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# IDENTIFYING THE RESONANCE CURVE OF A SYSTEM SUBJECTED TO LINEAR AND QUADRATIC PARAMETRIC EXCITATIONS (case with damping)

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In [3], a quasi linear oscillating system simultaneously subjected to linear and cubic parametric excitations has been studied. In the present paper, we examine the case in which the linear and quadratic parametric excitations are present.

The asymptotic method [1] is applied. We are interested in the method of identifying the resonance curve. The results obtained show that the "associated" equations can be used.

## §1. System under consideration - Original and Associated equations

Let us consider a quasilinear oscillating system described by the differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon\Big\{\Delta x - h\dot{x} - \gamma x^3 + 2px\cos 2\omega t + 2qx^2\cos\omega t\Big\} \tag{1.1}$$

where $2p > 0$, $2q > 0$ and $2\omega$, $\omega$ are intensities and frequencies of the linear and the quadratic parametric excitations; other notations have been explained in [2].

Introducing slowly varying amplitude $a$ and dephase $\theta$ by means of formulas

$$x = a\cos\psi, \quad \dot{x} = -\omega a\sin\psi, \quad \psi = \omega t + \theta \tag{1.2}$$

and applying the asymptotic method [1], the following averaged differential equations will be obtained:

$$\begin{cases} \dot{a} = \dfrac{-\varepsilon a}{2\omega} f_0 = \dfrac{-\varepsilon a}{2\omega}\Big\{h\omega + p\sin 2\theta + \dfrac{1}{2}qa\sin\theta\Big\}, \\ a\dot{\theta} = \dfrac{-\varepsilon a}{2\omega} g_0 = \dfrac{-\varepsilon a}{2\omega}\Big\{\Big(\Delta - \dfrac{3\gamma}{4}a^2\Big) + p\cos 2\theta + \dfrac{3}{2}qa\cos\theta\Big\}. \end{cases} \tag{1.3}$$

Stationary oscillations - their constant amplitude and dephase satisfy the equations

$$\begin{cases} f_0 = h\omega + p\sin 2\theta + \dfrac{1}{2}qa\sin\theta = 0, \\ g_0 = \left(\Delta - \dfrac{3\gamma}{4}a^2\right) + p\cos 2\theta + \dfrac{3}{2}qa\cos\theta = 0. \end{cases} \tag{1.4}$$

High harmonics $(\sin 2\theta, \cos 2\theta)$ and the term $h\omega$ make difficulty in analytically solving the system (1.4) and identifying the resonance curve. In order to eliminate mentioned harmonics, we use the transformation

$$\begin{cases} f = (2p\cos\theta + qa)f_0 - 2p\sin\theta g_0 = 0, \\ g = 2p\sin\theta f_0 + (2p\cos\theta - qa)g_0 = 0, \end{cases} \tag{1.5}$$

or in detail

$$\begin{cases} f = \left\{2p\left[p - \left(\Delta - \dfrac{3\gamma}{4}a^2\right)\right] + \dfrac{1}{2}q^2a^2\right\}\sin\theta + 2ph\omega\cos\theta + h\omega qa = 0, \\ g = 2ph\omega\sin\theta + \left\{2p\left[p + \left(\Delta - \dfrac{3\gamma}{4}a^2\right)\right] - \dfrac{3}{2}q^2a^2\right\}\cos\theta - \\ \quad - qa\left(\Delta - \dfrac{3\gamma}{4}a^2 - 2p\right) = 0. \end{cases} \tag{1.6}$$

The equations (1.4) and (1.6) are called respectively the "original" and the "associated" equations; and their corresponding resonance curves are called "original" and "associated" resonance curves and denoted by $C_0$ and $C$. The transformation $(f_0, g_0) \to (f, g)$ has matrix

$$\{T\} = \begin{Bmatrix} 2p\cos\theta + qa & -2p\sin\theta \\ 2p\sin\theta & 2p\cos\theta - qa \end{Bmatrix}. \tag{1.7}$$

Although $\{T\}$ depends on $a$ and $\theta$, its determinant depends only on $a$:

$$T = 4p^2 - q^2a^2. \tag{1.8}$$

In the (semi-upper) plane $R(\Delta; a^2 > 0)$, it is necessary to distinguish two domains: the equivalence domain and the non-equivalence one.

The equivalence domain satisfy

$$T = 4p^2 - q^2a^2 \neq 0 \quad \text{i,e.} \quad a^2 \neq a_*^2 = \frac{4p^2}{q^2}. \tag{1.9}$$

Obviously, in the equivalence domain, the original and associated equations are equivalent; consequently, corresponding parts of $C_0$ and $C$ coincide each with another.

The non-equivalence domain is the line

$$T = 0 \quad \text{i.e.} \quad a^2 = a_*^2. \tag{1.10}$$

Along the non-equivalence line, the equations $(f_0, g_0)$ and $(f, g)$ are not equivalent, $C_0$ differs from C. However, from (1.5), it follows that all solutions of $(f_0, g_0)$ also satisfy $(f, g)$. Hence $C_0 \subset C$, i.e. along the non-equivalence line the elements (representative point and dephase angle) of the original resonance curve $C_0$ must be and may be found among those of the associated resonance curve $C$. In other words, the non-equivalence line contains "strange" elements; By rejecting them, we obtain "exactly" $C_0$ from $C$.

## §2. The associated resonance curve

The associated equations are simple. They contain only and linearly $\sin\theta$; $\cos\theta$. The elimination of $\theta$ is elementary and we can easily obtain the required frequency-amplitude relationship.

Following the procedure in [2] (which has been used in [3]), we rewritte the associated equations in the form

$$\begin{cases} f = A\sin\theta + B\cos\theta - E = 0, \\ g = G\sin\theta + h\cos\theta - K = 0, \end{cases} \tag{2.1}$$

where:

$$\begin{aligned} A &= 2p\Big[p - \Big(\Delta - \frac{3\gamma}{4}a^2\Big)\Big] + \frac{1}{2}q^2a^2 = -\frac{1}{2}\Big(T + 4pX\Big), \\ B &= G = 2ph\omega; \quad E = -h\omega qa; \; K = qaX, \\ H &= 2p\Big[p + \Big(\Delta - \frac{3\gamma}{4}a^2\Big)\Big] - \frac{3}{2}q^2a^2 = \frac{1}{2}(3T + 4pX). \end{aligned} \tag{2.2}$$

Three characteristic determinants are:

$$\begin{aligned} D &= \begin{vmatrix} A & B \\ G & H \end{vmatrix} = -\Big\{\frac{1}{4}(T + 4pX)(3T + 4pX) + 4p^2h^2\omega^2\Big\}, \\ D_1 &= \begin{vmatrix} E & B \\ K & H \end{vmatrix} = -\frac{1}{2}h\omega qa(3T + 8pX), \\ D_2 &= \begin{vmatrix} A & E \\ G & K \end{vmatrix} = -\frac{1}{2}qa\Big\{TX + 4p(X^2 - h^2\omega^2)\Big\}, \end{aligned} \tag{2.3}$$

and the associated frequency - amplitude relationship is:

$$\begin{aligned} W(\Delta, a^2) &= D_1^2 + D_2^2 - D^2 = \\ &= \frac{1}{4}q^2a^2\Big\{h^2\omega^2(3T + 8pX)^2 + \big[TX + 4p(X^2 - h^2\omega^2)\big]^2\Big\} \\ &= \Big\{\frac{1}{4}(T + 4pX)(3T + 4pX) + 4p^2h^2\omega^2\Big\}^2 = 0. \end{aligned} \tag{2.4}$$

An important interesting property is that the function $W(\Delta, a^2)$ admits $T$ as a factor. In fact, we have:

$$\begin{aligned} & W(\Delta, a^2)\big|_{T=0} = \\ & = \left\{\frac{1}{4}q^2a^2\left[64p^2h^2\omega^2X^2 + 16p^2(X^2 - h^2\omega^2)^2\right] - 16p^4(X^2 + h^2\omega^2)\right\}_{T=0} \\ & = -4p^2\left\{(4p^2 - q^2a^2)(X^2 + h^2\omega^2)^2\right\}_{T=0} \\ & = -4p^2\left\{T(X^2 + h^2\omega^2)^2\right\}_{T=0} = 0. \end{aligned} \tag{2.5}$$

Therefore, the associated frequency amplitude relationship (2.4) can be factorized as

$$W(\Delta, a^2) = TW_0(\Delta, a^2) = 0 \tag{2.6}$$

where

$$\begin{aligned} W_0(\Delta, a^2) = & -\frac{9}{16}T^3 - \left(\frac{9}{4}h^2\omega^2 + 6pX + \frac{1}{4}X^2\right)T^2 \\ & - (3p^2h^2\omega^2 + 10ph^2\omega^2X + 21p^2X^2 + 2pX^3)T \\ & + [8p^3Xh^2\omega^2 - 24p^3X^3 - 4p^2(X^2 + h^2\omega^2)^2]. \end{aligned} \tag{2.7}$$

This means that the non-equivalence line $T = 0$ is a branch of the curve $W(\Delta, a^2) = 0$ (which, in general, does not coincide with the associated resonance curve).

The associated resonance curve C can be identified by the associated frequency amplitude relationship. It consists of two parts: - the ordinary part $C_1$ and - the critical one $C_2$. Let us separately analyze each of these two parts. The ordinary part $C_1$ is located in the ordinary region

$$R_1 : D \neq 0. \tag{2.8}$$

It is noted that, along $T = 0$, the determinant $D$ is negative:

$$D\big|_{T=0} = -4p^2(X^2 + h^2\omega^2) < 0, \tag{2.9}$$

i.e. the non - equivalence line is located in the ordinary region. On the other hand, since $T = 0$ satisfies (2.6), along $T = 0$, the algebraic solutions $u = \sin\theta$, $v = \cos\theta$ of the system (2.1) satisfy the trigonometrical identity $u^2 + v^2 = 1$.

Thus, the non-equivalence line $T = 0$ is an ordinary branch - a branch belonging to the ordinary part $C_1$ of the associated resonance curve C.

The critical part $C_2$ is located in the critical region:

$$R_2 : D = 0 \text{ i.e. } \frac{3}{4}T^2 + 4pTX + 4p^2(X^2 + h^2\omega^2) = 0 \tag{2.10}$$

and (since the rank of the matrix $D$ is equal to 1) satisfy:

- The compatibility conditions

$$D_1 = 0 \quad \text{i.e. } 3T + 8pX = 0, \tag{2.11}$$

$$D_2 = 0 \quad \text{i.e. } TX + 4p(X^2 - h^2\omega^2) = 0. \tag{2.12}$$

- The trigonometrical conditions

$$A^2 + B^2 \geq E^2 \quad \text{i.e. } \frac{1}{4}(T + 4pX)^2 + 4p^2h^2\omega^2 \geq h^2\omega^2q^2a^2, \tag{2.13}$$

$$G^2 + H^2 \geq K^2 \quad \text{i.e. } 4p^2h^2\omega^2 + \frac{1}{4}(3T + 4pX)^2 \geq q^2a^2X^2. \tag{2.14}$$

If $\omega^2$ is substituted by $(\Delta + 1)$, the equality (2.10) becomes a quadratic equation of unknown $\Delta$, admitting as solutions

$$\Delta = \frac{3\gamma}{4}a^2 + \frac{q^2a^2}{2p} - \frac{h^2}{2} \pm \sqrt{T^2 - 16p^2h^2 + \left[8pTh^2 - 16p^2h^2\left(\frac{3\gamma}{4}a^2 + 2p\right) + 2p^2h^2\right]} \tag{2..15}$$

or approximatively

$$\Delta = \frac{3\gamma}{4}a^2 + \frac{q^2a^2}{2p} \pm \frac{1}{4p}\sqrt{T^2 - 16p^2h^2}\ . \tag{2.16}$$

Thus, the critical region is an "hyperbola". It consists of two branches respectively located above and under the non-equivalence line $T = 0$, and admits as asymptotic two lines $\Delta = \frac{3\gamma}{4}a^2 + \frac{q^2a^2}{2p} \pm \frac{T}{4p}$ i.e. the lines $A = 0$ and $H = 0$.

The lower branch, denoted by $D'$, intersects the abscissa axis at $\Delta = \pm\sqrt{p^2 - h^2}$, reaches its maximum at $a^2 = a_*^2 - \frac{4ph}{q^2}$. As $h$ increases, $D'$ moves down then disappears when $h > p$ (it passes into the semi - lower plane $R(\Delta, a^2 < 0)$.

The upper branch, denoted by $D''$, has a minimum at $a^2 = a_*^2 + \frac{4ph}{q^2}$, moves up as $h$ increases

We try to determine the compatible ensemble in these two critical branches by two conditions (2.11), (2.12).

From (2.11), it follows:

$$\Delta = \frac{3\gamma}{4}a^2 + \frac{3q^2a^2}{8p} + \frac{p}{2}. \tag{2.17}$$

Substituting (2.17) into (2.12), we obtain a quadratic equation of unknown $a^2$

$$3q^4a^4 - 24(p^2q^2 + 2\gamma p^2h^2 + ph^2q^2)a^2 + 16(3p^4 - 4p^2h^2 - 2p^3h^2) = 0. \tag{2.18}$$

Approximatively, the two roots of (2.18) are

$$a^2_{1,2} = a^2_* \mp \frac{8ph\sqrt{3}}{3q^2}. \tag{2.19}$$

Thus, we obtain two compatible points which are intersection points of the line (2.17) and the hyperbola (2.10).

By $I'$, $I''$ we denote two compatible points. The lower point $I'$ (sign $-$) - located in $D'$ - exists when $h$ is small enough (if $h$ is large, $a_1^2 < 0$). The upper point $I''$ (sign $+$) - located in $D''$ - always exists. As $h$ increases, $I'$ moves down then disappears while $I''$ moves up.

It remains to verify the two trigonometrical conditions (2.13), (2.14). From (2.11), (2.12), it follows

$$X = -\frac{3T}{8p}, \quad h^2\omega^2 = \frac{3T^2}{64p^2}. \tag{2.20}$$

Substituting (2.20) into (2.13), (2.14) leads to

$$a^2_{1,2} \leq \frac{4}{3}a^2_* = \frac{16p^2}{3q^2}. \tag{2.21}$$

For small $h$, the two values $a^2_{1,2}$ (see (2.19)) are close to $\omega^2_*$, (2.21) is satisfied. Both I' and I" are critical (nodal) points. Increasing $h$, critical points I' and I" successively disappears: moving down, I' always satisfied (2.21). It is thus critical until it passes into the semi - lower plane ($a_1^2 < 0$); moving up, I" does not satisfy (2.21) when $h$ is large enough (it becomes an isolated trivial compatibility point).

Approximatively: - if $h \leq p\frac{\sqrt{3}}{6}$, both I' and I" are critical; - if $p\frac{\sqrt{3}}{6} < h < p\frac{\sqrt{3}}{2}$, only I' is critical; - if $h > p\frac{\sqrt{3}}{2}$, critical points do not exist.

For fixed values $\gamma = 0.04$, $p = 0.01$, $q = 0.03$ the resonance curves in fig. 1, 2, 3 correspond to $h = 0.0015, 0.0040, 0.0088$ respectively; the broken line represents the non-equivalence line).

We see that if $h$ is small enough, the resonance curve contains two critical nodal points (fig. 1). As $h$ increases, the upper critical point I" moves up then disappears first and the resonance curve has only one critical point - the lower I' (fig. 2). If $h$ is large enough, critical points do not exist (fig. 3).

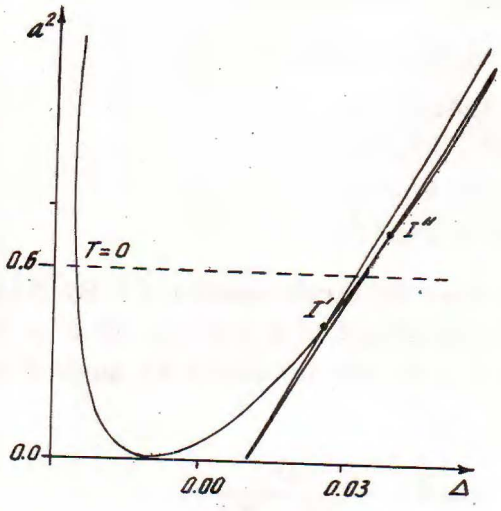


*Fig. 1*

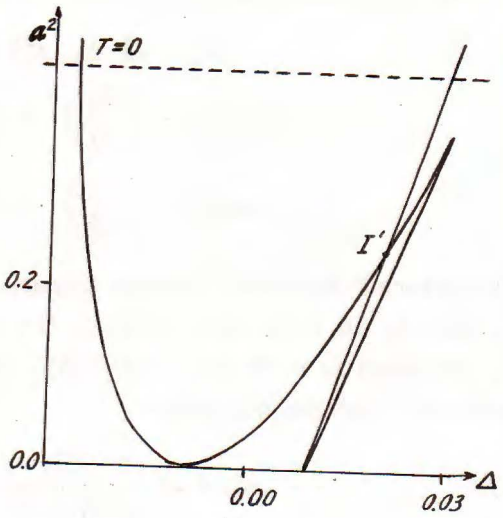


*Fig. 2*

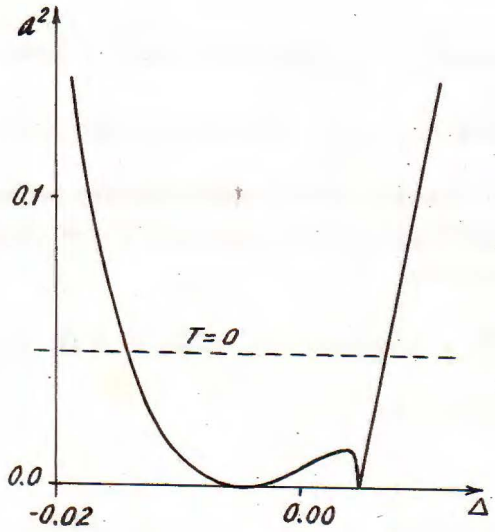


*Fig. 3*

## §3. The original resonance curve

As it has been noted in §1, the non-equivalence line contains strange elements. We shall demonstrate that for the case examined (system with damping, $h > 0$), the non-equivalence line $T = 0$ is strange, except its intersection points with the curve $W_0(\Delta, a^2) = 0$. By $I(\Delta_*, a_*^2)$ we denote the point of interest in $T = 0$ and

$\theta_*$ - its corresponding (associated) dephase. At I, we have

$$\begin{aligned} D_* &= D(\Delta_*, a_*^2) = -4p^2(X^2 + h^2\omega^2)_* < 0 \\ \sin\theta_* &= \Big(\frac{D_1}{D}\Big)_* = \Big(\frac{2ph\omega X}{X^2 + h^2\omega^2}\Big)_*, \\ \cos\theta_* &= \Big(\frac{D_2}{D}\Big)_* = \Big(\frac{X^2 - h^2\omega^2}{X^2 + h^2\omega^2}\Big)_* \end{aligned} \tag{3.1}$$

(the asterisk indicates that the quantity in question is calculated at I'). By $N(\Delta, a^2)$ we denote an arbitrary point in the neighbourhood of I, but out off $T = 0$. At $N$, we have $D < 0$ (by continuity) and $T \neq 0$. We introduce an angle $\bar{\theta}$ called artificial dephase defined as

$$\sin\bar{\theta} = \frac{-D_1}{\sqrt{D_1^2 + D_2^2}}, \quad \cos\bar{\theta} = \frac{-D_2}{\sqrt{D_1^2 + D_2^2}}. \tag{3.2}$$

Obviously, when $N$ tends to $I$, the artificial dephase $\bar{\theta}$ tends to the associated one $\theta_*$. At $N$, since $T \neq 0$, from (1.5), we can express $(f_0, g_0)$ as combinations of $(f, g)$:

$$\begin{cases} f_0(\Delta, a, \bar{\theta}) = \dfrac{1}{T}\Big\{(2p\cos\bar{\theta} - qa)f + 2p\sin\bar{\theta}, g\Big\}, \\ g_0(\Delta, a, \bar{\theta}) = \dfrac{1}{T}\Big\{ -2p\sin\bar{\theta} f + (2p\cos\bar{\theta} + qa)g\Big\}. \end{cases} \tag{3.3}$$

Using (2.1), (3.2) and regarding that, in the ordinary region (where the non-equivalence line and its neighbourhood are located) three characteristic determinants $D_1, D_2, D$ identically satisfy

$$AD_1 + BD_2 \equiv ED, \quad GD_1 + HD_2 \equiv KD, \tag{3.4}$$

we can write (3.3) in the form

$$\begin{cases} f_0(\Delta, a, \bar{\theta}) = \\ = \dfrac{-1}{T\sqrt{D_1^2 + D_2^2}}\Big\{(2p\cos\bar{\theta} - qa)E + 2p\sin\bar{\theta}\cdot K\Big\}\Big\{D + \sqrt{D_1^2 + D_2^2}\Big\} \\ = \dfrac{\{(2p\cos\bar{\theta} - qa)E + 2p\sin\bar{\theta}\cdot K\}\{D_1^2 + D_2^2 - D^2\}}{T\sqrt{D_1^2 + D_2^2}\{D - \sqrt{D_1^2 + D_2^2}\}}, \\ g_0(\Delta, a, \bar{\theta}) = \\ = \dfrac{-1}{T\sqrt{D_1^2 + D_2^2}}\Big\{ -2p\sin\bar{\theta}\cdot E + (2p\cos\bar{\theta} + qa)K\Big\}\Big\{D + \sqrt{D_1^2 + D_2^2}\Big\} \\ = \dfrac{\{ -2p\sin\bar{\theta}\cdot E + (2p\cos\bar{\theta} + qa)K\}\{D_1^2 + D_2^2 - D^2\}}{T\sqrt{D_1^2 + D_2^2}\{D - \sqrt{D_1^2 + D_2^2}\}}. \end{cases} \tag{3.5}$$

Recall that $D_1^2 + D_2^2 - D^2 = W(\Delta, a^2) = TW_0(\Delta, a^2)$, we obtain

$$\begin{cases} f_0(\Delta, a, \bar{\theta}) = \dfrac{\{(2p\cos\bar{\theta} - qa)E + 2p\sin\bar{\theta}\cdot K\}}{\sqrt{D_1^2 + D_2^2}(D - \sqrt{D_1^2 + D_2^2})} W_0(\Delta, a^2), \\ g_0(\Delta, a, \bar{\theta}) = \dfrac{\{-2p\sin\bar{\theta}\cdot E + (2p\cos\bar{\theta} + qa)K\}}{\sqrt{D_1^2 + D_2^2}(D - \sqrt{D_1^2 + D_2^2})} W_0(\Delta, a^2). \end{cases} \tag{3.6}$$

At limit ($\Delta \to \Delta_*$, $a \to a_*$, $\theta \to \theta_*$) we have

$$\begin{aligned} & f_0(\Delta, a, \bar{\theta}) \to f_0(\Delta_*, a_*, \theta_*), \quad g_0(\Delta, a, \bar{\theta}) \to g_0(\Delta_*, a_*, \theta_*), \\ & D \to D_* = -4p^2(X^2 + h^2\omega^2)_*, \quad \sqrt{D_1^2 + D_2^2} \to -D_*, \\ & E \to E_* = -(h\omega qa)_* = -2ph\omega_*, \quad K \to K_* = 2pX_*, \\ & W_0(\Delta, a^2) \to W_0(\Delta_*, a_*^2), \quad qa \to qa_* = 2p. \end{aligned} \tag{3.7}$$

Therefore

$$\begin{cases} f_0(\Delta_*, a_*, \theta_*) = \dfrac{-h\omega_*}{4p^2(X^2 + h^2\omega^2)_*^2} W_0(\Delta_*, a_*^2), \\ g_0(\Delta_*, a_*, \theta_*) = \dfrac{-X_*}{4p^2(X^2 + h^2\omega^2)_*^2} W_0(\Delta_*, a_*^2). \end{cases} \tag{3.8}$$

$I(\Delta_*, a_*^2)$ is a representative point of the original resonance curve $C_0$ and $\theta_*$ is the "original" dephase if and only if $f_0(\Delta_*, a_*, \theta_*)$ and $g_0(\Delta_*, a_*, \theta_*)$ simultaneously vanish. Since $h\omega_* \neq 0$, from (3.8), it follows that the required conditions lead to $W_0(\Delta_*, a_*^2) = 0$ i.e. $I(\Delta_*, a_*^2)$ is an intersection point of the non-equivalence line $T = 0$ and the curve $W_0(\Delta, a^2) = 0$. Summing up, the original resonance curve is given by the simplified frequency amplitude

$$W_0(\Delta, a^2) = \frac{W(\Delta, a^2)}{T} = 0. \tag{3.9}$$

(on condition that at compatible points ($D = D_1 = D_2 = 0$) the two trigonometrical conditions must be satisfied)

## Conclusion

We have identified the resonance curve of a quasi linear oscillating system subjected to linear and quadratic parametric excitations. The so-called associated equations has been used and the original resonance curve is obtained from the associated one by rejecting the non equivalence line.

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## REFERENCES

1. Mitropolski Yu. A., Nguyen Van Dao. Applied asymptotic methods in non-linear oscillations, Kluwer Academic Publisher, 1997.
2. Nguyen Van Dinh, Tran Kim Chi. Interaction between linear and cubic parametric excitations, Vietnam Journal of Mechanics. T.XX, 1998, No 4, 20-29.
3. Nguyen Van Dinh. Stationary oscillations in degenerated cases. Journal of Mechanics NCST of Vietnam, T. XVIII 1996, No 2, 13-19.

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## LẬP ĐƯỜNG CỘNG HƯỞNG CỦA HỆ CHỊU HAI KÍCH ĐỘNG THÔNG SỐ BẬC NHẤT VÀ BẬC 2

Xét hệ dao động á tuyến chịu hai kích động thông số bậc nhất và bậc hai ở trường hợp có cản. Hệ phương trình liên hợp được sử dụng và kết quả cho thấy đường cộng hưởng "gốc" được xác định từ đường cộng hưởng liên hợp bằng cách loại bỏ đường không tương đương.