based approximation sets in the incomplete information systems under the objects variation. At that time, the approximation sets can be formally based on the third type of rough membership function. When the objects vary, they lead to the variations of tolerance classes and decision classes. This makes the third type of rough membership function change. Based on the change of the third type of rough membership function, we suggest a way of maintenance approximation sets. The approximation sets can be updated by modifying the partial sets without recomputing sets from the very beginning when the objects vary. Our future work will focus on algorithm development, experimentation, evaluation and comparison in real databases for the validation of the proposed approaches.

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