# MODIFIED ELASTIC SOLUTION METHOD IN SOLVING ELASTOPLASTIC PROBLEMS OF STRUCTURE COMPONENTS SUBJECTED TO COMPLEX LOADING

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**SUMMARY.** Modified elastic solution method in the elastoplastic process theory has been proposed by the author [2] and was applied in solving some 2D and 3D elastoplastic problems of structure components subjected to complex loading. The method makes use of an algorithm in which a step is made in the loading process and iterations are carried out on this step. The performance of the method was fulfilled and the convergence of the method was considered numerically. In this paper the other performance of this method is presented and the convergence of the method is proven theoretically in the general case of a hardening body which obeys the elastoplastic process theory. The more complicated 3D problem of bodies of revolution subjected to non-axially symmetric load is investigated.

## 1. Boundary value problem of the elastoplastic process theory and modified elastic solution method

The formulation of the boundary value problem of the elastoplastic process theory and analysis of the existence and uniqueness theorems have been carried out in [3, 4].

Let $K_i(x, t)$ and $F_i(x, t)$ be external volume and surface forces that act on the body and let $\varphi_i(x, t)$ be displacement on the body's surface. It is necessary to find displacements $u_i(x, t)$, strain tensor $\varepsilon_{ij}(x, t)$ and stress tensor $\sigma_{ij}(x, t)$, where $t$ - the loading parameter, that satisfy the following equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho K_i = 0, \qquad\qquad x \in \Omega, \qquad (1.1)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad x \in \overline{\Omega}, \qquad (1.2)$$

$$\dot{S}_{ij} = \frac{2}{3} A\dot{e}_{ij} + (P - A)\frac{S_{k\ell}\dot{e}_{k\ell}}{\sigma_u^2} S_{ij}, \quad x \in \overline{\Omega}, \qquad (1.3)$$

$$\sigma = 3Ke = K\theta, \qquad\qquad x \in \overline{\Omega}, \qquad (1.4)$$

and the boundary conditions

$$\sigma_{ij}n_j = F_i, \qquad x \in S_\sigma, \tag{1.5}$$

$$u_i = \varphi_i, \qquad x \in S_u, \tag{1.6}$$

$$\bar{\Omega} = \Omega \cup S, \quad S_\sigma \cup S_u = S, \quad S_\sigma \cap S_u = \emptyset, \quad t \in [0,T],$$

where

$$A = \frac{\sigma_u}{s} + \left(3G - \frac{\sigma_u}{s}\right)\left(\frac{1-\cos\theta_1}{2}\right)^\alpha,$$

$$P = \phi'(s) - (3G - \phi'(s))\left(\frac{1-\cos\theta_1}{2}\right)^\beta \frac{1}{\cos\theta_1},$$

$$\cos\theta_1 = \frac{S_{ij}\dot{e}_{ij}}{\sigma_u v_u}, \quad s = \int_0^t v_u\,dt = \int_0^t \left(\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}\right)^{1/2}dt.$$

*Remark.* If we are concerned with the process theory of average curvature, then in relationship (1.3) we put

$$A = \frac{\sigma_u}{s}, \qquad P = \phi'(s).$$

For later use, for setting up the modified elastic solution method, we represent

$$A = 3G(1-\omega_1), \quad P = 3G(1-\omega_2), \quad 0 \leq \omega_1 \leq 1, \quad 0 \leq \omega_2 \leq 1,$$

then for the general elastoplastic process theory

$$\omega_1 = \left(1 - \frac{\sigma_u}{3Gs}\right)\left[1 - \left(\frac{1-\cos\theta_1}{2}\right)^\alpha\right],$$

$$\omega_2 = \left(1 - \frac{\phi'}{3G}\right)\left[1 + \frac{(1-\cos\theta_1)^\beta}{2^\beta \cos\theta_1}\right], \tag{1.7}$$

and for the process theory with average curvature

$$\omega_1 = 1 - \frac{\sigma_u}{3Gs}, \quad \omega_2 = 1 - \frac{\phi'}{3G}. \tag{1.8}$$

The stress-strain relationship (1.3), (1.4) can be rewritten as following

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{e}_{kl} = (E_{ijkl} - H_{ijkl})\dot{e}_{kl}, \tag{1.9}$$

where

$$E_{ijk\ell} = \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{k\ell} + G(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

$$H_{ijk\ell} = -\frac{2}{3}G\omega_1\delta_{ij}\delta_{k\ell} + G\omega_1(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + 3G(\omega_2 - \omega_1)\frac{S_{ij}S_{k\ell}}{\sigma_u^2}.$$

(1.10)

For any symmetric tensor $\varepsilon_{ij}$ we have

$$D_{ijk\ell}\varepsilon_{k\ell}\varepsilon_{ij} = 2G\varepsilon_{ij}\varepsilon_{ij} + \left(K - \frac{2}{3}G\right)\theta^2$$

$$- \left[2G\omega_1\left(\varepsilon_{ij}\varepsilon_{ij} - \frac{1}{3}(\varepsilon_{kk})^2\right) + 3G(\omega_2 - \omega_1)\frac{(S_{ij}\varepsilon_{ij})^2}{\sigma_u^2}\right].$$

Since $\omega_2 > \omega_1$, $\varepsilon_{ij}\varepsilon_{ij} - \frac{1}{3}\theta^2 > 0$, thus the expression in the square brackets is positive. On the other hand,

$$D_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} \geq \frac{2}{3}\phi'\varepsilon_{ij}\varepsilon_{ij} + K\theta^2 = 2G(1 - \omega_2)\varepsilon_{ij}\varepsilon_{ij} + K\theta^2$$

$$\geq (1 - \omega_2)\left[2G\varepsilon_{ij}\varepsilon_{ij} + \left(K - \frac{2}{3}G\right)\theta^2\right].$$

Consequently,

$$(1 - \omega_2)E_{ijk\ell}\varepsilon_{k\ell}\varepsilon_{ij} \leq D_{ijk\ell}\varepsilon_{k\ell}\varepsilon_{ij} \leq E_{ijk\ell}\varepsilon_{k\ell}\varepsilon_{ij}.$$

(1.11)

Now we subdivide the range of variation of the loading parameter $t$ into $N$ parts and denote $t$ at the nodes by $t_n$ $(n = 0, 1, 2, \ldots, N)$. Denote respectively

$$u_i(x, t_n) \equiv u_i^{(n)} = u_i^{(0)} + \sum_{m=1}^{n} \Delta u_i^{(m)} = u_i^{(n-1)} + \Delta u_i^{(n)},$$

$$\varepsilon_{ij}(x, t_n) \equiv \varepsilon_{ij}^{(n)} = \varepsilon_{ij}^{(0)} + \sum_{m=1}^{n} \Delta\varepsilon_{ij}^{(m)} = \varepsilon_{ij}^{(n-1)} + \Delta\varepsilon_{ij}^{(n)},$$

$$\sigma_{ij}(x, t_n) \equiv \sigma_{ij}^{(n)} = \sigma_{ij}^{(0)} + \sum_{m=1}^{n} \Delta\sigma_{ij}^{(m)} = \sigma_{ij}^{(n-1)} + \Delta\sigma_{ij}^{(n)},$$

$$K_i(x, t_n) = K_i^{(n)}, \quad F_i(x, t_n) = F_i^{(n)}, \quad \varphi_i(x, t_n) = \varphi_i^{(n)}.$$

At each step $n = 1, 2, \ldots, N$ of the change in the above - mentioned quantities, from (1.1) - (1.6) and taking into account (1.9) we set up the following system of equations

$$\frac{\partial \sigma_{ij}^{(n)}}{\partial x_j} + \rho K_i^{(n)} = 0, \qquad x \in \Omega,$$

$$\varepsilon_{ij}^{(n)} = \frac{1}{2}\left(\frac{\partial u_i^{(n)}}{\partial x_j} + \frac{\partial u_j^{(n)}}{\partial x_i}\right), \qquad x \in \overline{\Omega}, \qquad (1.12)$$

$$\sigma_{ij}^{(n)} = \sigma_{ij}^{(n-1)} + \Delta\sigma_{ij}^{(n)} = \sigma_{ij}^{(n-1)} + E_{ijk\ell}\Delta\varepsilon_{k\ell}^{(n)} - \langle H_{ijk\ell}\rangle^{(n)}\Delta\varepsilon_{k\ell}^{(n)}, \qquad x \in \overline{\Omega},$$

$$\sigma_{ij}^{(n)} n_j = F_i^{(n)}, \qquad x \in S_\sigma.$$

$$u_i^{(n)} = \varphi_i^{(n)}, \qquad x \in S_u,$$

where $\langle H_{ijk\ell}\rangle^{(n)}$ is an average quantity of $H_{ijk\ell}$ in the interval $(t_{n-1}, t_n)$ which can be taken as $\frac{1}{2}\left(H_{ijk\ell}^{(n-1)} + H_{ijk\ell}^{(n)}\right)$.

In approximation we take $\langle H_{ijk\ell}\rangle^{(n)} = H_{ijk\ell}^{(n-1)}$, the system of equations (1.12) can be considered as a system of equations for a certain inhomogeneous anisotropic elastic body with additional volume and surface forces. This system of equations is solved step by step, beginning from the first step $n = 1$. At the $n$-th step, $u_i^{(n-1)}$, $\varepsilon_{ij}^{(n-1)}$, $\sigma_{ij}^{(n-1)}$ are known functions, which have been determined at the $(n-1)$-th step, the problem leads to determine $\Delta u_i^{(n)}$, $\Delta\varepsilon_{ij}^{(n)}$ and $\Delta\sigma_{ij}^{(n)}$. At each step in the loading the problem generally is nonlinear, so we will solve it by using an iterative method - a modified elastic solution method [2, 3] - which is analogous to the elastic solution method in the deformation theory of plasticity [1, 5, 8]. Non-linearity of the problem is expressed in the constitutive equations, i.e. the third relation of (1.12). The procedure of the modified elastic solution method on this relation is written as follows

$$\sigma_{ij}^{(n,k)} = \sigma_{ij}^{(n-1)} + E_{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k)} - H_{ijk\ell}^{(n-1)}\Delta\varepsilon_{k\ell}^{(n,k-1)}, \qquad (1.13)$$

where $k = 1, 2, \ldots$ is the number of iteration on the $n$-th step of the change in the loading parameter. In the result at $n$-th step and $k$-th iteration, we can write the system of equations in the form

$$\frac{\partial}{\partial x_j}\left(E_{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k)}\right) - \frac{\partial}{\partial x_j}\left(H_{ijk\ell}^{(n-1)}\Delta\varepsilon_{k\ell}^{(n,k-1)}\right) + \frac{\partial\sigma_{ij}^{(n-1)}}{\partial x_j} + \rho K_i^{(n)} = 0, \quad x \in \Omega,$$

$$\Delta\varepsilon_{ij}^{(n,k)} = \frac{1}{2}\left(\frac{\partial\Delta u_i^{(n,k)}}{\partial x_j} + \frac{\partial\Delta u_j^{(n,k)}}{\partial x_i}\right), \qquad x \in \overline{\Omega},$$

$$(1.14)$$

and the boundary conditions

$$E_{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k)}n_j = F_i^{(n)} - \sigma_{ij}^{(n-1)}n_j + H_{ijk\ell}^{(n-1)}\Delta\varepsilon_{k\ell}^{(n,k-1)}n_j, \quad x \in S_\sigma,$$
$$\Delta u_i^{(n,k)} = \varphi_i^{(n)} - u_i^{(n-1)}, \quad x \in S_u,$$

(1.15)

with $H_{ijk\ell}^{(n-1)}\Delta\varepsilon_{k\ell}^{(n,0)} = 0$.

The system of equation (1.14) and the boundary conditions (1.15) represent a boundary value problem for an elastic body with the same elasticity constants $E_{ijk\ell}$ as the initial body but with changed volume and surface forces.

After the system of equations has been solved, i.e. $\Delta u_i^{(n)}$ known, the displacement is represented as $u_i^{(n)} = u_i^{(n-1)} + \Delta u_i^{(n)}$. The strains $\varepsilon_{ij}^{(n)}$ are determined from the Cauchy equations, these strains are then substituted into the constitutive equations (1.3), from where $\sigma_{ij}^{(n)}$ are obtained.

## 2. On the convergence of the modified elastic solution method

The modified elastic solution method was applied in considering stress and strain states of some 2D and 3D bodies subjected to complex loading [3, 9, 10, 11]. From obtained numerical results, we can talk about the convergence of the method. Generally, results of the third and fourth iterations are already closer to each other; they differ from each other with small errors.

Now we introduce the proof of the convergence of the method theoretically.

For this aim we bring into use the functional Hilbert space $H(\Omega)$ with the norm

$$\|u\|_H^2 = \int_\Omega 2G\Delta e_{ij}(u)\Delta e_{ij}(u)d\Omega + 9K\int_\Omega \Delta e^2(u)d\Omega$$
$$= \int_\Omega E_{ijk\ell}\Delta\varepsilon_{k\ell}(u)\Delta\varepsilon_{ij}(u)d\Omega.$$

(2.1)

Let $\Delta v$ be any smooth vector function such that

$$\Delta v \equiv \{\Delta v_i\} \quad \text{and} \quad \Delta v_i = 0 \quad \text{on} \quad S_u,$$

$\Delta v$ is considered as a variation of the displacement increment. Multiplying the first equation of (1.14) by $\Delta v_i$ and integrating which over the entire volume $\Omega$ of the body we obtain

$$\int_\Omega \frac{\partial}{\partial x_j}\left[(E_{ijk\ell} - H_{ijk\ell})\Delta\varepsilon_{k\ell}(u) + \sigma_{ij}^{(n-1)}\right]\Delta v_i d\Omega + \int_\Omega \rho K_i\Delta v_i d\Omega = 0.$$

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Using the divergence theorem and the boundary conditions (1.15) we have

$$\int_\Omega (E_{ijk\ell} - H_{ijk\ell})\Delta\varepsilon_{k\ell}(u)\Delta\varepsilon_{ij}(v)d\Omega =$$

$$= -\int_\Omega \sigma_{ij}^{(n-1)}\Delta\varepsilon_{ij}(v)d\Omega + \int_\Omega \rho K_i\Delta v_i d\Omega + \int_{S_\sigma} F_i\Delta v_i dS. \tag{2.2}$$

Analogously [3], we can show that the expression on the left hand side is a linear and continuous functional on $H(\Omega)$. It follows from Riesz's theorem that there exists an operator $A : H(\Omega) \to H^*(\Omega)$, where $H^*(\Omega)$ is the dual functional space of $H(\Omega)$, such that

$$(Au, v)_H = \int_\Omega (E_{ijk\ell} - H_{ijk\ell})\Delta\varepsilon_{ij}(u)\Delta\varepsilon_{ij}(v)d\Omega.$$

Let known function $\sigma_{ij}^{(n-1)} \in L_2$ and $K_i \in L_p$ $(p > 6/5)$, $F_i \in L_q$ $(q \geq 4/3)$, then the expression at the right hand side of (2.2) is also a linear continuous functional on $H(\Omega)$, and there exists an operator $L : H \to H^*$ such that

$$-\int_\Omega \sigma_{ij}^{(n-1)}\Delta v_i d\Omega + \int_\Omega \rho K_i\Delta v_i d\Omega + \int_{S_\sigma} F_i\Delta v_i dS = (L, v)_H.$$

The equation (2.2) reduces to an equivalent operator equation

$$Au = L, \quad u \in H(\Omega). \tag{2.3}$$

A generalized solution of the boundary value problem (1.14) - (1.15) is also a solution of the operator equation (2.3) and conversely.

In the case, when iterations are carried out, we put $H_{ijk\ell} = H_{ijk\ell}^{(n-1)}$. Using the inequality (1.11) we can prove that, the operator equation (2.3) has a unique solution (similarly [3, 7]).

Now consider the convergence of the above mentioned method. Because of the scalar product in $H(\Omega)$

$$\int_\Omega E_{ijk\ell}\Delta\varepsilon_{k\ell}(u)\Delta\varepsilon_{ij}(v)d\Omega = (u, v)_H,$$

and from the existence of the fundamental operator $A$, then

$$\int_\Omega H_{ijk\ell}^{(n-1)}\Delta\varepsilon_{k\ell}(u)\Delta\varepsilon_{ij}(v)d\Omega = (B^*u, v)_H,$$

the equation (2.3) is rewritten in another form

$$Au = u - B^*u = L,$$
$$\text{or} \quad u = B^*u + L = Qu. \tag{2.4}$$

The algorithm of the modified elastic solution method for solving the equation (2.4) is as follows: on the $k$-th iteration to seek $u^{(k)}$ from the equation

$$u^{(k)} = Qu^{(k-1)}. \tag{2.5}$$

From (2.4), (2.5) we have

$$(u^{(k+1)} - u^{(k)}, v)_H = (Qu^{(k)} - Qu^{(k-1)}, v)_H = (B^*u^{(k)} - B^*u^{(k-1)}, v)_H$$
$$= \int_\Omega (H^{(n-1)}_{ijk\ell} \Delta \varepsilon^{(k)}_{k\ell}(u) - H^{(n-1)}_{ijk\ell} \Delta \varepsilon^{(k-1)}_{k\ell}(u)) \Delta \varepsilon_{ij}(v) d\Omega$$
$$= \int_\Omega H^{(n-1)}_{ijk\ell} \Delta \varepsilon_{k\ell}(u^{(k)} - u^{(k-1)}) \Delta \varepsilon_{ij}(v) d\Omega.$$

Using the inequality (1.11) into the last equation, it follows

$$\left(u^{(k+1)} - u^{(k)}, v\right)_H \leq \max(\omega_2) \int_\Omega E_{ijk\ell} \Delta \varepsilon_{k\ell}(u^{(k)} - u^{(k-1)}) \Delta \varepsilon_{ij}(v) d\Omega.$$

By putting $v = u^{(k+1)} - u^{(k)}$ the obtained result reduces to

$$\left\| u^{(k+1)} - u^{(k)} \right\|^2_H \leq \max \omega_2 \int_\Omega E_{ijk\ell} \Delta \varepsilon_{k\ell}(u^{(k)} - u^{(k-1)}) \Delta \varepsilon_{ij}(u^{(k+1)} - u^{(k)}) d\Omega. \tag{2.6}$$

Further, applying the Bunhiakovsky, Cauchy-Schwarz inequalities into functional

$$I = \int_\Omega E_{ijk\ell} \Delta \varepsilon_{k\ell}(u^{(k)} - u^{(k-1)}) \Delta \varepsilon_{ij}(u^{(k+1)} - u^{(k)}) d\Omega$$
$$= \int_\Omega 2G \Delta e_{ij}(u^{(k)} - u^{(k-1)}) \Delta e_{ij}(u^{(k+1)} - u^{(k)}) d\Omega$$
$$+ 9K \int_\Omega \Delta e(u^{(k)} - u^{(k-1)}) \Delta e(u^{(k+1)} - u^{(k)}) d\Omega$$

we obtain

$$I \leq \left( \int_\Omega 2G\Delta e_{ij}(u^{(k)} - u^{(k-1)})\Delta e_{ij}(u^{(k)} - u^{(k-1)})d\Omega \right)^{1/2}$$

$$\times \left( \int 2G\Delta e_{mn}(u^{(k+1)} - u^{(k)})\Delta e_{mn}(u^{(k+1)} - u^{(k)})d\Omega \right)^{1/2}$$

$$+ \left( \int 9K\Delta e^2(u^{(k)} - u^{(k-1)})d\Omega \right)^{1/2} \left( \int 9K\Delta e^2(u^{(k+1)} - u^{(k)})d\Omega \right)^{1/2}$$

$$\leq \left( \int 2G\Delta e_{ij}(u^{(k)} - u^{(k-1)})\Delta e_{ij}(u^{(k)} - u^{(k-1)})d\Omega + 9K \int \Delta e^2(u^{(k)} - u^{(k-1)})d\Omega \right)^{1/2}$$

$$\times \left( \int 2G\Delta e_{mn}(u^{(k+1)} - u^{(k)})\Delta e_{mn}(u^{(k+1)} - u^{(k)})d\Omega \right.$$

$$\left. + 9K \int \Delta e^2(u^{(k+1)} - u^{(k)})d\Omega \right)^{1/2}$$

$$\leq \left\| u^{(k)} - u^{(k-1)} \right\|_H \left\| u^{(k+1)} - u^{(k)} \right\|_H.$$

Substituting the expression I into (2.6) we have

$$\left\| u^{(k+1)} - u^{(k)} \right\|_H^2 \leq \max \omega_2 \left\| u^{(k)} - u^{(k-1)} \right\|_H \left\| u^{(k+1)} - u^{(k)} \right\|_H,$$

or

$$\left\| u^{(k+1)} - u^{(k)} \right\|_H \leq \max \omega_2 \left\| u^{(k)} - u^{(k-1)} \right\|_H. \tag{2.7}$$

Since $\max(\omega_2) < 1$, the operator $Q$ is compressible, from (2.7) one can lead the convergence of the iterative method. The condition $\max \omega_2 = \max \left( 1 - \dfrac{\phi'(s)}{3G} \right) < 1$ is equivalent to $\phi'(s) > 0$, i.e. the material must be hardening.

## 3. Performance of the modified elastic solution method for the problem in curvilinear coordinates

On curvilinear coordinates the system of equations (1.14) is of the form

$$\nabla_j \left( E^{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k)} \right) + \nabla_j \sigma_{(n-1)}^{ij} - \nabla_j \left( H_{(n-1)}^{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k-1)} \right) + \rho K_{(n)}^i = 0, \tag{3.1}$$

$$\Delta\varepsilon_{ij}^{(n,k)} = \frac{1}{2} \left[ \nabla_j(\Delta u_i^{(n,k)}) + \nabla_i(\Delta u_j^{(n,k)}) \right], \quad x \in \Omega, \tag{3.2}$$

and boundary conditions

$$E^{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k)} n_j = F_{(n)}^i - \sigma_{(n-1)}^{ij} n_j + H_{(n-1)}^{ijk\ell}\Delta\varepsilon_{k\ell}^{(n,k-1)} n_j, \quad x \in S_\sigma,$$

$$\Delta u_i^{(n,k)} = \varphi_i^{(n)} - u_i^{(n-1)}, \quad x \in S_u, \tag{3.3}$$

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where

$$E^{ijk\ell} = \lambda g^{ij}g^{k\ell} + G(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}),$$

$$H^{ijk\ell} = -\frac{2}{3}G\omega_1 g^{ij}g^{k\ell} + G\omega_1(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}) + 3G(\omega_2 - \omega_1)\frac{S^{ij}S^{k\ell}}{\sigma_u^2},$$

$\nabla_j$ denotes the covariant derivative with respect to $x^j$;

$g^{ij}$ - metric tensor of curvilinear coordinate.

For investigation of the body of revolution subjected to complex loading, we usually consider the problem in a cylindrical coordinate

$$(r, \varphi, z): \quad g^{11} = 1, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = 1, \quad g^{ij} = 0 \quad (i \neq j).$$

Denote $\Delta u_r = \Delta u$, $\Delta u_\varphi = \Delta v$, $\Delta u_z = \Delta w$, the strain increment components are determined by Cauchy equations (3.2):

$$\Delta\varepsilon_{rr} = \frac{\partial \Delta u}{\partial r}, \quad \Delta\varepsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial \Delta v}{\partial \varphi} + \frac{\Delta u}{r}, \quad \Delta\varepsilon_{zz} = \frac{\partial \Delta w}{\partial z},$$

$$\Delta\varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial \Delta u}{\partial \varphi} + \frac{\partial \Delta v}{\partial r} - \frac{\Delta v}{r}\right), \quad \Delta\varepsilon_{\varphi z} = \frac{1}{2}\left(\frac{\partial \Delta v}{\partial z} + \frac{1}{r}\frac{\partial \Delta w}{\partial \varphi}\right), \qquad (3.4)$$

$$\Delta\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial \Delta w}{\partial r} + \frac{\partial \Delta u}{\partial z}\right).$$

The system of equations (3.1) in this case reduces to

$$\nabla^2 \Delta w^{(n,k)} + \frac{1}{1-2\nu}\frac{\partial \Delta\theta^{(n,k)}}{\partial z} = -\frac{\rho K_z^{(n)}}{G} - \frac{R_z^{(n-1)}}{G} + \frac{R_z^{*(n,k-1)}}{G},$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)\Delta u^{(n,k)} - \frac{2}{r^2}\frac{\partial \Delta v^{(n,k)}}{\partial \varphi} + \frac{1}{1-2\nu}\frac{\partial \Delta\theta^{(n,k)}}{\partial r} =$$

$$= -\frac{\rho K_r^{(n)}}{G} - \frac{R_r^{(n-1)}}{G} + \frac{R_r^{*(n,k-1)}}{G}, \qquad (3.5)$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)\Delta v^{(n,k)} + \frac{2}{r^2}\frac{\partial \Delta u^{(n,k)}}{\partial \varphi} + \frac{1}{1-2\nu}\cdot\frac{1}{r}\frac{\partial \Delta\theta^{(n,k)}}{\partial \varphi} =$$

$$= -\frac{\rho K_\varphi^{(n)}}{G} - \frac{R_\varphi^{(n-1)}}{G} + \frac{R_\varphi^{*(n,k-1)}}{G},$$

where

$$\Delta\theta = \frac{\partial \Delta w}{\partial z} + \frac{\partial \Delta u}{\partial r} + \frac{\Delta u}{r} + \frac{1}{r}\frac{\partial \Delta v}{\partial \varphi},$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2},$$

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$$R_z^{(n-1)} = \frac{\partial \sigma_{zz}^{(n-1)}}{\partial z} + \frac{1}{r}\frac{\partial \left(r\sigma_{rz}^{(n-1)}\right)}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi z}^{(n-1)}}{\partial \varphi}, \tag{3.6}$$

$$R_r^{(n-1)} = \frac{\partial \sigma_{rz}^{(n-1)}}{\partial z} + \frac{1}{r}\frac{\partial \left(r\sigma_{rr}^{(n-1)}\right)}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\varphi}^{(n-1)}}{\partial \varphi} - \frac{\sigma_{\varphi\varphi}^{(n-1)}}{r},$$

$$R_\varphi^{(n-1)} = \frac{\partial \sigma_{\varphi z}^{(n-1)}}{\partial z} + \frac{1}{r}\frac{\partial \left(r\sigma_{r\varphi}^{(n-1)}\right)}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi\varphi}^{(n-1)}}{\partial \varphi} + \frac{\sigma_{r\varphi}^{(n-1)}}{r},$$

$R_z^*$, $R_r^*$, $R_\varphi^*$ have similar forms to (3.6), where

$$\sigma_{ij}^* = 2G\omega_1\left(\Delta\varepsilon_{ij} - \frac{1}{3}\Delta\theta\delta_{ij}\right) + 2G(\omega_2 - \omega_1)\frac{S_{ij}S_{k\ell}\Delta\varepsilon_{k\ell}}{\sigma_u^2} \quad (i,j = r,\varphi,z). \tag{3.7}$$

By combining with boundary conditions, we have the solving equations. Consequently, at each iteration it is necessary to solve a problem of linear elasticity with new supplementary volume and surface forces.

## 4. Particular case. Numerical example

A short cylinder of radius $R$ and length $L$ rotates with angle velocity $\omega(t)$ and is subjected to axesymmetrically with respect to axis $z$ tangential and normal forces. Introduce variables $\varsigma = \frac{z}{R}$, $\rho = \frac{r}{R}$, $\varphi$ and $t$, the stress and strain components do not depend on variable $\varphi$, furthermore, there are no torsional forces, the stress state is determined by $\sigma_{zz}$, $\sigma_{rr}$, $\sigma_{\varphi\varphi}$, $\sigma_{rz}$, the strain state - by $\varepsilon_{zz}$, $\varepsilon_{rr}$, $\varepsilon_{\varphi\varphi}$, $\varepsilon_{rz}$ and radial displacement - $Ru$, axial displacement - $Rw$.

The system of solving equations (3.5) becomes

$$\nabla^2\Delta w^{(n,k)} + \frac{1}{1-2\nu}\frac{\partial\Delta\theta^{(n,k)}}{\partial\varsigma} = -\frac{R_z^{(n-1)}}{G} + \frac{R_z^{*(n,k-1)}}{G},$$

$$\left(\nabla^2 - \frac{1}{\rho^2}\right)\Delta u^{(n,k)} + \frac{1}{1-2\nu}\frac{\partial\Delta\theta^{(n,k)}}{\partial\rho} = -\rho\Omega^{(n)} - \frac{R_r^{(n-1)}}{G} + \frac{R_r^{*(n,k-1)}}{G}, \tag{4.1}$$

where

$$\Delta\theta = \frac{\partial\Delta u}{\partial\rho} + \frac{\Delta u}{\rho} + \frac{\partial\Delta w}{\partial\varsigma},$$

$$\Omega = \frac{\omega^2 R^2}{G}\frac{\gamma}{g},$$

$$R_z^{(n-1)} = \frac{\partial\sigma_{rz}^{(n-1)}}{\partial\rho} + \frac{\partial\sigma_{zz}^{(n-1)}}{\partial\varsigma} + \frac{\sigma_{rz}^{(n-1)}}{\rho}, \tag{4.2}$$

$$R_r^{(n-1)} = \frac{\partial\sigma_{rr}^{(n-1)}}{\partial\rho} + \frac{\partial\sigma_{rz}^{(n-1)}}{\partial\varsigma} + \frac{\sigma_{rr}^{(n-1)} - \sigma_{\varphi\varphi}^{(n-1)}}{\rho},$$

$R_z^{*(n,k-1)}$, $R_r^{*(n,k-1)}$ have similar forms to (4.2) where

$$\sigma_{ij}^{*(n,k-1)} = 2G\omega_1^{(n-1)}\left(\Delta\varepsilon_{ij}^{(n,k-1)} - \frac{1}{3}\Delta\theta^{(n,k-1)}\delta_{ij}\right)$$

$$+ 3G\left(\omega_2^{(n-1)} - \omega_1^{(n-1)}\right)\frac{S_{ij}^{(n-1)}S_{k\ell}^{(n-1)}\Delta\varepsilon_{k\ell}^{(n,k-1)}}{\left(\sigma_u^{(n-1)}\right)^2}, \quad (i,j = r,\varphi,z).$$

Boundary conditions are of the form:

a) with $\rho = 1$

$$\frac{1-\nu}{1-2\nu}\frac{\partial\Delta u^{(n,k)}}{\partial\rho} + \frac{\nu}{1-2\nu}\left(\frac{\Delta u^{(n,k)}}{\rho} + \frac{\partial\Delta w^{(n,k)}}{\partial\varsigma}\right) = \frac{F_1^{(n)}}{2G} - \frac{\sigma_{rr}^{(n-1)}}{2G} + \frac{\sigma_{rr}^{*(n,k-1)}}{2G},$$

$$\frac{1}{2}\left(\frac{\partial\Delta w^{(n,k)}}{\partial\rho} + \frac{\partial\Delta u^{(n,k)}}{\partial\varsigma}\right) = \frac{F_2^{(n)}}{2G} - \frac{\sigma_{rz}^{(n-1)}}{2G} + \frac{\sigma_{rz}^{*(n,k-1)}}{2G}; \qquad (4.3)$$

b) with $\varsigma = \dfrac{L}{R} = \ell$

$$\frac{1-\nu}{1-2\nu}\frac{\partial\Delta w^{(n,k)}}{\partial\varsigma} + \frac{\nu}{1-2\nu}\left(\frac{\partial\Delta u^{(n,k)}}{\partial\rho} + \frac{\Delta u^{(n,k)}}{\rho}\right) = \frac{F_3^{(n)}}{2G} - \frac{\sigma_{zz}^{(n-1)}}{2G} + \frac{\sigma_{zz}^{*(n,k-1)}}{2G},$$

$$\frac{1}{2}\left(\frac{\partial w^{(n,k)}}{\partial\rho} + \frac{\partial\Delta u^{(n,k)}}{\partial\varsigma}\right) = \frac{F_4^{(n)}}{2G} - \frac{\sigma_{rz}^{(n-1)}}{2G} + \frac{\sigma_{rz}^{*(n,k-1)}}{2G}; \qquad (4.4)$$

c) with $\varsigma = 0$

$$\frac{1-\nu}{1-2\nu}\frac{\partial\Delta w^{(n,k)}}{\partial\varsigma} + \frac{\nu}{1-2\nu}\left(\frac{\partial\Delta u^{(n,k)}}{\partial\rho} + \frac{\Delta u^{(n,k)}}{\rho}\right) = \frac{F_5^{(n)}}{2G} - \frac{\sigma_{zz}^{(n-1)}}{2G} + \frac{\sigma_{zz}^{*(n,k-1)}}{2G},$$

$$\frac{1}{2}\left(\frac{\partial\Delta w^{(n,k)}}{\partial\rho} + \frac{\partial\Delta u^{(n,k)}}{\partial\varsigma}\right) = \frac{F_6^{(n)}}{2G} - \frac{\sigma_{rz}^{(n-1)}}{2G} + \frac{\sigma_{rz}^{*(n,k-1)}}{2G}, \qquad (4.5)$$

where $F_i$ $(i = 1,2)$ - normal and tangential forces acting on cylinder surface $\rho = 1$; $F_i$ $(i = 3 \div 6)$ - forces acting on butt-ends of cylinder; $\sigma_{ij}^{(n-1)}$ - known values of stress components at preceding step $(n-1$-th step); $\sigma_{ij}^{*(n,k-1)}$ - also known values at considering step $(n$-th step) but on preceding iteration $(k-1$-th iteration).

Following [6] in order to solve the homogeneous system of equations (4.1) we express surface forces $F_1^{(n)}$, $F_2^{(n)}$ and

$$-\frac{\sigma_{rr}^{(n-1)}}{2G} + \frac{\sigma_{rr}^{*(n,k-1)}}{2G} \equiv F_{rr}^{(k-1)}(\varsigma,1), \qquad -\frac{\sigma_{rz}^{(n-1)}}{2G} + \frac{\sigma_{rz}^{*(n,k-1)}}{2G} \equiv F_{rz}^{(k-1)}(\varsigma,1)$$

in Fourier series

$$F_1^{(n)} = \sum_{i=1}^{\infty} F_{1i} \cos k_i \varsigma, \quad F_{rr}^{(k-1)} = \sum_{i=1}^{\infty} F_{1i}^* \cos k_i \varsigma,$$

$$F_2^{(n)} = \sum_{i=0}^{\infty} F_{2i} \sin k_i \varsigma, \quad F_{rz}^{(k-1)} = \sum_{i=1}^{\infty} F_{2i}^* \sin k_i \varsigma,$$
(4.6)

and the solution of the homogeneous system of equations corresponding to periodic surface forces is of the form

$$\Delta w^{(T)} = -\sum_{i=1}^{\infty} \left[ C_1 \rho I_1(k_i \rho) + \left( C_1 + 4 \frac{1-\nu}{k_i} C_2 \right) I_0(k_i \rho) \right] \sin k_i \varsigma,$$

$$\Delta u^{(T)} = \sum_{i=1}^{\infty} \left[ C_1 \rho I_0(k_i \rho) + C_2 I_1(k_i \rho) \right] \cos k_i \varsigma,$$
(4.7)

where $k_i = \dfrac{\pi i}{\ell}$, $I_i$ - modified Bessel functions of first type, $i$-degree.

A particular solution with respect to volume force has the form

$$\Delta w^{(\Omega)} = 0,$$

$$\Delta u^{(\Omega)} = -\frac{\Omega}{16} \frac{1-2\nu}{1-\nu} \rho^3.$$
(4.8)

For seeking a particular solution with respect to supplementary volume forces

$$-\frac{R_z^{(n-1)}}{G} + \frac{R_z^{*(n,k-1)}}{G} \equiv \frac{K_z^*(\rho,\varsigma)}{G}; \quad -\frac{R_r^{(n-1)}}{G} + \frac{R_r^{*(n,k-1)}}{G} \equiv \frac{K_r^*(\rho,\varsigma)}{G},$$

we express them into series

$$\frac{K_z^*}{G} = \sum_{i=1}^{\infty} \left\{ a_{i0}^{(1)} + \sum_{j=1}^{\infty} a_{ij}^{(1)} J_0(\lambda_j \rho) \right\} \sin k_i \varsigma,$$

$$\frac{K_r^*}{G} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{(2)} J_1(\lambda_j \rho) \cos k_i \varsigma,$$
(4.9)

where

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$$a_{i0}^{(1)} = \frac{\displaystyle\int_0^1 \int_0^\ell \frac{K_z^*(\varsigma,\rho)}{G} \sin k_i\varsigma\, d\rho\, d\varsigma}{\displaystyle\int_0^1 \int_0^\ell \rho \sin^2 k_i\varsigma\, d\rho\, d\varsigma},$$

$$a_{ij}^{(1)} = \frac{\displaystyle\int_0^1 \int_0^\ell \frac{K^*(\varsigma,\rho)}{G} \rho J_0(\lambda_j\rho) \sin k_i\varsigma\, d\rho\, d\varsigma}{\displaystyle\int_0^1 \int_0^\ell J_0^2(\lambda_j\rho)\rho \sin^2 k_i\varsigma\, d\rho\, d\varsigma}, \quad (i,j = 1,2,\ldots)$$

$$a_{ij}^{(2)} = \frac{\displaystyle\int_0^1 \int_0^\ell \frac{K_r^*(\varsigma,\rho)}{G} \rho J_1(\lambda_j\rho) \cos k_i\varsigma\, d\rho\, d\varsigma}{\displaystyle\int_0^1 \int_0^\ell J_1^2(\lambda_j\rho)\rho \cos^2 k_i\varsigma\, d\rho\, d\varsigma}, \quad (i = 0,1,2,\ldots j = 1,2,3,\ldots)$$

here $\lambda_j$ is a solution of the equation

$$J_1(\lambda_j) = 0,$$

$J_i$ - Bessel function of $i$-degree.

A particular solution with respect to these forces has the form

$$\Delta w^{(K)} = \sum_{i=1}^\infty \left\{ b_{i0}^{(1)} + \sum_{j=1}^\infty b_{ij}^{(1)} J_0(\lambda_j\rho) \right\} \sin k_i\varsigma,$$

$$\Delta u^{(K)} = \sum_{i=0}^\infty \sum_{j=1}^\infty b_{ij}^{(2)} J_1(\lambda_j\rho) \cos k_i\varsigma,$$

(4.10)

where

$$b_{ij}^{(1)} = \frac{k_i\lambda_j a_{ij}^{(2)} - \left[(1-2\nu)k_i^2 + 2(1-\nu)\lambda_j^2\right]a_{ij}^{(1)}}{2(1-\nu)(k_i^2 + \lambda_j^2)}, \quad (i = 1,2,3,\ldots;\ j = 0,1,2,\ldots),$$

$$b_{ij}^{(2)} = \frac{k_i\lambda_j a_{ij}^{(1)} - \left[(1-2\nu)\lambda_j^2 + 2(1-\nu)k_i^2\right]a_{ij}^{(2)}}{2(1-\nu)(k_i^2 + \lambda_j^2)}, \quad (i = 0,1,2,\ldots;\ j = 1,2,3,\ldots).$$

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The general solution (4.7) contains undetermined constants, which can satisfy boundary conditions on $\rho = 1$, taking into account expansion (4.6) with $k_i \neq 0$. But with $k_i = 0$ it must be to seek a part of the solution of homogeneous (4.1) in elementary functions

$$\Delta w^{(S)} = A_1 \varsigma,$$
$$\Delta u^{(S)} = A_2 \rho. \tag{4.11}$$

Hence, the general solution of the system (4.1) as follows

$$\Delta w = \Delta w^{(T)} + \Delta w^{(S)} + \Delta w^{(\Omega)} + \Delta w^{(K)},$$
$$\Delta u = \Delta u^{(T)} + \Delta u^{(S)} + \Delta u^{(\Omega)} + \Delta u^{(K)}, \tag{4.12}$$

which contains 4 arbitrary constants $C_1, C_2, A_1, A_2$. They are sufficient to satisfy exactly boundary conditions at $\rho = 1$ and integrally boundary conditions at butt-ends of the cylinder.

Numerical calculation was carried out for cylinder made of steel 15X18H12C4TIO with the following characteristics: $\dfrac{L}{R} = 4$, $\sigma_s = 800\,\mathrm{MPa}$, $\dfrac{\phi'}{E} = 0.32$, $\gamma = \dfrac{1}{2}$ (incompressible material) and under external loading

$$F_1(z,t) = 0, \quad F_2(z,t) = T_2(t)\sigma_s \sin \pi\Big(\frac{2z}{L} - 1\Big),$$
$$F_3(r,t) = F_5(r,t) = P_3(t)\sigma_s,$$
$$F_4(\sigma,t) = -F_6(r,t) = T_4(t)\sigma_s J_1(\lambda_1 r),$$

where $T_2(t)$, $P_z(t)$, $T_4(t)$ may depend arbitrary on a parameter $t$. It means that the loading process may be complicated.

Calculations in solving the problem have been fulfilled on PC with PASCAL programme [9]. Subdivide loading parameter $t$ into steps, increasing from 0 to 40 and solve the problem step by step. At each step 4 iterations were carried out.

From the results it can be seen that

a) The error between two successive approximations decreases when the number of iterations increases, i.e. the condition (2.7) is satisfied. It is shown that the modified elastic solution method can be applied to this problem and its convergence has been proved.

b) When the cylinder is subjected to the same loading process, if we subdivide the loading process into smaller steps, the error between two iterations of all quantities is also smaller, i.e. the error decreases.

c) With the same value of load, the plastic deformation region in the cylinder appears differently depending on the character of the loading process which reaches

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that value: the loading process is more complicated, so the plastic region is more enlarging. Under complex loading the body works more weakly.

d) Established calculations may give a picture of elasto-plastic states of the cylinder under axesymmetrical loads. Further we can consider elasto-plastic problems of the cylinder under non-axesymmetrical loads by the above mentioned method.

## 5. Conclusions

a) Another performance of the modified elastic solution method in theory of elastoplastic process is presented.

b) The convergence of this method is proved theoretically in the general case of a hardening body with a supplementary assumption in approximation.

c) The application of the method to the more complicated 3D problem of bodies of revolution is considered.

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## PHƯƠNG PHÁP BIẾN THỂ NGHIỆM ĐÀN HỒI GIẢI CÁC BÀI TOÁN ĐÀN HỒI DẺO CỦA CÁC KẾT CẤU CHỊU TẢI PHỨC TẠP

Phương pháp biến thể nghiệm đàn hồi trong lý thuyết quá trình đàn dẻo do tác giả đề xuất đã được ứng dụng để giải một số bài toán đàn hồi dẻo 2 chiều và 3 chiều của các kết cấu chịu tải phức tạp. Phương pháp này cho ta thuật toán giải tiến bước theo tham số tải và tại mỗi bước (giai đoạn đặt tải) thực hiện phép lặp gần đúng liên tiếp, tại mỗi gần đúng ta có bài toán đàn hồi tuyến tính thuần nhất nhưng với lực khối và lực mặt thay đổi. Đã tiến hành cách thể hiện phương pháp và khảo sát sự hội tụ của phương pháp qua các thí dụ bằng số. Trong bài báo này trình bày một cách thể hiện khác của phương pháp và chứng minh chặt chẽ sự hội tụ của phương pháp trong trường hợp vật liệu tái bền và tuân theo lý thuyết quá trình đàn dẻo. Bài toán ba chiều phức tạp hơn đối với vật thể tròn xoay chịu tải không đối xứng trục đã được đề cập đến.