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ASSOCIATED EQUATIONS AND THEIR CORRESPONDING RESONANCE CURVE

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In the theory of nonlinear oscillations, in order to identify the resonance curve we usually try to eliminate the dephase θ in the equations of stationary oscillations. We obtain thus a certain frequency-amplitude relationship.

In simple cases when the mentioned equations contain only and linearly the first harmonics ($\sin\theta, \cos\theta$) the elimination of θ is elementary, by using the trigonometrical identity $\sin^2\theta + \cos^2\theta = 1$.

In general, high harmonics ($\sin 2\theta$, $\cos 2\theta$, etc.) are present. Consequently the expressions of $\sin\theta$, $\cos\theta$ are cumbersome or do not exist and the analytical elimination of θ is quite inconvenient or impossible. For this reason, to identify the resonance curve of complicated systems, we use the numerical method.

Below, intending to develop the analytical method, we shall propose a procedure enabling us to transform the "original" complicated equations of stationary oscillations into the so-called associated ones, only and linearly containing $\sin\theta$, $\cos\theta$. The equivalence of the original and associated equations will be treated and the associated resonance curve-that is determined by the associated equations-will be analyzed

The discussion will be restricted to a simple practical case in which, beside $\sin\theta$ and $\cos\theta$, only $\sin 2\theta$ and $\cos 2\theta$ are present. Nevertheless, the method proposed and the results obtained can be generalized.

§1. System under consideration. The elimination of 2θ

Let

$$\begin{aligned} \dot{a} &= \varepsilon f_0(\omega, a, \theta) = \varepsilon\Big\{P_0 + S_{01}\sin\theta + C_{01}\cos\theta + M\sin 2\theta\Big\}, \\ a\dot{\theta} &= \varepsilon g_0(\omega, a, \theta) = \varepsilon\Big\{Q_0 + R_{01}\sin\theta + K_{01}\cos\theta + M\cos 2\theta\Big\}, \end{aligned} \tag{1.1}$$

be the averaged differential equations governing the oscillating system of interest, where: a, θ are amplitude and dephase angle, respectively; ω is the frequency; overdots denote the derivation relative to time t; $\varepsilon > 0$ is a small formal parameter; P_0, Q_0, S_0, C_{01}, R_{01}, K_{01} are polynomials in ω, a.

Constant amplitude and dephase of stationary oscillations satisfy the equations:

$$\begin{aligned} f_0 &= P_0 + S_{01}\sin\theta + C_{01}\cos\theta + M\sin 2\theta = 0, \\ g_0 &= Q_0 + Q_{01}\sin\theta + K_{01}\cos\theta + M\cos 2\theta = 0. \end{aligned} \tag{1.2}$$

The equations (1.2) will be called "original" ones. They determine the "true" "original" resonance curve-denoted by C_0.

We use the following two step procedure to eliminate $(\sin 2\theta, \cos 2\theta)$: First, we form the equations, equivalent to (1.2) and of the same structure as (1.2)

$$\begin{aligned} f_1 &= f_0\cos\theta - g_0\sin\theta = \\ &= P_1 + S_{11}\sin\theta + C_{11}\cos\theta + S_{12}\sin 2\theta + C_{12}\cos 2\theta = 0, \\ g_1 &= f_0\sin\theta + g_0\cos\theta = \\ &= Q_1 + R_{11}\sin\theta + K_{11}\cos\theta + R_{12}\sin 2\theta + K_{12}\cos 2\theta = 0, \end{aligned} \tag{1.3}$$

where:

$$\begin{aligned} P_1 &= \frac{1}{2}(C_{01} - R_{01}); \quad S_{11}M - Q_0; \quad C_{11} = P_0, \\ S_{12} &= \frac{1}{2}(S_{01} - K_{01}); \quad C_{12} = \frac{1}{2}C_{01} + R_{01}), \\ Q_1 &= \frac{1}{2}(S_{01} + K_{01}); \quad R_{11} = P_0; \quad K_{11} = M + Q_0; \\ R_{12} &= \frac{1}{2}(C_{01} + R_{01}); \quad K_{12} = \frac{1}{2}(K_{01} - S_{01}). \end{aligned} \tag{1.4}$$

Then, we choose suitable combinations of the form:

$$\begin{aligned} f &= p_{10}f_0 + q_{10}g_0 + p_{11}f_1 + q_{11}g_1 = 0, \\ g &= p_{20}f_0 + q_{20}g_0 + p_{21}f_1 + q_{21}g_1 = 0. \end{aligned} \tag{1.5}$$

Evidently, f does not contain $\sin 2\theta$, $\cos 2\theta$ if:

$$\begin{aligned} M \cdot p_{10} + S_{12} \cdot p_{11} + R_{12} \cdot q_{11} = 0, \\ M \cdot q_{10} + C_{12} \cdot P_{11} + K_{12} \cdot q_{11} = 0. \end{aligned} \tag{1.6}$$

We can choose, for instance:

$$p_{10} = S_{12}; \quad p_{11} = -M; \quad q_{10} = C_{12}; \quad q_{11} = 0. \tag{1.7}$$

Similarly, g does not contain $\sin 2\theta$, $\cos 2\theta$ if we choose:

$$p_{20} = C_{12}; \quad p_{21} = 0; \quad q_{20} = -S_{12}; \quad q_{21} = -M. \tag{1.8}$$

Finally, we obtain the following equations, which do not contain $\sin 2\theta$, $\cos 2\theta$:

$$\begin{aligned} f &= S_{12}f_0 + C_{12}g_0 - Mf_1 = \\ &= (S_{12} - M\cos\theta)f_0 + (C_{12} + M\sin\theta)g_0 = 0, \\ g &= C_{12}f_0 - S_{12}g_0 - Mg_1 = \\ &= (C_{12} - M\sin 2\theta)f_0 - (S_{12} + M\cos 2\theta)g_0 = 0. \end{aligned} \tag{1.9}$$

or:

$$\begin{aligned} f &= A\sin\theta + B\cos\theta - E = 0, \\ g &= G\sin\theta + H\cos\theta - K = 0. \end{aligned} \tag{1.10}$$

where:

$$\begin{aligned} A &= S_{12}S_{01} + C_{12}R_{01} - MS_{11}; \quad & B &= S_{12}C_{01} + C_{12}K_{01} - MC_{11}, \\ E &= MP_1 - S_{12}P_0 - C_{12}Q_0; \quad & K &= MQ_1 - C_{12}P_0 + S_{12}Q_0, \\ G &= C_{12}S_{01} - S_{12}R_{01} - MR_{11}; \quad & H &= C_{12}C_{01} - S_{12}K_{01} - MK_{11}. \end{aligned} \tag{1.11}$$

The equations (1.10) will be called associated ones. They determine the so-called "associated" resonance curve-denoted by C.

§2. The equivalence and the non equivalence domains

Naturally, a question arises: The original and the associated equations, are they equivalent? C_0 and C, do they coincide each with another?

It is noted that, the transformation $(f_0, g_0) \to (f, g)$ has matrix:

$$\left\{ \mathcal{T} \right\} = \begin{Bmatrix} (S_{12} - M\cos\theta) & (C_{12} + M\sin\theta) \\ (C_{12} - M\sin\theta) & -(S_{12} + M\cos\theta) \end{Bmatrix} \tag{2.1}$$

Although, in general, the matrix of transformation depends on ω, a and also on θ, its determinant T depends only on ω, a:

$$T = \begin{vmatrix} (S_{12} - M\cos\theta) & (C_{12} + M\sin\theta) \\ (C_{12} - M\sin\theta & -(C_{12} + M\cos\theta) \end{vmatrix} = M^2 - (S_{12}^2 + C_{12}^2) \tag{2.2}$$

Thus, in the (semi upper) plane $\mathcal{R}(\omega, a > 0)$, it is necessary to distinguish two domains: the equivalence domain and the non equivalence one.

The equivalence domain satisfies the inequality:

$$T = M^2 - (S_{12}^2 + C_{12}^2) \neq 0. \tag{2.3}$$

Evidently, in this domain, original and associated equations are equivalent and, consequently, corresponding parts of the original resonance curve C_0 and of the associated one C coincide. It means that, together with the associated resonance curve C (in the equivalence domain) we simultaneously obtain the original-the "true"-resonance curve C_0 (in the same domain).

The non equivalence line (domain) is determined by the equality:

$$T = M^2 - (S_{12}^2 + C_{12}^2) = 0. \tag{2.4}$$

In the non equivalence line, the original and the associated equations are not equivalent, C_0 differs from C. However, from (1.9) we see that (a, θ) satisfy the original equations (f_0, g_0). They also satisfy the associated ones (f, g). This means that $C_0 \subset C$, the elements of the "original"-the "true"-resonance curve C_0 (in the non equivalence domain) must be and may be found among those of the associated resonance curve C. In other words, C contains "strange" "superfluous" "extraneous" elements-those belonging to C but not to C_0.

§3. The associated resonance curve C

We apply the procedure presented in [1] to examine the associated equations (1.10), trying to identify the associated resonance curve C.

Choosing P_0, Q_0, P_1, Q_1, S_{12}, C_{12}, M as "basic" coefficients, we can express other coefficients as:

$$\begin{aligned}
&S_{01} = S_{12} + Q_1; \quad C_{01} = C_{12} + P = 1; \quad R_{01} = C_{12} - P_1; \quad K_{01} = Q_1 - S_{12},\\
&S_{11} = M - Q_0; \quad K_{11} = M + Q_0; \quad C_{11} = R_{11} = P_0.
\end{aligned} \tag{3.1}$$

Then, inserting T, we have:

$$\begin{aligned}
A &= S_{12}S_{01} + C_{12}R_{01} - MS_{11} =\\
&= S_{12}(S_{12} + Q_1) + C_{12}(C_{12} - P_1) - M(M - Q_0) =\\
&= (S_{12}Q_1 - C_{12}P_1 + MQ_0) - (M^2 - S_{12}^2 - C_{12}^2) = X - T,\\
H &= C_{12}C_{01} - S_{12}K_{01} - MK_{11} =\\
&= C_{12}(C_{12} + P_1) - S_{12}(Q_1 - S_{12}) - M(M + Q_0) =\\
&= -(S_{12}Q_1 - C_{12}P_1 + MQ_0) - (M^2 - S_{12}^2 - C_{12}^2) = -(X + T),
\end{aligned}$$

$$\begin{aligned} B &= S_{12}C_{01} + C_{12}K_{01} - MC_{11} = \\ &= S_{12}(C_{12} + P_1) + C_{12}(Q_1 - S_{12}) - MP_0 = \\ &= S_{12}P_1 + C_{12}Q_1 - MP_0 = G, \\ G &= C_{12}S_{01} - S_{12}R_{01} - MR_{11} = \\ &= C_{12}(S_{12} + Q_1) - S_{123}(C_{12} - P_1) - MP_0 = B, \\ E &= MP_1 - S_{12}P_0 - C_{12}Q_0, \\ K &= MQ_1 - C_{12}P_0 + S_{12}Q_0, \\ T &= M^2 - (S_{12}^2 + C_{12}^2), \\ X &= S_{12}Q_1 - C_{12}P_1 + MQ_0. \end{aligned} \tag{3.2}$$

Three characteristic determinants of the associated equations can be written on the basis of (3.2):

$$\begin{aligned} D &= \begin{vmatrix} A & B \\ G & H \end{vmatrix} = \begin{vmatrix} (X+T) & B \\ B & -(X+T) \end{vmatrix} = T_1^2 - (X^2 + B^2), \\ D_1 &= \begin{vmatrix} E & B \\ K & H \end{vmatrix} = \begin{vmatrix} E & B \\ K & -(X+T) \end{vmatrix} = -\Big\{ET + (EX + BK)\Big\}, \\ D_2 &= \begin{vmatrix} A & E \\ G & K \end{vmatrix} = \begin{vmatrix} (X-T) & E \\ B & K \end{vmatrix} = -KT + (KX - EB). \end{aligned} \tag{3.3}$$

The associated frequency-amplitude relationship is:

$$\begin{aligned} W(\omega, a) &= D_1^2 + D_2^2 - D^2 = \\ &= \Big\{ET + (EX + BK)\Big\}^2 + \Big\{-KT + (KX - EB)\Big\}^2 \\ &\quad - \Big\{T^2 - (X^2 + B^2)\Big\}^2 = 0. \end{aligned} \tag{3.4}$$

An important property: the function $W(\omega, a)$ admit T as a factor. Indeed, along $T = 0$, we have:

$$\begin{aligned} W(\omega, a)\big|_{T=0} &= \Big\{(EX + BK)^2 + (KX - EB)^2 - (X^2 + B^2)^2\Big\}_{T=0} = \\ &= \Big\{(E^2 + K^2 - X^2 - B^2)(X^2 + B^2)\Big\}_{T=0}. \end{aligned} \tag{3.5}$$

Using the expressions E, K, X, B in (3.2), we successively obtain:

$$\begin{aligned} E^2 + K^2 &= M^2(P_1^2 + Q_1^2) + (S_{12}^2 + C_{12}^2)(P_0^2 + Q_0^2) \\ &\quad - 2M(S_{12}P_0P_1 + C_{12}P_0Q_1 + C_{12}P_1Q_0 - S_{12}Q_1Q_0), \\ X^2 + B^2 &= M^2(P_0^2 + Q_0^2) + (S_{12}^2 + C_{12}^2)(P_1^2 + Q_1^2) \\ &\quad - 2M(S_{12}P_0P_1 + C_{12}P_0Q_1 + C_{12}P_1Q_0 - S_{12}Q_)Q_1), \end{aligned} \tag{3.6}$$

$$E^2 + K^2 - X^2 - B^2 = (M^2 - S_{12}^2 - C_{12}^2)(P_1^2 + Q_1^2 - P_0^2 - Q_0^1) =$$
$$= T(P_1^2 + Q_1^2 - P_0^2 - Q_0^2).$$

Therefore:

$$W(\omega, a)\big|_{T=0} = \Big\{T(P_1^2 + Q_1^2 - P_0^2 - Q_0^2)(X^2 + B^2)\Big\} = 0. \tag{3.7}$$

Thus, the associated frequency-amplitude relationship can be written as:

$$W(\omega, a) = T \cdot W_0(\omega, a) = 0, \tag{3.8}$$

where:

$$W_0(\omega, a) = -T^3 + \Big\{2(X^2 + B^2) + E^2 + K^2\Big\}T + \tag{3.9}$$
$$+ \Big\{2E(EX + BK) + 2K(EB - KX) + (P_1^2 + Q_1^2 - P_0^2 - Q_0^2)(X^2 + B^2)\Big\}.$$

In other words, the non equivalence line $T = 0$ is a branch of the curve $W(\omega, a) = 0$

Is $T = 0$ a branch of the associated resonance curve C and if it is, does it belong to the ordinary part C_1 or to the critical part C_2?

We know that the resonance curve is defined as the locus of those points (ω, a), at each one, the equations of stationary oscillations (which become trigonometrical since ω, a already fixed) are solvable.

From the results obtained in [1], for an arbitrary point $I(\omega, a)$ of the curve (3.4): $W(\omega, a) = 0$, the given definition can be translated as follows:

- If $D(\omega, a) \neq 0$, I "automatically" belongs to C,

- If $D(\omega, a) = 0$ and $\text{rank}\{D\} = 1$, I belongs to C_2 on the condition that the trigonometrical restrictions ($A^2 + B^2 \geq E^2$, $G^2 + H^2 \geq K^2$) are satisfied,

- If $D(\omega, a) = 0$ and $\text{rank}\{D\} = 0$, I belongs to C_2 on the condition that $E = 0$, $K = 0$; in this case, the dephase is arbitrary.

Let us calculate the determinant D along $T = 0$. We have

$$D\big|_{T=0} = -(X^2 + B^2) = 0. \tag{3.10}$$

Thus, in practice, the "whole" non equivalence line $T = 0$ or most of its points (at which $D < 0$) belong to C_1. It remains to examine some particular points satisfying $T = 0$, $X = 0$, $B = 0$. Form (3.2), it follows $A = B = G = H = 0$ and then, $\text{rank}\{D\} = 0$. For last two coefficients E, K we note that:

- If $M = 0$, from $T = M^2 - (S_{12}^2 + C_{12}^2) = 0$, it follows $S_{12} = C_{12} = 0$ and $E = 0$, $K = 0$ are evident

- If $M \neq 0$, from $X = 0$, $B = 0$, it follows:

$$Q_0 = \frac{1}{M}(C_{12}P_1 - S_{12}Q_1), \quad P_0 = \frac{1}{M}(C_{12}Q_1 + S_{12}P_1). \tag{3.11}$$

Substituting (3.11) into the expressions of E, K we obtain $E = K = 0$. Now, we can conclude that the non equivalence line forms a branch of the associated resonance curve C. Often, it belongs to the ordinary part C_1; in particular case, it may contain some critical points.

§4. Example

As an illustration, we consider a system of Van der Pol type [2, 3]:

$$\ddot{x} + \omega^2 x = \varepsilon\Big\{\omega\Delta x + \big[1 - (x + q\cos\omega t)^2\big]\dot{x} \tag{4.1}$$

(all the notations have been explained in [3]).

The original equations are:

$$\begin{cases} f_0 &= \Big(\frac{1}{4}a^2 + \frac{1}{2}q^2 - 1\Big) + \frac{1}{2}qa\cos\theta - \frac{1}{4}q^2\cos 2\theta = 0, \\ g_0 &= \Delta + \frac{1}{2}qa\sin\theta + \frac{1}{4}q^2\sin 2\theta = 0 \end{cases} \tag{4.2}$$

(light differences on the order of the equations and the signs of the second harmonics in comparison with (1.2)).

The matrix of transformation is:

$$\Big\{\mathcal{T}\Big\} = \begin{Bmatrix} 2a + q\cos\theta & q\sin\theta \\ -q\sin\theta & 2a - q\cos\theta \end{Bmatrix} \tag{4.3}$$

and the associated equations are:

$$\begin{cases} f &= 2a\Big(\frac{1}{4}a^2 + \frac{1}{2}q^2 - 1\Big) - q\Delta\sin\theta + q\Big(\frac{5}{4}a^2 + \frac{1}{4}q^2 - 1\Big)\cos\theta = 0, \\ g &= 2a\delta + q\Big(\frac{3}{4}a^2 - \frac{3}{4}q^2 + 1\Big)\sin\theta - q\Delta\cos\theta = 0. \end{cases} \tag{4.4}$$

The non equivalence line is:

$$T = 4a^2 - q^2 = 0 \quad \text{i.e.} \quad a^2 = a_0^2 = \frac{q^2}{4}. \tag{4.5}$$

Inserting T in the expressions of the coefficients of the associated equations, we have:

$$\begin{aligned}
&A = -q\Delta; \quad H = -q\Delta; \quad K = -2a\Delta \\
&B = q\Big(\frac{5}{4}a^2 + \frac{1}{4}q^2 - 1\Big) = \frac{9}{16}(5T + X); \\
&G = q\Big(\frac{3}{4}a^2 - \frac{3}{4}q^2 + 1\Big) = \frac{9}{16}(3T - X) \\
&E = -2a\Big(\frac{1}{4}a^2 + \frac{1}{2}q^2 - 1\Big) = -\frac{2a}{6}(T + X); \\
&X = 9q^2 - 16.
\end{aligned} \tag{4.6}$$

Three characteristic determinants are:

$$\begin{aligned}
D &= \begin{vmatrix} A & B \\ G & H \end{vmatrix} = q^2\Big\{\Delta^2 - \frac{1}{256}(5T + X)(3T - X)\Big\}; \\
D_1 &= \begin{vmatrix} E & B \\ K & H \end{vmatrix} = \frac{2qa\Delta}{16}(6T + 2X); \\
D_2 &= \begin{vmatrix} A & E \\ G & K \end{vmatrix} = 2aq\Big\{\Delta^2 + \frac{1}{256}(T + X)(3T - X)\Big\}.
\end{aligned} \tag{4.7}$$

The frequency-amplitude relationship is:

$$\begin{aligned}
W(\Delta, a^2)\big|_{T=0} &= D_1^2 + D_2^2 - D^2 = \\
&= \frac{4a^2q^2\Delta^2}{256}(6T + 2X)^2 + 4a^2q^2\Big\{\Delta^2 + \frac{1}{256}(T + X)(3T - X)\Big\}^2 - \\
&\quad - q^4\Big\{\Delta^2 - \frac{1}{256}(5T + X)(3T - X)\Big\}^2 = 0.
\end{aligned} \tag{4.8}$$

Along the non equivalence line $T = 0$ we have:

$$\begin{aligned}
W(\Delta, a^2)\big|_{T=0} &= \Big\{\frac{16a^2q^2\Delta^2}{256}X^2 + 4a^2q^2\Big(\Delta^2 - \frac{X^2}{256}\Big)^2 - q^4\Big(\Delta^2 + \frac{X^2}{256}\Big)^2\Big\}_{T=0} = \\
&= \Big\{(4a^2 - q^2)\Big(q^2\Delta^4 + \frac{2\Delta^2X^2}{256} + \frac{q^2X^4}{256}\Big)\Big\}_{T=0} = 0,
\end{aligned} \tag{4.9}$$

$$D(\Delta, a^2)\big|_{T=0} = q^2\Big(\Delta^2 + \frac{X^2}{256}\Big)_{T=0} \geq 0. \tag{4.10}$$

Therefore:

- If $X \neq 0$ i.e. $q^2 \neq \dfrac{16}{9}$, the non-equivalence line $T = 0$ is an ordinary branch of the associated resonance curve C.

- If $X = 0$, i.e. $q^2 = \frac{16}{9}$, the non-equivalence line $T = 0$ is also an ordinary branch of the associated resonance curve C except the point I ($\Delta = 0$, $a^2 = a_0^2$).

Conclusion

The method of elimination of the dephase θ in the equations containing $\sin 2\theta$, $\cos 2\theta$ has been presented. The original equations can be transformed into the associated ones, which contain only and linearly $\sin\theta$, $\cos\theta$. The two systems of equations are not equivalent in the non-equivalence line. The latter is a particular branch of the associated resonance curve.

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HỆ LIÊN HỢP VÀ ĐƯỜNG CỘNG HƯỞNG CỦA NÓ

Vấn đề khử pha θ trong các phương trình dao động dừng được quan tâm. Trường hợp các phương trình chứa các ác mônic thứ hai $\sin 2\theta$, $\cos 2\theta$ được xem xét. Đã cho thấy các phương trình gốc có thể biến đổi thành các phương trình liên hợp chỉ chứa ở bậc nhất các ác mônic $\sin\theta$, $\cos\theta$. Các phương trình gốc và liên hợp không tương đương nên đường không tương đương; đường này là một nhánh của đường cộng hưởng của hệ liên hợp.