

## SOME RESULTS IN STUDYING MULTILAYERED COMPOSITE PLATES

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### 1. Introduction

Multilayered composite plates have wide applications in modern engineering: civil engineering, transportation, aerospace, aviation, ocean engineering ... At present, research problems are centered on the calculation and design of composite structures, including: solution to static, dynamic and stability problems of multilayered composite structures; analysis of the affects of the connection and laminar alignments of materials on the plate working capacity; the optimization of structures of multilayered composite plates... This paper presents some results in studying the static and dynamic problems of multilayered composite plates, of which individual layer is made of unidirectional composite material. Calculations are based on the technical theory of laminar plates combined with using finite element method.

### 2. Formulation of the problem and method of solution

Consider a  $n$ -layered thin plate of which every lamina is composed of unidirectional composite materials (Fig. 1).

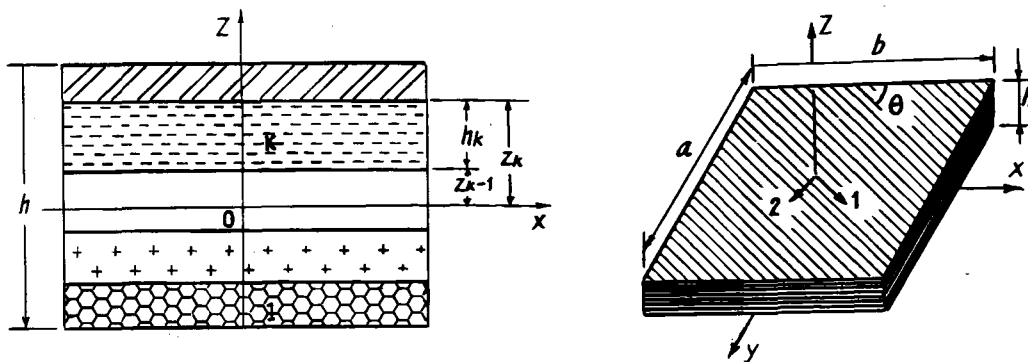


Fig. 1

We have a general vibration equation:

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F(t)\}. \quad (2.1)$$

For the problem of free vibration, without damping, (2.1) can be written as:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0. \quad (2.2)$$

For the static problem, the equilibrium equation is of the form:

$$[K]\{q\} = F \quad (2.3)$$

where

$[M]$ ,  $[C]$ ,  $[K]$  - the mass, damping and stiffness matrices of plate, respectively;  
 $\{q\}$ ,  $\{\dot{q}\}$ ,  $\{\ddot{q}\}$  - the vectors of nodal displacements, velocities and accelerations, respectively.

$\{F(t)\}$  - the nodal force vector.

To solve the above problems, matrices  $[K]$ ,  $[M]$ ,  $[C]$  must be defined. These matrices are built on stiffness matrix  $[K^e]$ , mass matrix  $[M^e]$  of element. Using rectangular elements for composite plate problems (Fig. 2), at node  $i$  there are five degrees of freedom:

$$\{q\}_i = \{u^i v^i w^i \varphi_x^i \varphi_y^i\}^T.$$

A corresponding force vector is:

$$\{P\}_i = \{R_x^i R_y^i R_z^i M_x^i M_y^i\}^T$$

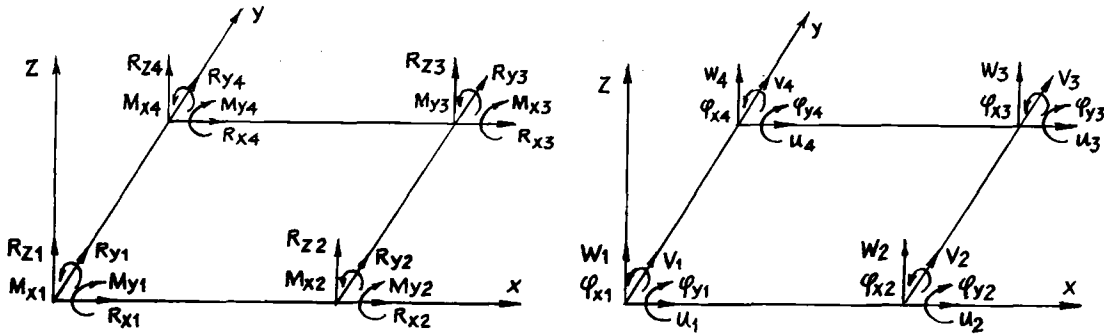


Fig. 2

The shape functions of displacements are chosen in form of polynomial expressions as in plate flexure problem and plane elasticity problem of homogeneous materials [3, 5]. We have a relationship between displacements within element and nodal displacements of element as follows:

$$\{u^*\} = [f]\{q\}_e \quad (2.4)$$

where:

\*  $\{u^*\} = \{u \ v \ w\}^T$  - vector of displacements within element.

\*  $[f]$  - matrix of shape function.

$$[f] = \begin{bmatrix} f_1 & 0 & 0 & 0 & 0 & f_6 & 0 & 0 & 0 & 0 & f_{11} & 0 & 0 & \dots \\ 0 & f_2 & 0 & 0 & 0 & 0 & f_7 & 0 & 0 & 0 & 0 & f_{12} & 0 & \dots \\ 0 & 0 & f_3 & f_4 & f_5 & 0 & 0 & f_8 & f_9 & f_{10} & 0 & 0 & f_{13} & \dots \\ & & & & & \dots & 0 & 0 & f_{16} & 0 & 0 & 0 & 0 \\ & & & & & \dots & 0 & 0 & 0 & f_{17} & 0 & 0 & 0 \\ & & & & & \dots & f_{14} & f_{15} & 0 & 0 & f_{18} & f_{19} & f_{20} \end{bmatrix}$$

where,

$$f_1 = f_2 = 1 - \frac{y}{b} - \frac{x}{a} + \frac{xy}{ab}$$

$$f_3 = 1 - \frac{3x^2}{a^2} - \frac{3y^2}{b^2} - \frac{xy}{ab} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab} + \frac{2x^3}{a^3} + \frac{2y^3}{b^2} - \frac{2x^3y}{a^3b} - \frac{2xy^3}{ab}$$

$$f_4 = x - \frac{2x^2}{a} + \frac{x^3}{a^2} - \frac{xy}{ab} + \frac{2x^2y}{a^2b} - \frac{x^3y}{a^2b}$$

$$f_5 = y - \frac{2y^2}{b} + \frac{y^3}{b^2} - \frac{xy}{ab} + \frac{2xy^2}{ab} - \frac{xy^3}{ab^2}$$

$$f_6 = f_7 = \frac{x}{a} + \frac{xy}{ab}$$

$$f_8 = \frac{3x^2}{a^2} + \frac{xy}{ab} - \frac{3x^2y}{a^2b} - \frac{3xy^2}{ab^2} - \frac{2x^3}{a^3} + \frac{2x^3y}{a^3b} + \frac{2xy^3}{ab^3}$$

$$f_9 = -\frac{x^2}{a} + \frac{x^2y}{ab} + \frac{x^3}{a^2} - \frac{x^3y}{a^2b}$$

$$f_{10} = \frac{xy}{a} - \frac{2xy^2}{ab} + \frac{xy^3}{ab^2}$$

$$f_{11} = f_{12} = \frac{xy}{ab}$$

$$f_{13} = -\frac{xy}{ab} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2} - \frac{2x^3y}{a^3b} - \frac{2xy^3}{ab^3}$$

$$f_{14} = -\frac{x^2y}{ab} + \frac{x^3y}{a^2b}$$

$$f_{15} = -\frac{xy^2}{ab} + \frac{xy^3}{ab^2}$$

$$f_{16} = f_{17} = \frac{y}{b} - \frac{xy}{ab}$$

$$f_{18} = \frac{3y^2}{b^2} + \frac{xy}{ab} - \frac{3x^2y}{a^2b} - \frac{3xy^2}{ab^2} - \frac{2y^3}{b^3} + \frac{2x^3y}{a^3b} + \frac{2xy^3}{ab^3}$$

$$f_{19} = \frac{xy}{b} - \frac{2x^2y}{ab} + \frac{x^3y}{a^2b}$$

$$f_{20} = -\frac{y^2}{b} + \frac{xy^2}{ab} + \frac{y^3}{b^2} - \frac{xy^3}{ab}$$

\*  $\{q\}_e$  - vector of nodal displacement of element:

$\{q\}_e = \{u_1 v_1 w_1 \varphi_{x_1} \varphi_{y_1} u_2 v_2 w_2 \varphi_{x_2} \varphi_{y_2} u_3 v_3 w_3 \varphi_{x_3} \varphi_{y_3} u_4 v_4 w_4 \varphi_{x_4} \varphi_{y_4}\}^T$   
 If now notations could be used as:

$$\{\varepsilon^*\} = \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad 2 \frac{\partial^2 w}{\partial x \partial y} \right\}^T$$

$$\{\sigma^*\} = \{N_x \quad N_y \quad N_{xy} \quad M_x \quad M_y \quad M_{xy}\}^T$$

then components of the element stiffness matrix are determined by:

$$K_{ij}^e = \int_0^a \int_0^b \{\varepsilon^*\}_i^T \{\sigma^*\}_j dx dy. \quad (2.5)$$

The stress-strain relationship for the  $k^{th}$  layer as follows

$$\{\sigma^k\} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \{\varepsilon^k\}, \quad (2.6)$$

in which:

$[A]$ ,  $[B]$ ,  $[D]$  - the inplane, bending-inplane and bending matrices, respectively

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}; \quad [B] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix};$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}$$

with

$$[A] = \sum_{k=1}^n \int_{z^{k-1}}^{z^k} [G]^k dz; \quad [B] = \sum_{k=1}^n \int_{z^{k-1}}^{z^k} [G]^k z dz; \quad [D] = \sum_{k=1}^n \int_{z^{k-1}}^{z^k} [G]^k z^2 dz, \quad (2.7)$$

where  $[G]^k$  - a  $k^{th}$  lamina stiffness matrix:

$$[G]^k = \begin{bmatrix} g_{11} & g_{12} & g_{16} \\ g_{12} & g_{22} & g_{26} \\ g_{16} & g_{26} & g_{66} \end{bmatrix} \quad (2.8)$$

The components  $g_{ij}^k$  ( $i, j = 1, 2, 6$ ) are determined from elastic constants of fibre and matrix materials of the  $k^{th}$  layer.

If all coefficients of the matrix  $[G]^k$  are constants with respect to individual laminar thickness then we have:

$$[A] = \sum_{k=1}^n [G]^k (z_k - z_{k-1}); \quad [B] = \frac{1}{2} \sum_{k=1}^n [G]^k (z_k^2 - z_{k-1}^2)$$

$$[D] = \frac{1}{3} \sum_{k=1}^n [G]^k (z_k^3 - z_{k-1}^3) \quad (2.9)$$

The components element mass matrix can be determined as following

$$[M_e] = \int_v [f]^T \rho [f] dV. \quad (2.10)$$

By homogenization method [1] , we have:

$$[M_e] = \int_v [f]^T \langle \rho \rangle [f] dV. \quad (2.11)$$

Components of matrix  $[G]^k$  and  $\langle \rho \rangle$  (average density of composite material) are determined by homogenization method in mechanics of composite materials [1, 5].

From the above described expressions and equations, the matrices  $[K]$ ,  $[M]$ ,  $[C]$  of plate can be established for solution to the static, natural vibration and forced vibration problems of multilayered composite plate using finite element technique

### 3. Investigation of the affect of geometric and physical factors on working capacity of mulilayered composite materials

A thin laminated plate is considered as shown in Fig. 3

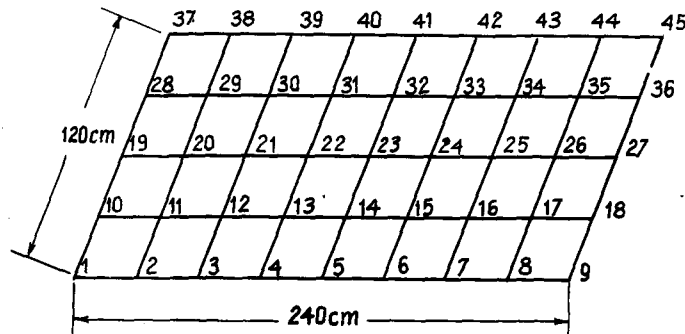


Fig. 3

Material properties of layers for composite plate as shown in table 1

Table 1

Material type	$E_c$ ( $kG/cm^2$ )	$E_n$ ( $kG/cm^2$ )	$V_c$	$V_n$	$\rho_c$ ( $kg/cm^3$ )	$\rho_n$ ( $kg/cm^3$ )	$\psi$
1	$39 \cdot 10^5$	$7 \cdot 10^5$	0.3	0.3	0.00185	0.0027	0.4
2	$13 \cdot 10^5$	$7 \cdot 10^5$	0.3	0.3	0.028	0.0027	0.4

Introducing some notations:

- Uniform distributed loads on over plate  $p_z = 0.1kG/cm^2$ ,
- All edges of plate are clamped,
- $\sigma_{td} = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$  - equivalent stress,
- $K_d$  : dynamic coefficient,
- $\omega_0$  : 1<sup>th</sup> natural frequency of vibration.

And using following algorithms for:

- Solving static problem by Gauss Elimination Method.
- Solving natural vibration problems by Subspace Iteration Method.
- Solving forced vibration problems by Mode Superposition Method.

We can investigate a series of specific problems. We represent some results of calculations in following aspects:

### 1.1. The affect of the laminar arrangement order

Note that:

- Laminar material with fibre angle 0 has label 1
- Laminar material with fibre angle 45 has label 2
- Individual laminar thickness equals to 0.75 cm
- Layers are aligned in different orders (table 2)

Table 2

Layer label	Cases			
	1	2	3	4
1	0	45	0	0
2	45	0	45	0
3	45	0	0	45
4	0	45	45	45

Results:

- 1<sup>th</sup> case:  $W_{\max} = 0.0212$  cm;  $U_{\max} = 0$ ;  $V_{\max} = 0$ ;  
 $\omega_0 = 4.795$  Hz;  $\sigma_{td} = 225.26$  kG/cm<sup>2</sup>;  $K_d = 1.180$
- 2<sup>th</sup> case:  $W_{\max} = 0.0242$  cm;  $U_{\max} = 0$ ;  $V_{\max} = 0$ ;  
 $\omega_0 = 4.556$  Hz;  $\sigma_{td} = 235.25$  kG/cm<sup>2</sup>;  $K_d = 1.193$

3<sup>th</sup> case:  $W_{\max} = 0.0226$  cm;  $U_{\max} = 0.0000427$  cm;  $V_{\max} = 0.0000174$  cm;  
 $\omega_0 = 4.740$  Hz ;  $\sigma_{td} = 241.02$  kG/cm<sup>2</sup>;  $K_d = 1.183$   
 4<sup>th</sup> case:  $W_{\max} = 0.0228$  cm;  $U_{\max} = 0.0000599$  cm;  $V_{\max} = 0.0000351$  cm;  
 $\omega_0 = 4.722$  Hz;  $\sigma_{td} = 242.00$  kG/cm<sup>2</sup>;  $K_d = 1.184$ .

*Remark:* For arrangement of layers in 1<sup>th</sup> case, the multilayered composite plate stiffness is obtained more than other cases. Arrangement of layers in 3<sup>th</sup> case and 4<sup>th</sup> gives values of displacement  $u$  and  $v$  different from zero. This shows that there is inplane-bending effect.

### 1.2. The affect of the fibrous directions

In order to investigate this factor, the 5 - layered composite plate is chosen, of which layers are aligned by rule as shown in table 3.

Table 3

Laminar	Material type	Fibrous angle $\varphi$ (degree)	Thickness (cm)
1	1	$+\varphi$	0.6
2	2	$-\varphi$	0.6
3	2	$+\varphi$	0.6
4	2	$-\varphi$	0.6
5	1	$+\varphi$	0.6

Let  $\varphi$  angle between fibrous direction and  $x$ -axis at the plate vary from 0° to 180° we investigate the variation of static displacement  $w_{\max}^t$ , 1<sup>th</sup> natural frequency  $\omega_0$  and dynamic coefficient  $K_d$ .

*Results:* the relations of those parameters with angle  $\varphi$  are shown by diagrams in fig. 4 ÷ 6.

*Remark:* analysing these graphics we can see that: in cases  $\varphi = 75^\circ$  and  $\varphi = 105^\circ$  obtained deflections are minimum and natural frequency  $\omega_0$  are maximum. It shows that in these examples, the composite plate has maximum stiffness values. In addition, dynamic coefficient  $K_d$  is minimum corresponding with these 2 fibrous angles.

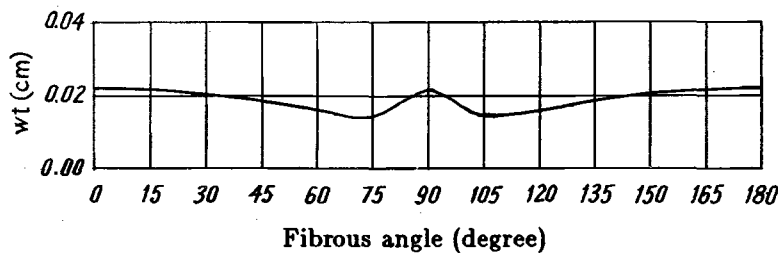


Fig. 4. Relation of deflection  $w_{\max}^t$  and fibrous angle  $\varphi$

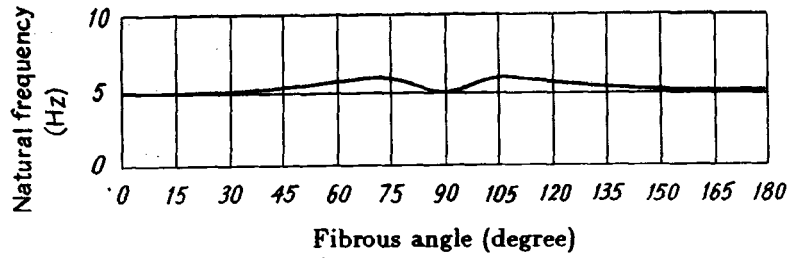


Fig. 5. Relation of 1<sup>th</sup> natural frequency  $\omega_0$  and fibrous angle  $\varphi$

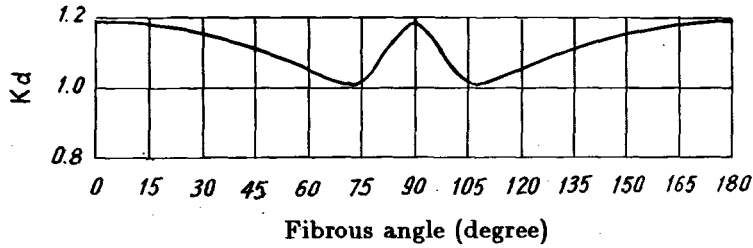


Fig. 6. Relation of dynamic coefficient  $K_d$  and fibrous angle  $\varphi$

### 1.3. The affects of the material characteristic parameters

The ratios  $E_c/E_n$  and  $\rho_c/\rho_n$  characterise the properties of fibre and matrix materials. Affects of those ratios on working capacity of composite can be seen by investigating the following problem:

Multilayered composite plate consisting of  $n = 5$  layers are aligned as shown in table 4.

*Table 4*

Laminar	Fibrous angle $\varphi$ (degree)	Thickness (cm)
1	45	0.6
2	-45	0.6
3	45	0.6
4	45	0.6
5	-45	0.6

In order to research the variation of displacements, stresses, natural frequencies we solve static, natural vibration and forced vibration problems when varying ratios  $E_c/E_n$  and  $\rho_c/\rho_n$ . Using obtained results we build diagrams between  $w_{\max}^t$ ,  $\omega_0$ ,  $K_d$  and ratios  $E_c/E_n$ ,  $\rho_c/\rho_n$  (see fig. 7 ÷ 11).

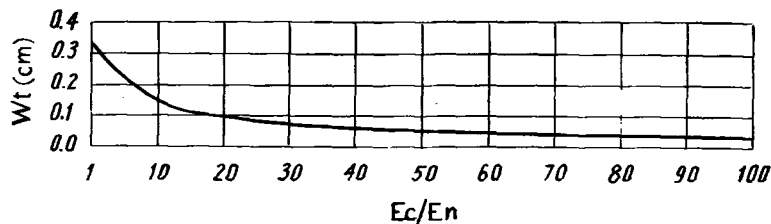


Fig. 7. Relation of deflection  $w_{\max}^t$  and  $E_c/E_n$



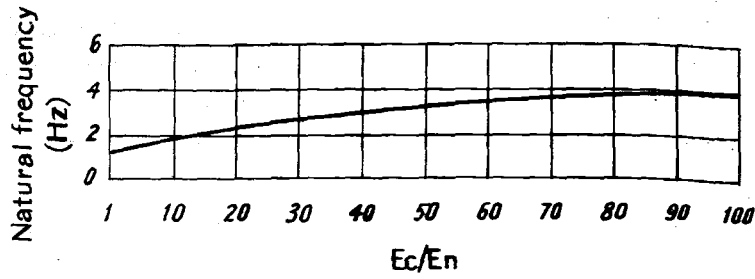


Fig. 8. Relation of natural frequency  $\omega_0$  and  $E_c/E_n$

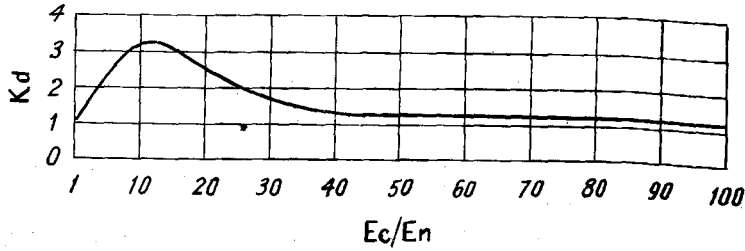


Fig. 9. Relation of dynamic coefficient  $K_d$  and  $E_c/E_n$

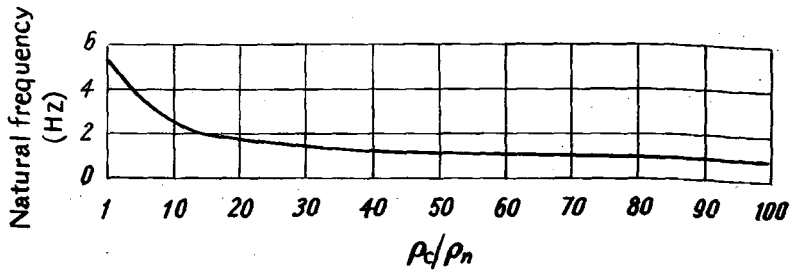


Fig. 10. Relation of natural frequency  $\omega_0$  and  $\rho_c/\rho_n$

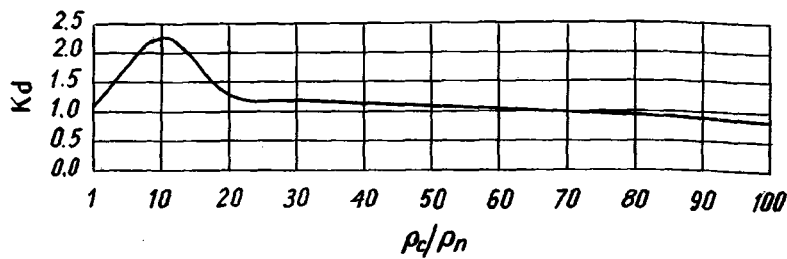


Fig. 11. Relation of dynamic coefficient  $K_d$  and  $\rho_c/\rho_n$

#### 4. Conclusion

On the basis of laminar plate theory and using finite element technique we can solve static and dynamic problems of multilayered composite plates subjected to arbitrary loading and boundary conditions. From obtained results some conclusions can be drawn as follows:

- Fibrous directions with respect to common coordinate of plate have a clear effect on working capacity of the plates. In the example of fibre angle  $\varphi$  of about  $75^\circ$  and  $105^\circ$ , composite plates have maximum stiffness.

- The relations between elastic modul and density of the fibre and matrix materials have a clear effect on the working of the plates.

- We can arrange layers with appropriate fibrous angles and chosen material for obtaining multilayered composite plates satisfying application purposes.

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#### MỘT SỐ KẾT QUẢ NGHIÊN CỨU TẤM COMPOSIT NHIỀU LỚP

Bài báo trình bày một số kết quả tính toán tấm composite nhiều lớp bằng phương pháp thuần nhất hóa. Các tác giả đã sử dụng thuật toán của phương pháp phần tử hữu hạn để giải các bài toán tĩnh và động tấm composite nhiều lớp có cấu trúc phức tạp. Qua khảo sát một số ví dụ tính toán, các tác giả đã nghiên cứu ảnh hưởng của các yếu tố hình học và vật lý của các pha vật liệu đến khả năng làm việc của tấm composite nhiều lớp.