Vietnam Journal of Mechanics, NCST of Vietnam Vol. 21, 1999, No 2, (75 - 88)

NONLINEAR OSCILLATORS UNDER DELAY CONTROL

NGUYEN VAN DAO

Vietnam National University, Hanoi
19 Le Thanh Tong, Hanoi, Vietnam

ABSTRACT. In this paper, oscillations and stability of nonlinear oscillators with time delay are studied by means of the asymptotic method of nonlinear mechanics. Harmonic, superharmonic, subharmonic and parametric resonances of a Duffing's oscillator are analyzed. Analytical method in combination with a computer is used.

1. Introduction

The harmonically forced Duffing's oscillator with time delay state feedback has been investigated in [1] by using the method of multiple scales [2]. Both primary and 1/3 subharmonic resonances of the Duffing's oscillator with weak nonlinearity and weak delay feedback have been examined. As shown in [1] the simplest model for various controlled nonlinear systems, e.g., active vehicle suspension systems when the nonlinearity in tires is taken into account, is described by a second order differential equation with time delay in the form

$$\frac{d^2x(t)}{dt^2} + x(t) = -2\xi\frac{dx(t)}{dt} - \mu x^3(t) + 2ux(t-\Delta) + 2v\frac{dx(t-\Delta)}{dt} + 2p\cos\lambda t, \quad (1.1)$$

where ξ, μ, u, v and Δ are constants.

To study all possible simple resonances in the dynamic system governed by equation (1.1), in the present paper it is supposed that between the external frequency λ and the natural frequency 1 there exists a relationship of the form

$$\lambda = n + \varepsilon\sigma, \quad (1.2)$$

where $n = \frac{p}{q}$ is a rational number, p and q are integers. We suppose that parameters ξ, μ, u, v are small. The smallness of these parameters is insured by introducing small positive parameter ε.

2. Harmonic Resonance

Assuming that $n = 1$ and f is a small quantity of ε - order, we can rewrite equation (1.1) in the form

$$\frac{d^2x(t)}{dt^2} + \lambda^2 x(t) = \varepsilon(2\sigma x + F), \tag{2.1}$$

where

$$F = -2\xi\frac{dx(t)}{dt} - \mu x^3(t) + 2ux(t-\Delta) + 2v\frac{dx(t-\Delta)}{dt} + 2p\cos\lambda t. \tag{2.2}$$

The solution of equation (2.1) is found in the form

$$x(t) = a\cos\Psi(t), \quad \Psi(t) = \lambda t + \theta.$$

$$\frac{dx(t)}{dt} = -a\lambda\sin\Psi(t), \tag{2.3}$$

where a and θ are unknown functions. By substituting these expressions into (2.1) and solving for $\frac{da}{dt}\ \frac{d\theta}{dt}$ we obtain the following equations for a and θ:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{\lambda}(2\sigma x + F)\sin\Psi, \\ a\frac{d\theta}{dt} &= -\frac{\varepsilon}{\lambda}(2\sigma x + F)\cos\Psi. \end{aligned} \tag{2.4}$$

In the first approximation we can replace the right hand sides of (2.4) by their averaged values:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{\lambda}(aL + p\sin\theta), \\ a\frac{d\theta}{dt} &= -\frac{\varepsilon}{\lambda}\left[a\left(M - \frac{3}{8}\mu a^2\right) + p\cos\theta\right], \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} L &= \xi\lambda + u\sin(\lambda\Delta) - \lambda v\cos(\lambda\Delta), \\ M &= \sigma + u\cos(\lambda\Delta) + \lambda v\sin(\lambda\Delta). \end{aligned} \tag{2.6}$$

The stationary solution of (2.5) is $a = a_0 =$ const, $\theta = \theta_0 =$ const which satisfy the relationships:

$$\begin{aligned} a_0L + p sin\theta_0 &= 0, \\ a_0\left(M - \frac{3}{8}\mu a_0^2\right) + p\cos\theta_0 &= 0. \end{aligned} \tag{2.7}$$

From here we obtain:

$$W(a_0^2, \lambda) = a_0^2\Big[L^2 + \Big(M - \frac{3}{8}\mu a_0^2\Big)^2\Big] - p^2 = 0, \tag{2.8}$$

$$\mathrm{tg}\theta_0 = \frac{L}{M - \frac{3}{8}\mu a_0^2}\,. \tag{2.9}$$

The resonance curves are presented in Fig. 1 for the parameters: $p = 0.05$, $\xi = 0.05$, $\mu = 1.5$ and for various values of u, v: $u = v = 0.05$, $\Delta = 0$ (curve 1), $u = v = 0$ (curve 2), $u = -v = 0.05$ and $\Delta = 0$ (curve 3), $\Delta = 0.5$ (curve 4), $\Delta = 1$ (curve 5). Curve 1 corresponds to the case of an ordinary Duffing's oscillator without friction. Curve 2 corresponds to the well-known Duffing's oscillator without time delay. Curve 3 also represents the Duffing's oscillator without time delay and with a viscous friction $2(\xi - v)\dot{x}$ two times larger than in the previous case. Hence, the maximum of the amplitudes strongly decreases. By increasing time delay (curves 4, 5), the resonance curves lean toward the right and the maximum of the amplitudes slightly decreases.

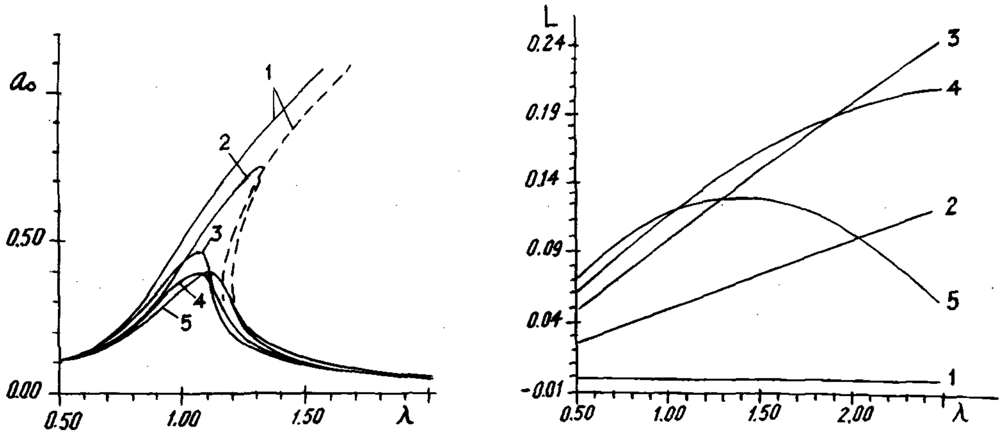


Figure 1a *Figure. 1b*

Figure 1. Resonance curves in the case of harmonic resonance

To study the stability of stationary solution a_0, θ_0 we use the variational equations obtained from (2.5) by letting $a = a_0 + \delta a$, $\theta = \theta_0 + \delta\theta$. Thus, we have

$$\begin{aligned} \frac{d\delta a}{dt} &= -\frac{\varepsilon}{\lambda}\Big[L\delta a - a_0\Big(M - \frac{3}{8}\mu a_0^2\Big)\delta\theta\Big], \\ a_0\frac{d\delta\theta}{dt} &= -\frac{\varepsilon}{\lambda}\Big[\Big(M - \frac{9}{8}\mu a_0^2\Big)\delta a + a_0 L.\delta\theta\Big]. \end{aligned} \tag{2.10}$$

The characteristic equation for this system of equations is

$$a_0\rho^2 + \frac{2\varepsilon a_0}{\lambda}L\rho + \frac{\varepsilon^2}{\lambda^2}a_0\Big[L^2 + \Big(M - \frac{3}{8}\mu a_0^2\Big)\Big(M - \frac{9}{8}\mu a_0^2\Big)\Big] = 0.$$

Taking the expression (2.8) into account we can write this equation in the form:

$$a_0\rho^2 + \frac{2\varepsilon}{\lambda}a_0 L \cdot \rho + \frac{\varepsilon^2}{\lambda^2}a_0 \cdot \frac{\partial W}{\partial a_0^2} = 0.$$

Hence, the stability conditions will be

1) $L > 0$ (2.11)

2) $\dfrac{\partial W}{\partial a_0} > 0.$ (2.12)

It is easy to identify the stability zone by using the rule stated in [3]. In Figure 1 the stable branches are represented by solid lines, while the unstable branches - by broken lines. It seems that time delay plays the same role as friction, decreasing the amplitudes and stabilizing the oscillations.

3. Superharmonic resonance of third order

Supposing that $n = \dfrac{1}{3}$, we have the equation (1.1) in the form:

$$\frac{d^2x(t)}{dt^2} + 9\lambda^2 \cdot x(t) = \varepsilon\left[6\varepsilon\sigma x(t) + F_0\right] + 2p\cos\lambda t, \tag{3.1}$$

where

$$F_0 = -2\xi\frac{dx(t)}{dt} - \mu x^3(t) + 2ux(t-\Delta) + 2v\frac{dx(t-\Delta)}{dt}. \tag{3.2}$$

We transform equation (3.1) into a system of two equations of the first order relative to the amplitude a and phase θ as follows:

$$\begin{aligned} x(t) &= a\cos(3\lambda t + \theta) + 2p_*\cos\lambda t, \\ \frac{dx(t)}{dt} &= -3\lambda a\sin(3\lambda t + \theta) - 2\lambda p_*\sin\lambda t, \\ p_* &= \frac{f}{8\lambda^2}\,. \end{aligned} \tag{3.3}$$

It is easy to find the equations for a and θ :

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{3\lambda}(6\sigma a\cos\Psi + F_0)\sin\Psi, \\ a\frac{d\theta}{dt} &= -\frac{\varepsilon}{3\lambda}(6\sigma a\cos\Psi + F_0)\cos\Psi, \end{aligned} \tag{3.4}$$

where

$$\Psi = 3\lambda t + \theta. \tag{3.5}$$

In the first approximation we can replace the right hand sides of (3.4) by their averaged values. Hence, we have the following averaged equations:

$$\begin{cases} \dfrac{da}{dt} = \dfrac{\varepsilon}{3\lambda}(-aL_1 + \mu p_*^3 \sin\theta), \\ a\dfrac{d\theta}{dt} = \dfrac{\varepsilon}{3\lambda}\Big[-a\Big(M_1 - \dfrac{3}{8}\mu a^2\Big) + \mu p_*^3 \cos\theta\Big], \end{cases} \tag{3.6}$$

where

$$\begin{cases} L_1 = 3\xi\lambda + u\sin 2\lambda\Delta - 3v\lambda\cos 3\lambda\Delta, \\ M_1 = 3\sigma - 3\mu f_*^2 + u\cos(3\lambda\Delta) + 3v\lambda\sin(3\lambda\Delta), \\ \varepsilon\sigma = \lambda - \dfrac{1}{3}\,. \end{cases} \tag{3.7}$$

Stationary solution $a = a_0$, $\theta = \theta_0$ of equations (3.6) satisfies relationships:

$$\begin{aligned} a_0 L_1 &= \mu p_*^3 \sin\theta_0, \\ a_0\Big(M_1 - \frac{3}{8}\mu a_0^2\Big) &= \mu p_*^3 \cos\theta_0. \end{aligned} \tag{3.8}$$

Eliminating the phase θ_0 we obtain the following equation for the resonance curves:

$$W_1(a_0^2, \lambda) = a_0^2\Big[L_1^2 + \Big(M_1 - \frac{3}{8}\mu a_0^2\Big)^2\Big] - \mu^2 p_*^6 = 0. \tag{3.9}$$

4. Subharmonic resonance of order one third (1/3)

Now, we consider the case when $n = 3$ in the relationship (1.2) and when the equation (1.1) has the form:

$$\frac{d^2x(t)}{dt^2} + \frac{\lambda^2}{9}x(t) = \varepsilon\Big[\frac{2\varepsilon}{3}\sigma x(t) + F_0\Big] + 2p\cos\lambda t, \tag{4.1}$$

where $\varepsilon\sigma = \lambda - 3$ and F_0 is the same as in (3.2). The solution of equation (4.1) is found in the form:

$$\begin{cases} x(t) = a\cos\Big(\dfrac{\lambda}{3}t + \theta\Big) + 2p_{1*}\cos\lambda t, \\ \dfrac{dx(t)}{dt} = -\dfrac{a\lambda}{3}\sin\Big(\dfrac{\lambda}{3}t + \theta\Big) - 2\lambda p_{1*}\sin\lambda t, \end{cases} \tag{4.2}$$

where a and θ are new variables and

$$p_{1*} = -\frac{9f}{8\lambda^2}\,. \tag{4.3}$$

Substituting expressions (4.2) into (4.1) and solving relative to the derivatives of a and θ we obtain:

$$\begin{aligned} \frac{da}{dt} &= -\frac{3\varepsilon}{\lambda}\Big(\frac{2\sigma}{3}a\cos\varphi + F_0\Big)\sin\varphi, \\ a\frac{d\theta}{dt} &= -\frac{3\varepsilon}{\lambda}\Big(\frac{2\sigma}{3}a\cos\varphi + F_0\Big)\cos\varphi, \end{aligned} \tag{4.4}$$

where $\varphi = \dfrac{\lambda}{3}t + \theta$.

Averaging the right hand sides of (4.4) over time, we have the following averaged equations:

$$\begin{cases} \dfrac{da}{dt} = \dfrac{3\varepsilon}{\lambda}a\Big(-L_2 + \dfrac{3}{4}\mu a p_{1*}\sin 3\theta\Big), \\ a\dfrac{d\theta}{dt} = \dfrac{3\varepsilon}{\lambda}a\Big(-M_2 + \dfrac{3}{8}\mu a^2 + \dfrac{3}{4}\mu a p_{1*}\cos 3\theta\Big), \end{cases} \tag{4.5}$$

where,

$$\begin{aligned} L_2 &= \frac{1}{3}\Big[\xi\lambda + 3u\sin\Big(\frac{\lambda\Delta}{3}\Big) - v\lambda\cos\Big(\frac{\lambda\Delta}{3}\Big)\Big], \\ M_2 &= \frac{1}{3}\Big[\sigma - 9\mu p_{1*}^2 + 3u\cos\Big(\frac{\lambda\Delta}{3}\Big) + v\lambda\sin\Big(\frac{\lambda\Delta}{3}\Big)\Big]. \end{aligned} \tag{4.6}$$

The stationary solutions $a = a_0$, $\theta = \theta_0$ of equations (4.5) are determined by the relationships:

$$\begin{aligned} a_0\Big(L_2 - \frac{3}{4}\mu a_0 p_{1*}\sin 3\theta_0\Big) &= 0, \\ a_0\Big(M_2 - \frac{3}{8}\mu a_0^2 - \frac{3}{4}\mu a_0 p_{1*}\cos 3\theta_0\Big) &= 0. \end{aligned} \tag{4.7}$$

By eliminating θ_0 we obtain:

$$W_2(a_0^2, \lambda) = a_0^2\Big\{L_2^2 + \Big(M_2 - \frac{3}{8}\mu a_0^2\Big)^2 - \frac{9}{16}\mu^2 p_{1*}^2 a_0^2\Big\} = 0. \tag{4.8}$$

Using the last equation we can construct the resonance curves, giving the dependence of the amplitude a_0 on frequency λ of external force. In Figure 2 the resonance curves are drawn for the parameters $p_{1*} = 0.8$, $\xi = 0.01$, $\mu = 0.02$ and $u = v = 0$ (curve 1), $u = 0.01$, $v = -0.01$ and $\Delta = 0$ (curve 2), $\Delta = 0.05$ (curve 3), $\Delta = 0.1$ (curve 4). The abscissa - axis λ corresponds to the zero solution $a = 0$ of equations (4.5). Curve 1 is the resonance curve in the ordinary Duffing's oscillator without time delay. With the presence of delay elements (u, v) the resonance curve moves up. The larger the time delay Δ, the higher the resonance curve (see curves 2, 3, 4 for $\Delta = 0$, $\Delta = 0.05$ and $\Delta = 0.1$ respectively).

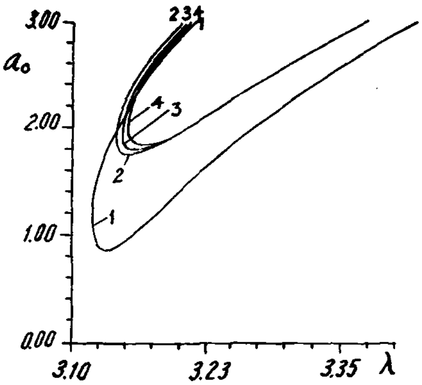


Figure 2a

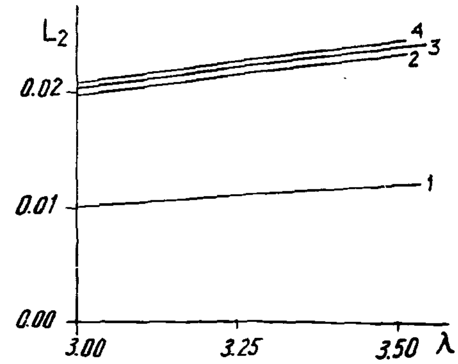


Figure 2b

Figure 2. **Resonance curves in the case of subharmonic resonance**

To study the stability of the stationary oscillations we use the variational equations:

$$\begin{aligned} \frac{d\delta a}{dt} &= -\frac{3\varepsilon}{\lambda}\Big[-L_2\delta a + a_0\Big(-3M_2 + \frac{9}{8}\mu a_0^2\Big)\delta\theta\Big], \\ a_0\frac{d\delta\theta}{dt} &= -\frac{3\varepsilon}{\lambda}\Big[-\Big(M_2 + \frac{3}{8}\mu a_0^2\Big)\delta a + 3a_0L_2\delta\theta\Big]. \end{aligned} \tag{4.9}$$

The characteristic equation of this system of equations is of the form:

$$a_0\rho^2 + \frac{6\varepsilon}{\lambda}a_0L_2\rho + \frac{27\varepsilon^2}{\lambda^2}a_0\frac{\partial W_2}{\partial a_0^2} = 0. \tag{4.10}$$

Hence, the stability conditions are

$$\begin{aligned} &1) \quad L_2 > 0 \\ &2) \quad \frac{\partial W_2}{\partial a_0} > 0. \end{aligned} \tag{4.11}$$

From the expression (4.8) it is seen that for very small values of a_0 the function $W_2(a_0^2, \lambda)$ is positive. Hence, in Figure 3, this function is positive outside of the "parabola" - resonance curve and is negative inside of this "parabola". If moving upwards along a straight line which paralells the ordinate axis a_0 and cuts the resonance curve, we go from the zone where W_2 is negative to the zone where W_2 is positive, then at the intersection point of the straight line with the resonance curve the derivative $\frac{\partial W_2}{\partial a_0}$ is positive. In the opposite case $\frac{\partial W_2}{\partial a_0}$ is negative. Using this rule and taking the conditions (4.11) into consideration we can see that the upper (lower) branch of the resonance curve corresponds to the stability (instability) of stationary oscillation. In Figure 3a the stable branches are shown by solid lines and the unstable branches - by broken lines. Figure 3b shows the dependence of

L_2 on λ. It is seen that for values interested of $\lambda(\lambda = 3 \div 3.5)$ the expression L_2 is positive, and the first stability condition (4.11) is satified.

It is easy to see that the zero solution $a = 0$ of equations (4.5) is stable, because the corressponding variational equation is:

$$\frac{d\delta a}{dt} = -\frac{3\varepsilon}{\lambda} L_2 \delta a,$$

where L_2 is positive in the interval interested of $\lambda(3 \div 3.5)$ (see Figure 2b).

5. Parametric resonance

Let us consider a dynamic system described by the differential equation:

$$\frac{d^2 x(t)}{dt^2} + x(t) = \varepsilon F_1, \tag{5.1}$$

where

$$F_1 = -2\xi \frac{dx(t)}{dt} - \mu x^3(t) + 2ux(t-\Delta) + 2v\frac{dx(t-\Delta)}{dt} + 2px(t)\cos\lambda t. \tag{5.2}$$

Different from (1.1), here the external force appears as parametric excitation in the form $2fx(t) \cdot \cos \lambda t$. It is supposed that $\lambda = 2 + \varepsilon\sigma$, so that the equation (5.1) can be written in the form:

$$\frac{d^2x(t)}{dt^2} + \frac{\lambda^2}{4} x(t) = \varepsilon[\sigma x(t) + F_1]. \tag{5.3}$$

Introducing the amplitude a and the phase θ as new variables, associated with x and $\dot{x}$ by the formulae:

$$\begin{aligned} x &= a\cos(\frac{\lambda}{2}t + \theta), \\ \frac{dx}{dt} &= -\frac{\lambda}{2} a \sin\left(\frac{\lambda}{2}t + \theta\right), \end{aligned} \tag{5.4}$$

we have the following equations which are equivalent to (5.3):

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{2\varepsilon}{\lambda}(\sigma a \cos\eta + F_1)\sin\eta, \\ a\dfrac{d\theta}{dt} = -\dfrac{2\varepsilon}{\lambda}(\sigma a \cos\eta + F_1)\cos\eta, \end{cases} \tag{5.5}$$

$$\eta = \frac{\lambda}{2}t + \theta.$$

In the first approximation, equations (5.5) can be replaced by the following averaged equations:

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{\varepsilon a}{\lambda}(L_3 + p\sin 2\theta), \\ a\dfrac{d\theta}{dt} = -\dfrac{\varepsilon a}{\lambda}\left(M_3 - \dfrac{3}{4}\mu a^2 + p\cos 2\theta\right), \end{cases} \tag{5.6}$$

where

$$\begin{aligned} L_3 &= \xi\lambda + 2u\sin\left(\frac{\lambda\Delta}{2}\right) - \lambda v\cos\left(\frac{\lambda\Delta}{2}\right), \\ M_3 &= \sigma + 2u\cos\left(\frac{\lambda\Delta}{2}\right) + \lambda v\sin\left(\frac{\lambda\Delta}{2}\right). \end{aligned} \tag{5.7}$$

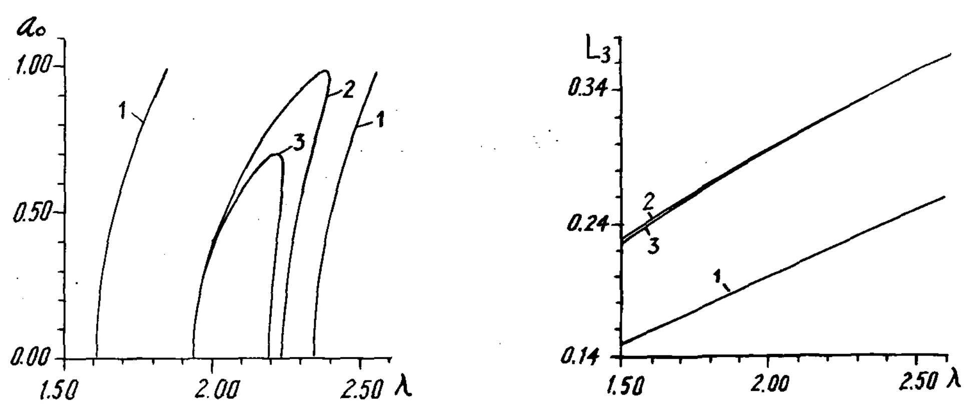
The stationary solution $a = a_0 \neq 0$, $\theta = \theta_0$ of equations (5.6) is determined by the relationships:

$$\begin{aligned} &L_3 + p\sin 2\theta_0 = 0, \\ &M_3 - \frac{3}{4}\mu a_0^2 + p\cos 2\theta_0 = 0. \end{aligned} \tag{5.8}$$

Eliminating θ_0 we get:

$$W_3(a_0^2, \lambda) = L_3^2 + \left(M_3 - \frac{3}{4}\mu a_0^2\right)^2 - p^2 = 0. \tag{5.9}$$

The resonance curves are plotted in Figure 3 for the parameters $p = 0.42$, $\xi = 0.1$, $\mu = 0.3$, $u = v = 0$ (curve 1), $u = 0.05$, $v = -0.1$ and $\Delta = 1$, (curve 2), $\Delta = 0.95$ (curve 3). By decreasing the delay parameter, the amplitude of oscillation decreases. The maximum of the amplitudes is very sensitive to a change in the delay parameter.



Figgure 3a *Figure 3b*

Figure 3. Resonance curves for parametric oscillations

The stabilility of the stationary oscillations obtained is examined by using the corresponding variational equations:

$$\begin{aligned} \frac{d\delta a}{dt} &= -\frac{2\varepsilon}{\lambda} p a_0 \cos 2\theta_0 \cdot \delta\theta, \\ a_0 \frac{d\delta\theta}{dt} &= \frac{3\varepsilon}{2\lambda}\mu a_0^2 \cdot \delta a + \frac{2\varepsilon}{\lambda} p a_0 \sin 2\theta_0 \cdot \delta\theta. \end{aligned} \tag{5.10}$$

where, $\delta a = a - a_0$, $\delta\theta = \theta - \theta_0$. The characteristic equation of equations (5.10) is

$$\rho^2 + \frac{2\varepsilon}{\lambda} L_3 + \frac{3\varepsilon^2}{\lambda^2}\mu a_0^2 \Big(\frac{3}{4}\mu a_0^2 - M_3\Big) = 0. \tag{5.11}$$

It is easy to see that

$$\frac{\partial W_3}{\partial a_0^2} = \frac{3\mu}{2}\Big(\frac{3}{4}\mu a_0^2 - M_3\Big).$$

Hence, the stability conditions of stationary solution ($a_0 \neq 0$) are

1) $L_3 > 0$

2) $\dfrac{\partial W_3}{\partial a_0} > 0.$ (5.12)

Since L_3 is positive in the interval interested of $\lambda(1.5 \div 2.5)$, we then consider only the second stability condition (5.12). It is easy to identify the sign of the function W_3 in the plane (λ, a_0), because for $a_0 = 0$ and for very large values of λ the function W_3 (5.9) is positive. This function vanishes on the resonance curve and changes sign when crossing the resonance curve. According to the well-known rule [3] we can see that the upper branch of the resonance curve is stable and the lower branch is unstable.

In order to study the stabilility of the trivial solution $a = 0$ of the equations (5.6) it is convenient to use the cartesian coordinate (u, v) instead of the polar coordinates a and θ as follows:

$$y = a\cos\theta, \qquad z = a\sin\theta, \tag{5.13}$$

which gives

$$\begin{aligned} \frac{dy}{dt} &= -\frac{\varepsilon}{\lambda}\Big[L_3 y + \Big(M_3 - \frac{3}{4}\mu a^2 - p\Big) z\Big], \\ \frac{dz}{dt} &= -\frac{\varepsilon}{\lambda}\Big[\Big(M_3 - \frac{3}{4}\mu a^2 + p\Big) y + L_3 z\Big]. \end{aligned} \tag{5.14}$$

The characteristic equation of this system is

$$\rho^2 + \frac{2\varepsilon}{\lambda} L_3 \rho + \frac{\varepsilon^2}{\lambda^2} W_3(a_0^2, \lambda) = 0. \tag{5.15}$$

Hence, the abcissa axis λ ($a = 0$ or $y = z = 0$) is stable where

$$L_3 > 0 \quad \text{and} \quad W_3(a_0^2, \lambda) > 0. \tag{5.16}$$

In other word, on the abscissa - axis λ the segments lying outside the interval, from which the resonance curve is growing up, are stable and the interval lying inside the resonance curve is unstable (Figure 3a).

6. Parametric excitation of the second degree

Changing the structure of the parametric excitation, we consider the system described by the differential equation of the form (5.1) with the function F in the form:

$$F_2 = -2\xi \frac{dx(t)}{dt} - \mu x^3(t) + 2ux(t-\Delta) + 2v\frac{dx(t-\Delta)}{dt} + 2px^2(t)\cos\lambda t, \tag{6.1}$$

here the parametric excitation appears with second degree in x. Supposing that

$$\lambda = 1 + \varepsilon\sigma, \tag{6.2}$$

we can write equation (5.1) as

$$\frac{d^2x(t)}{dt^2} + \lambda^2 x(t) = \varepsilon[2\sigma x(t) + F_2]. \tag{6.3}$$

The solution of equation (6.3) is taken in the form (2.3). The averaged equations are now:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon a}{\lambda}\left(L + \frac{1}{4}pa\sin\theta\right), \\ a\frac{d\theta}{dt} &= -\frac{\varepsilon a}{\lambda}\left(M - \frac{3}{8}\mu a^2 + \frac{3}{4}pa\cos\theta\right), \end{aligned} \tag{6.4}$$

where, L and M are of the form (2.6). Stationary solutions of equations (6.4) are:

1)$a_0 = 0$

2)$a_0 \neq 0$ determied by

$$W_4 = 9L^2 + \left(M - \frac{3}{8}\mu a_0^2\right)^2 - \frac{9}{16}p^2a_0^2 = 0. \tag{6.5}$$

The stability of stationary solutions $a = a_0 \neq 0$, $\theta = \theta_0$ is studied by using the variational equation:

$$\begin{aligned} \frac{d\delta a}{dt} &= \frac{\varepsilon}{\lambda}\left[L\delta a + \frac{a_0}{3}\left(M - \frac{3}{8}\mu a_0^2\right)\delta\theta\right], \\ a_0\frac{d\delta\theta}{dt} &= \frac{\varepsilon}{\lambda}\left[\left(M + \frac{3}{8}\mu a_0^2\right)\delta a - 3La_0\delta\theta\right], \end{aligned} \tag{6.6}$$

where $\delta a = a - a_0$, $\delta\theta = \theta - \theta_0$, $a_0 \neq= 0$. The characteristic equation of this system is

$$\rho^2 + \frac{2\varepsilon}{\lambda}L\rho + \frac{\varepsilon^2 a_0^2}{3\lambda^2}\cdot\frac{\partial W_4}{\partial a_0^2} = 0, \tag{6.7}$$

where $\frac{\partial W_4}{\partial a_0^2} = \frac{-1}{a_0^2}\left(9L^2 + M^2 - \frac{9}{64}\mu^2 a_0^4\right)$. The stability conditions are

1) $L > 0$

2) $\frac{\partial W_4}{\partial a_0} > 0.$ (6.8)

To study the stability of zero solution $a = 0$ of equations (6.4) we use the transformation (5.13). The equations for y and z are:

$$\begin{aligned} \frac{dy}{dt} &= \frac{\varepsilon}{\lambda}(-Ly + Mz) + \ldots, \\ \frac{dz}{dt} &= -\frac{\varepsilon}{\lambda}(My + Lz) + \ldots, \end{aligned} \tag{6.9}$$

where the nonwritten terms are the terms with high degree relative to y and z. The characteristic equation of the linear part of equations (6.9) is:

$$\rho^2 + \frac{3\varepsilon}{\lambda}L\rho + \frac{\varepsilon^2}{\lambda^2}(L^2 + M^2) = 0. \tag{6.10}$$

Hence, the stability condition of the zero solution is $L > 0$.

7. Parametric excitations of the third degree

Suppose the following form of equation (5.1):

$$\frac{d^2x}{dt^2}(t) + x(t) = \varepsilon F_3, \tag{7.1}$$

where

$$F_3 = -2\xi\frac{dx(t)}{dt} - \mu x^3(t) + 2ux(t-\Delta) + 2v\frac{dx(t-\Delta)}{dt} + 2qx^3(t)\cos 2\lambda t, \tag{7.2}$$

and

$$\lambda^2 = 1 + \varepsilon 2\sigma. \tag{7.3}$$

In this case we have

$$\frac{d^2x(t)}{dt^2} + \lambda^2 x(t) = \varepsilon(2\sigma x(t) + F_3). \tag{7.4}$$

The solution of equation (7.2) will be found in the form (2.3). In the first approximation, the amplitude a and the phase θ are determined by the equations:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon a}{\lambda}\left(L + q\frac{a^2}{4}\sin 2\theta\right), \\ a\frac{d\theta}{dt} &= \frac{-\varepsilon a}{\lambda}\left(M - \frac{3}{8}\mu a^2 + q\frac{a^2}{2}\cos 2\theta\right), \end{aligned} \tag{7.5}$$

where L and M are of the form (2.6). The amplitude and phase of stationary oscillations satisfy the equations:

$$
\begin{gathered}
4L + qa_0^2 \sin 2\theta_0 = 0, \\
2\left(M - \frac{3}{8}\mu a_0^2\right) + qa_0^2 \cos 2\theta_0 = 0.
\end{gathered}
\tag{7.6}
$$

Eliminating the phase θ_0 we obtain:

$$
V_5 = 16L^2 + 4\left(M - \frac{3}{8}\mu a_0^2\right)^2 - q^2 a_0^4 = 0. \tag{7.7}
$$

The stability of stationary oscillations obtained is examined by means of variational equations:

$$
\begin{cases}
\dfrac{d\delta a}{dt} = \dfrac{\varepsilon}{\lambda}\left[2L\delta a + a_0\left(M - \dfrac{3}{8}\mu a_0^2\right)\delta\theta\right], \\
a_0 \dfrac{d\delta\theta}{dt} = \dfrac{\varepsilon}{\lambda}\left(2M\delta a - 4a_0 L\delta\theta\right),
\end{cases}
\tag{7.8}
$$

where $\delta a = a - a_0$, $\delta\theta = \theta - \theta_0$, $a_0 \neq 0$. The characteristic equation of system (7.8) is

$$
\rho^2 + \frac{3\varepsilon}{\lambda} L\rho + \frac{\varepsilon^2 a_0^2}{4\lambda^2} \cdot \frac{\partial W_5}{\partial a_0^2} = 0, \tag{7.9}
$$

where

$$
\frac{\partial W_5}{\partial a_0^2} = -\frac{8}{a_0^2}\left[4L^2 + M\left(M - \frac{3}{8}\mu a_0^2\right)\right].
$$

Hence, the stability conditions are

$$
\begin{aligned}
&1) \quad L > 0 \\
&2) \quad \frac{\partial W_5}{\partial a_0^2} > 0.
\end{aligned}
\tag{7.10}
$$

8. Conclusion

The nonlinear oscillators under delay control described by differential equations of types (2.1), (3.1), (4.1), (5.1), (6.1), (7.1) have been examined. When the delay parameter Δ vanishes we have the corresponding classical nonlinear oscillators. The appearance of a delay parameter makes the systems under examination change qualitatively and quantitatively. See, for example, curves 4 and 5 of Figure 1a for the case of increasing time delay. The resonance curves lean toward the right and the maximum of the amplitudes slightly decreases. In Figure 2, by increasing the delay parameter the resonance curve moves up. The larger the time delay Δ, the higher the resonance curve. In the case of parametric oscillation (Figure

3a), by decreasing the delay parameter decreases the amplitude of oscillation. The maximum of the amplitudes is very sensitive to a change in the delay parameter. In general, the dependence of the maximum of the amplitudes of oscillations on the delay parameter Δ is complicated because this parameter appears under the functions sinus and cosinus in the expressions L, M (2.6), L_1, M_1 (3.7), L_2, M_2 (4.6), L_3, M_3 (5.7). Increasing Δ sometimes leads to decreasing the maximum of the amplitudes of oscillation, but other times leads to increasing this maximum.

The author is indebted to Dr. Tran Kim Chi for mathematical modeling on computer and for generating the figures.

This work was supported by the Council for Natural Science of Vietnam.

REFERENCES

1. Haiyan Hu, Earl H. Dowell, Lawrence N. Virgin. Resonances of a harmonically forced Duffing oscillator with time delay state feedback. Nonlinear Dynamics 15 : 311 - 327. *Kluwer Academic Publishers*, 1998.
2. Nayfeh A. H. and Mook D. T. Nonlinear oscillations, *Wiley, New York*, 1979.
3. Mitropolskii Yu. A., Nguyen Van Dao. Applied asymptotic methods in nonlinear oscillations. *Kluwer Academic Publishers*, 1997.

Received June 5, 1999

CÁC CHẤN TỬ PHI TUYẾN DƯỚI SỰ ĐIỀU KHIỂN TRỄ

Trong bài báo đã nghiên cứu các chấn tử phi tuyến dưới sự điều khiển trễ mô tả bởi các phương trình vi phân dạng (2.1), (3.1), (4.1), (5.1), (6.1) và (7.1). Sự xuất hiện yếu tố trễ Δ đã làm cho các kết quả nghiên cứu cổ điển thay đổi cả về lượng và chất. Chẳng hạn, quan sát các đường cong 4, 5 trên hình 1a khi tăng Δ. Các đường cong này ngả về bên phải và biên độ cực đại giảm. Còn trên hình 2, việc tăng Δ đã làm cho đường cộng hưởng dịch chuyển lên cao. Trong trường hợp dao động thông số (H. 3a), giảm Δ sẽ làm giảm biên độ cực đại. Nói chung, sự phụ thuộc của biên độ cực đại vào thông số trễ rất phức tạp do thông số này nằm dưới các hàm sin, cosin qua các biểu thức L, M (2.6), L_1, M_1 (3.7), L_2, M_2 (4.6), L_3, M_3 (5.7). Việc tăng Δ không nhất thiết dẫn tới giảm, mà có khi lại làm tăng biên độ cực đại.