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LINEARIZATION OF RANDOMLY EXCITED NONLINEAR SYSTEMS BY USING LOCAL MEAN SQUARE ERROR CRITERION

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ABSTRACT. The paper presents the analysis of three non-linear systems under random excitation by using Local Mean Square Error Criterion which is an extension of Gaussian Equivalent Linearization proposed by Caughey. The obtained results shows that the new technique allows to get much more accurate solutions than that using Caughey Criterion. The paper leads out some new conclusions which have not been found yet by the previous researches. The new conclusions more clarify the significance of this technique.

1. Introduction

One of the known approximate techniques is Gaussian equivalent linearization (GEL) which was first proposed by Caughey [1] and has been developed by many authors (see, e.g. [4, 6-9]). It has been shown that the Caughey method is presently the simplest tool widely used for analysis of non-linear stochastic problems. However, a major limitation of this method is perhaps that its accuracy decreases as the non-linearity increases, and it can lead to unacceptable errors in the second moments [3, 5, 10, 11]. Further, if one needs more accurate approximate solution, there is no way to obtain them using the conventional version of Gaussian equivalent linearization.

N. D. Anh and M. Dipaola proposed "Local Mean Square Error Criterion" (LOMSEC) which is an extension of GEL. The proposed technique is then just applied to Duffing and Van der Pol oscillators under a zero mean Gaussian white noise to show significant improvement over the accuracy of the classical GEL [8]. The publications of Anh and Dipaola [8] as well as of the previous others however have not considered other diverse systems by using the proposed technique, their comments and conclusions are therefore limited.

L. X. Hung carried out a further investigation [10, 11], which analyzes a series of non-linear systems under zero mean Gaussian random excitation by using the proposed technique. The obtained results make LOMSEC's significance more clarified, add more comprehensive and reliable remarks.

To enrich the investigation of the new technique. The paper presents the analysis of three more non-linear systems by using the proposed technique, and joins with the publications [10, 11] to lead out some new comments and conclusions

more plentiful and more improved.

2. Local Mean Square Error Criterion (LOMSEC)

Consider the generalized non-linear stochastic system:

$$e(z,\dot{z}) \equiv \ddot{z} + 2h\dot{z} + \omega_0^2 z + \varepsilon g(z,\dot{z}) - f(t) = 0. \tag{2.1}$$

The symbols in (2.1) have their customary meanings; g is a non-linear function that can be expanded into a polynomial series form, the excitation $f(t)$ is a zero mean Gaussian stationary process with the correlation function and spectral density given, respectively by:

$$R_f(\tau) = \langle f(t) f(t+\tau) \rangle,$$

where $\langle \ldots \rangle$ denotes the expectation,

$$S_f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_f(\tau) e^{i\omega\tau} d\tau. \tag{2.2}$$

We restrict ourselves to the case of stationary response of equation (2.1) if it exists. Following the linearization method, we put new linear terms in the expression of $e(z,\dot{z})$:

$$e(z,\dot{z}) = \ddot{z} + (2h+\mu)\dot{z} + (\omega_0^2 + \lambda)z + \varepsilon g(z,\dot{z}) - \mu\dot{z} - \lambda z - f(t). \tag{2.3}$$

Let $x(t)$ be a stationary solution of the linearized equation:

$$\ddot{x} + (2h+\mu)\dot{x} + (\omega_0^2 + \lambda)x - f(t) = 0. \tag{2.4}$$

Substitute (2.4) into (2.3) one gets:

$$e(x,\dot{x}) = \varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x, \tag{2.5}$$

which is an equation error of (2.1) after linearized. Using the criterion of the mean square error, the coefficients of linearization μ, λ are determined as follows:

By the method of Caughey

$$\langle e^2(x,\dot{x}) \rangle = \langle (\varepsilon g(x,\dot{x}) - \mu\dot{x} - \lambda x)^2 \rangle \longrightarrow \min_{\mu,\lambda}. \tag{2.6}$$

It follows:

$$\mu = \varepsilon \frac{\langle g\dot{x} \rangle}{\langle \dot{x}^2 \rangle}; \quad \lambda = \varepsilon \frac{\langle gx \rangle}{\langle x^2 \rangle}. \tag{2.7}$$

Since the process $x(t)$ is a solution of the linearized equation (2.4) under Gaussian process excitation, one gets that $x(t)$ and $\dot{x}(t)$ are Gaussian or normal processes. Hence, all higher moments $\langle x^{2n}\rangle$, $\langle \dot{x}^{2n}\rangle$ can be expressed in terms of the second moments $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$ by the generalized formula as follows:

$$\langle x^{2n}\rangle = 1\cdot 3\cdot 5\cdots(2n-1)\langle x^2\rangle^n. \tag{2.8}$$

To close the system (2.7), other two equations for $\langle x^2\rangle$, $\langle \dot{x}^2\rangle$ can be derived from (2.4):

$$\begin{aligned}\langle \dot{x}^2\rangle &= \int_{-\infty}^{+\infty} \frac{\omega^2 S_f(\omega)d(\omega)}{(2h+\mu)^2\omega^2 + (\omega^2-\omega_0^2-\lambda)^2};\\ \langle x^2\rangle &= \int_{-\infty}^{+\infty} \frac{S_f(\omega)d(\omega)}{(2h+\mu)^2\omega^2 + (\omega^2-\omega_0^2-\lambda)^2}.\end{aligned} \tag{2.9}$$

By the method of Local Mean Square Error Criterion (LOMSEC)

Denote $p(x,\dot{x})$ the joint probability density function of the response Gaussian processes $x(t)$ and $\dot{x}(t)$. The criterion (2.6) can be rewritten in the form:

$$\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty} e^2(x,\dot{x})p(x,\dot{x})dx\Big)d\dot{x} \longrightarrow \min_{\mu,\lambda}. \tag{2.10}$$

Since the integration is taken over all the phase space x, $\dot{x} \in (-\infty;+\infty)$, the criterion (2.6) or (2.10) is called here "Global Mean Square Error Criterion". An extension of the concept, which supposes that the criterion (2.6) or (2.10) can lead to a large error for some non-linear systems, especially as strong non-linearity. To increase the accuracy, the expected integration should be taken only in a domain where the original response processes $x(t)$, $\dot{x}(t)$ are concentrated; LOMSEC requires [8]:

$$\big[e^2(x,\dot{x})\big]_{x_0,\dot{x}_0}^{x_1,\dot{x}_1} \equiv \int_{\dot{x}_0}^{\dot{x}_1}\Big(\int_{x_0}^{x_1} e^2(x,\dot{x})p(x,\dot{x})dx\Big)d\dot{x} \longrightarrow \min_{\mu,\lambda}. \tag{2.11}$$

To distinguish the LOMSEC from the Caughey's criterion, we use the symbol $[\ \ ]$ instead of $\langle\ \ \rangle$. One gets from (2.11):

$$\mu = \varepsilon\frac{\big[g\dot{x}\big]_{\dot{x}_0}^{\dot{x}_1}}{\big[\dot{x}^2\big]_{\dot{x}_0}^{\dot{x}_1}}; \quad \lambda = \varepsilon\frac{\big[gx\big]_{x_0}^{x_1}}{\big[x^2\big]_{x_0}^{x_1}}. \tag{2.12}$$

In the cases $\langle x \rangle = 0$, all moment $[x^{2n}]$, $[\dot{x}^{2n}]$ in LOMSEC can also be expressed in terms of the second moments $\langle x^2 \rangle$; $\langle \dot{x}^2 \rangle$ by the formula (which is easily provable after changing variable $x = t\sigma_x$; where $\sigma_x^{2n} = \langle x^2 \rangle^n$) [8]:

$$[x^{2n}]_0^{r\sigma_x} = T(n,r) \cdot \langle x^2 \rangle^n \quad \text{or} \quad [x^{2n}]_{-r\sigma_x}^{+r\sigma_x} = 2T(n,r) \cdot \langle x^2 \rangle^n; \quad n = 1, 2, \ldots \tag{2.13}$$

where

$$T(n,r) = \int_0^r t^{2n} n(t) dt; \quad n(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}. \tag{2.14}$$

Input to n, r concrete values, one gets $T(n,r)$ is a positive constant.
For the case of odd order moment, one gets:

$$[x^{2n+1}] = \langle x^{2n+1} \rangle \equiv 0. \tag{2.15}$$

It is also easily provable that the character of linear combination in the Caughey's process is quite applied for LOMSEC's process, it means:

$$[\alpha g(x) + \beta f(x)]_a^b = \int_a^b (\alpha g + \beta f) p(x) dx = \alpha [g(x)]_a^b + \beta [f(x)]_a^b. \tag{2.16}$$

The formulas (2.12) show that it is possible to get a lot of various approximate solutions depended on the chosen values of finite integration domain. The key question is to choose a possible integration domain for LOMSEC in order to get a better approximate solution.

3. Application of LOMSEC to some non-linear stochastic systems

We proceed analysis of three nonlinear systems under Gaussian white noise random excitation. These systems have exact solutions found by solving the Fokker-Planck equation [2], or known thanks to the method of equivalent non-linearization [6]. To find a possible integration domain, we solve the inverse problem using the formula of LOMSEC's solution, but $\langle x^2 \rangle_{LG}$ replaced by the exact solution $\langle x^2 \rangle_e$, then one gets the integration interval r_e corresponding with the exact solution $\langle x^2 \rangle_e$. The value of r_e varies following nonlinearity, therefore a mean value $\bar{r}$ for calculation of LOMSEC solution should be chosen from the series of values r_e. Next step, put $\bar{r}$ in the formula of LOMSEC's solution one gets LOMSEC solution $\langle x^2 \rangle_{LG}$.

3.1. The oscillator with non-linear damping

Consider the system:

$$\ddot{x} + 2\varepsilon\dot{x} + 2\varepsilon\gamma\dot{x}^3 + \omega_0^2 x = \sigma w(t). \tag{3.1}$$

The analytic solution of this equation has been not found yet. The solution found by the method of equivalent non-linearization $\langle x \rangle_{ENLE}$ is regarded as the exact solution [6]. Roberts and Spanos solved the problem corresponding with the case: $\sigma = \sqrt{4\varepsilon}$; $\omega_0^2 = 1$; $\varepsilon = 0.05$; and γ varies.

The equivalent linearized equations of (3.1) with $2\varepsilon\dot{x}^3 = \mu\dot{x}$:

$$\ddot{x} + (2\varepsilon + \mu)\dot{x} + \omega_0^2 x = \sigma w(t). \tag{3.2}$$

The solution of the linearized equation (3.2):

$$\langle x^2 \rangle = \frac{\sigma^2}{2(2\varepsilon + \mu)\omega_0^2}. \tag{3.3}$$

The coefficient of linearization μ is determined by the criterion of the mean square error (by Caughey or by LOMSEC).

By the criterion of Caughey:

$$\langle e^2(\dot{x}) \rangle = \langle (2\varepsilon\gamma\dot{x}^3 - \mu\dot{x})^2 \rangle \to \min_{\mu} \Rightarrow \left\langle \frac{\partial}{\partial\mu}(2\varepsilon\gamma\dot{x}^3 - \mu\dot{x})^2 \right\rangle = 0. \tag{3.4}$$

Using (2.8) and (3.4) yields

$$\mu = 2\varepsilon\gamma\frac{\langle \dot{x}^4 \rangle}{\langle \dot{x}^2 \rangle} = 6\varepsilon\gamma\langle \dot{x}^2 \rangle. \tag{3.5}$$

Put (3.5) in (3.3) and denote $\langle x \rangle_G$ as the solution obtained by Caughey criterion, after some calculation steps with an attention of that $\langle \dot{x}^2 \rangle = \langle x^2 \rangle \omega_0^2$ one gets the Caughey solution:

$$\langle x^2 \rangle_G = \frac{\sigma^2}{4\varepsilon(1 + 3\gamma\langle x^2 \rangle\omega_0^2)\omega_0^2}. \tag{3.6}$$

By LOMSEC

$$\left[e^2(\dot{x})\right]_{\dot{x}_0}^{\dot{x}_1} = \left[(2\varepsilon\gamma\dot{x}^3 - \mu\dot{x})^2\right]_{\dot{x}_0}^{\dot{x}_1} \to \min_{\mu} \Rightarrow \left[\frac{\partial}{\partial\mu}(2\varepsilon\gamma\dot{x}^3 - \mu\dot{x})^2\right]_{\dot{x}_0}^{\dot{x}_1} = 0. \tag{3.7}$$

Using (2.13), (2.14) and (3.7) yields

$$\mu = 2\varepsilon\gamma\frac{\left[\dot{x}^4\right]_{\dot{x}_0}^{\dot{x}_1}}{\left[\dot{x}^2\right]_{\dot{x}_0}^{\dot{x}_1}} = 2K_r\varepsilon\gamma\langle \dot{x}^2 \rangle, \tag{3.8}$$

where

$$K_r = \frac{\int_0^r t^4 n(t)dt}{\int_0^r t^2 n(t)dt}; \quad n(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}. \tag{3.9}$$

Put (3.8) in (3.3), after some calculation steps one gets the formula of the LOMSEC solution:

$$\langle x^2 \rangle_{LG} = \frac{\sigma^2}{4\varepsilon(1 + \gamma\langle \dot{x}^2 \rangle K_r)\omega_0^2}. \tag{3.10}$$

The choice of the possible integration interval $\bar{r}$ to find the LOMSEC solution: Firstly use the formula (3.10) but impute to $\langle x^2 \rangle_{LG}$ the value of the found exact solution $\langle x^2 \rangle_{ENLE}$, after the calculation one gets:

$$K_r = \frac{1}{\gamma \langle x^2 \rangle_{ENLE}^2 \omega_0^2} \left(\frac{\sigma^2}{4\varepsilon\omega_0^2} - \langle x^2 \rangle_{ENLE} \right). \tag{3.11}$$

Combine (3.11) with (3.9) to find the integration interval r_e corresponding with the value of $\langle x^2 \rangle_{ENLE}$ (this means the solving of the contrary problem). Then a mean value $\bar{r}$ chosen is as the middle value of the series of value r_e (it means $\bar{r} = \frac{r_{e_1} + r_{e_n}}{2}$). Use (3.9) and (3.10) with the integration interval $\bar{r}$ one gets LOMSEC solution.

Consider the case: $\sigma = \sqrt{4\varepsilon}$; $\omega_0^2 = 1$; $\varepsilon = 0.05$; mean while γ varies. The calculation result is given in the Table 1. The Fig. 1, 2 show clearly the variation of r_e depending on γ, and the error level of the two methods, where *Dg*, *Dlg* denote the error of $\langle x^2 \rangle_G$ and $\langle x^2 \rangle_{LG}$ versus $\langle x^2 \rangle_{ENLE}$.

Table 1. The calculation result of the case
$\sigma = \sqrt{4\varepsilon}$; $\omega_0^2 = 1$; $\varepsilon = 0.05$; γ varies

No	γ	$\langle x^2 \rangle_{ENLE}$	$\langle x^2 \rangle_G$	Dg(%)	K_r	r	$\langle x^2 \rangle_{LG}$	Dlg(%)
1	1	0.4603	0.4342	-5.7	2.5472	2.65	0.4633	+0.65
2	2	0.3584	0.3333	-7.0	2.4974	2.59	0.3582	-0.05
3	3	0.3058	0.2824	-7.6	2.4745	2.59	0.3045	-0.42
4	4	0.2720	0.2500	-8.1	2.4600	2.58	0.2701	-0.70
5	5	0.2476	0.2270	-8.3	2.4546	2.58	0.2456	-0.81
6	6	0.2294	0.2095	-8.7	2.4406	2.57	0.2270	-1.00
7	7	0.2147	0.1957	-8.8	2.4337	2.57	0.2122	-1.16
8	8	0.2025	0.1843	-9.0	2.4310	2.56	0.2000	-1.23
9	9	0.1923	0.1748	-9.1	2.4629	2.56	0.1898	-1.30
10	10	0.1835	0.1667	-9.1	2.4248	2.55	0.1810	-1.36

The possible integration interval $\bar{r} = 2.60$; The probability $P(\bar{r}) = 0.9907$

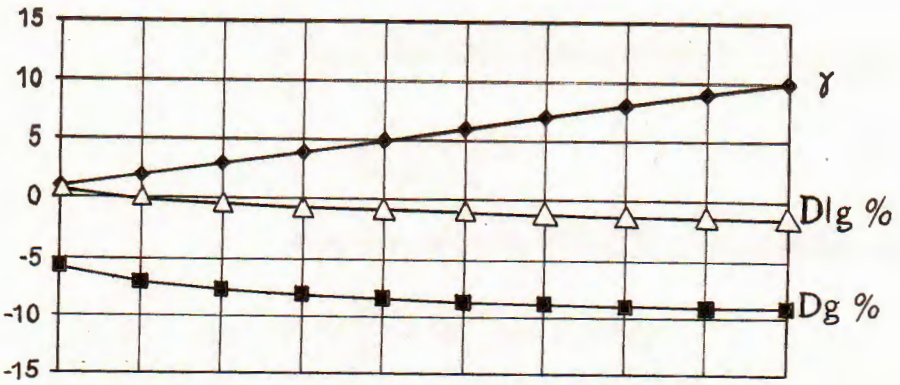

Fig. 1. Error of Caughey and LOMSEC solutions versus the nonlinearity γ

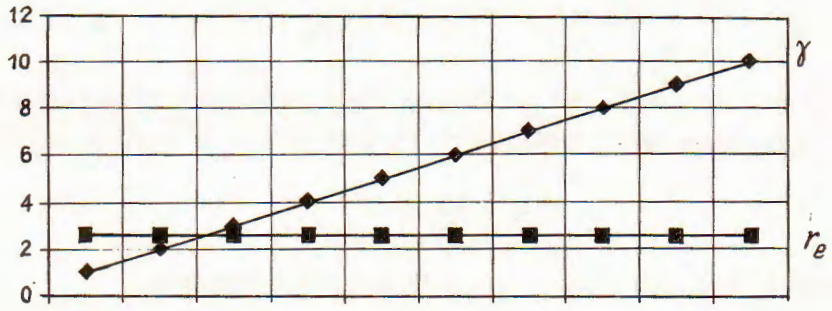

Fig. 2. The variation of r_e versus the nonlinearity γ

3.2. The oscillator with nonlinear damping following x and $\dot{x}$

Consider the oscillator:

$$\ddot{x} + 4h\Big(\frac{\dot{x}^2}{2} + \omega_0^2\frac{x^2}{2}\Big)\dot{x} + \omega_0^2 x = \sigma w(t). \tag{3.12}$$

By solving the Fokker-Planck equation, one finds the probability density function $p(x;\dot{x})$, then gets the exact solution as follows:

$$\langle x^2\rangle_e = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 p(x;\dot{x})dxd\dot{x} \Rightarrow$$

$$\langle x^2\rangle_e = \frac{\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 \exp\Big\{-\frac{4h}{\sigma^2}\Big(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2\Big)^2\Big\}dxd\dot{x}}{\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\Big\{-\frac{4h}{\sigma^2}\Big(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2\Big)^2\Big\}dxd\dot{x}}. \tag{3.13}$$

The equivalent linearized equation of (3.12) with $4h\Big(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2\Big)\dot{x} = \mu\dot{x}$ is

$$\ddot{x} + \mu\dot{x} + \omega_0^2 x = \sigma w(t). \tag{3.14}$$

The solution of the linearized equation (3.14) is

$$\langle x^2 \rangle = \frac{\sigma^2}{2\mu\omega_0^2}. \tag{3.15}$$

The equation error of (3.12) after linearized is

$$e(x;\dot{x}) = 4h\Big(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2\Big)\dot{x} - \mu\dot{x}. \tag{3.16}$$

By the criterion of Caughey:

$$\langle e^2(x;\dot{x})\rangle \to \min_{\mu} \Rightarrow \Big\langle e\frac{\partial e}{\partial \mu}\Big\rangle = 0. \tag{3.17}$$

Solve the system (3.17) to determine the coefficients of linearization; after some calculation steps with an attention of that $\langle \dot{x}^2 \rangle = \langle x^2 \rangle \omega_0^2$, it follows:

$$\mu = 8h\omega_0^2\langle x^2 \rangle. \tag{3.18}$$

Substitute (3.18) into (3.15), one gets Caughey solution:

$$\langle x^2 \rangle_G = \frac{\sigma}{4\sqrt{h}\omega_0^2}. \tag{3.19}$$

By LOMSEC

$$\big[e^2(x;\dot{x})\big]_{x_0,\dot{x}_0}^{x_1,\dot{x}_1} \Rightarrow \Big[e\frac{\partial e}{\partial \mu}\Big]_{x_0,\dot{x}_0}^{x_1,\dot{x}_1} = 0. \tag{3.20}$$

By the calculation steps which are similar to the above, one gets

$$\mu = 2h\big(K_r + H_r\big)\omega_0^2\langle x^2 \rangle, \tag{3.21}$$

where

$$K_r = \frac{\int_0^r t^4 n(t)dt}{\int_0^r t^2 n(t)dt}, \quad H_r = \frac{\int_0^r t^2 n(t)dt}{\int_0^r n(t)dt}, \quad n(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}. \tag{3.22}$$

Substituting (3.21) into (3.15) we have:

$$\langle x^2 \rangle_{LG} = \frac{\sigma}{2\omega_0^2\sqrt{h(K_r + H_r)}}. \tag{3.23}$$

The method of choosing the possible integration interval $\bar{r}$ to find the LOMSEC solution is similar to the above. Use the equations (3.22) and (3.23) with the integration interval $\bar{r}$ one gets LOMSEC solution $\langle x\rangle_{LG}$.

Table 2. The calculation result of the case $\omega_0^2 = 1$; $h = 1$; σ varies

No	σ	$\langle x^2\rangle_e$	$\langle x^2\rangle_G$	Dg(%)	$K_r + H_r$	r_e	$\langle x^2\rangle_{LG}$	Dlg(%)
1	0.5	0.14227	0.12500	-12.140	3.08774	2.23611	0.13672	-3.905
2	1.0	0.28388	0.25000	-11.934	3.10222	2.27500	0.27343	-3.680
3	2.0	0.56677	0.50000	-11.780	3.11310	2.27500	0.54687	-3.511
4	5.0	1.41382	1.25000	-11.587	3.12675	2.27500	1.36717	-3.299
5	10.0	2.59896	2.50000	-3.808	3.70117	2.81944	2.73434	5.209

The possible integration interval $\bar{r} = 2.52778$. The probability $P(\bar{r}) = 0.98515$

3.3. The non-linear dynamic systems in the form of $\dot{x} + g(x) = \sigma w(t)$; $g(x)$ is a polynomial

Consider the system:

$$\dot{x} + \alpha x + \varepsilon |x| x = \sigma w(t). \tag{3.24}$$

By solving the Fokker-Planck equation, one finds the probability density function $p(x)$, then gets the exact solution as follows:

$$\langle x^2\rangle_e = \int_{-\infty}^{+\infty} x^2 p(x)dx \Rightarrow \langle x^2\rangle_e = \frac{\displaystyle\int_{-\infty}^{+\infty} x^2 \exp\left\{-\frac{2}{\sigma^2}\left(\frac{\alpha}{2}x^2 + \frac{\varepsilon}{3}x^3\right)\right\}dx}{\displaystyle\int_{-\infty}^{+\infty} \exp\left\{-\frac{2}{\sigma^2}\left(\frac{\alpha}{2}x^2 + \frac{\varepsilon}{3}x^3\right)\right\}dx}. \tag{3.25}$$

The equivalent linearized equation of (3.24) with $\varepsilon|x|x = \lambda x$ is

$$\dot{x} + (\alpha + \lambda)x = \sigma w(t). \tag{3.26}$$

The solution of the linearized equation (3.26) is

$$\langle x^2\rangle = \frac{\sigma^2}{2(\alpha + \lambda)}. \tag{3.27}$$

By the criterion of Caughey:

$$\langle e^2(x)\rangle = \langle(\varepsilon|x|x - \lambda x)^2\rangle \to \min_{\lambda} \Rightarrow \left\langle \frac{\partial}{\partial\lambda}(\varepsilon|x|x - \lambda x)^2 \right\rangle = 0. \tag{3.28}$$

Solve the equation (3.28) to determine the coefficients of linearization:

$$\lambda = \varepsilon \frac{\langle |x| x^2 \rangle}{\langle x^2 \rangle}. \tag{3.29}$$

Since $|x|x^2$ is an even function, one gets:

$$\langle |x| x^2 \rangle = 2 \int_0^\infty |x| x^2 P(x) dx = 2 \int_0^\infty x^3 P(x) dx, \tag{3.30}$$

where $P(x)$ is the normal distribution:

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{ - \frac{x^2}{2\sigma_x^2} \right\}. \tag{3.31}$$

Substituting (3.31) into (3.30) with taking $\sigma_x^2 = \langle x^2 \rangle$ in (3.31) into account, it follows:

$$\langle |x| x^2 \rangle = \frac{4 \langle x^2 \rangle \sqrt{\langle x^2 \rangle}}{\sqrt{2\pi}}. \tag{3.32}$$

From (3.32) and (3.29), one gets:

$$\lambda = \frac{4\varepsilon \sqrt{\langle x^2 \rangle}}{\sqrt{2\pi}}. \tag{3.33}$$

Substituting (3.33) into (3.27), one gets the equation to find Caughey solution:

$$\dot{\alpha} \langle x^2 \rangle_G + \frac{4\varepsilon}{\sqrt{2\pi}} \langle x^2 \rangle_G \sqrt{\langle x^2 \rangle_G} - \frac{\sigma^2}{2} = 0. \tag{3.34}$$

By LOMSEC

$$\left[e^2(x) \right]_{x_0}^{x_1} = \left[(\varepsilon |x| x - \lambda x)^2 \right]_{x_0}^{x_1} \to \min_{\lambda} \Rightarrow \left[\frac{\partial}{\partial \lambda} (\varepsilon |x| x - \lambda x)^2 \right]_{x_0}^{x_1} = 0. \tag{3.35}$$

Solve the equation (3.35) to determine the coefficients of linearization:

$$\lambda = \varepsilon \frac{\left[|x| x^2 \right]_{x_0}^{x_1}}{\left[x^2 \right]_{x_0}^{x_1}}. \tag{3.36}$$

Let $x_0 = -r\sigma_x$, $x_1 = r\sigma_x$, $x = t\sigma_x$ one gets:

$$\left[|x| x^2 \right]_{-r\sigma_x}^{+r\sigma_x} = 2 \int_0^{r\sigma_x} x^3 P(x) dx = 2 \langle x^2 \rangle \sqrt{\langle x^2 \rangle} K_{1r} \tag{3.37}$$

$$\text{and} \quad \left[x^2 \right]_{-r\sigma_x}^{+r\sigma_x} = 2 \int_0^{r\sigma_x} x^2 P(x) dx = 2 \langle x^2 \rangle K_{2r} \tag{3.38}$$

where

$$K_{1r} = \int_0^r t^3 n(t)dt, \quad K_{2r} = \int_0^r t^2 n(t)dt, \quad n(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Substituting (3.37) and (3.38) into (3.36), one gets the coefficient of linearization:

$$\lambda = \varepsilon\sqrt{\langle x^2\rangle}\frac{K_{1r}}{K_{2r}} = \varepsilon\sqrt{\langle x^2\rangle}K_r, \tag{3.39}$$

where

$$K_r = \frac{K_{1r}}{K_{2r}} = \frac{\int_0^r t^3 n(t)dt}{\int_0^r t^2 n(t)dt}. \tag{3.40}$$

Substituting (3.39) into (3.27), one gets the equation to find LOMSEC solution:

$$\alpha\langle x^2\rangle_{LG} + \varepsilon K_r \langle x^2\rangle_{LG}\sqrt{\langle x^2\rangle_{LG}} - \frac{\sigma^2}{2} = 0. \tag{3.41}$$

Choice of the possible integration interval $\bar{r}$ to find the LOMSEC solution: use the formula (3.41) but impute to $\langle x^2\rangle_{LG}$ the value of the found exact solution $\langle x^2\rangle_e$, after the calculation one gets:

$$K_r = \frac{1}{\varepsilon\langle x^2\rangle_e^{3/2}}\left(\frac{\sigma^2}{2} - \alpha\langle x^2\rangle_e\right). \tag{3.42}$$

Combine (3.42) with (3.40) to find the integration interval r_e corresponding with the value of $\langle x^2\rangle_e$. Then a mean value $\bar{r}$ chosen is as the middle value of the series of value r_e. Use (3.40) and (3.41) with the integration interval $\bar{r}$ one gets LOMSEC solution $\langle x\rangle_{LG}$.

Table 3. The calculation result of the case $\alpha = 1$, $\sigma = 1.4$, ε varies

No	ε	$\langle x^2\rangle_e$	$\langle x^2\rangle_G$	Dg(%)	K_r	r_e	$\langle x^2\rangle_{LG}$	Dlg(%)
1	0.5	0.61257	0.60475	-1.277	1.53274	2.91594	0.61632	0.612
2	1	0.47861	0.46840	-2.132	1.51428	2.79575	0.48003	0.297
3	2	0.35253	0.34190	-3.015	1.49886	2.71028	0.35203	-0.143
4	3	0.28811	0.27805	-3.491	1.49134	2.67233	0.28691	-0.416
5	5	0.21911	0.21033	-4.008	1.48378	2.63594	0.21751	-0.729
6	10	0.14711	0.14040	-4.555	1.47621	2.60132	0.14552	-1.079
7	20	0.09661	0.09183	-4.944	1.47104	2.57861	0.09531	-1.339
8	50	0.05421	0.05136	-5.272	1.46682	2.56055	0.05337	-1.564
9	80	0.04006	0.03790	-5.382	1.46542	2.55463	0.03940	-1.641
10	100	0.03446	0.03278	-5.422	1.46493	2.55260	0.03408	-1.668

The possible integration interval $\bar{r} = 2.73427$; The probability $P(\bar{r}) = 0.99380$

4. Conclusions

Through the non-linear systems considered, the LOMSEC technique allows to get better solutions than that using Caughey Criterion (in many cases the Caughey's error unacceptable as showed in [11]). That is a significant advantage of LOMSEC.

The obtained results show that for the considered systems (including [10, 11]) there exist values of integration interval r_e allowing to get the exact solution when using LOMSEC technique. It means that in principle, it is possible for LOMSEC method to find exact solution, meanwhile this is impossible for the Caughey technique.

By the way of changing the limitation of integration domain, the LOMSEC provides with a lot of different approximate solutions (the case of $r = \infty$ LOMSEC gives Caughey solution). This is an important base to create the method of continuous approximation for finding a better solution than the previous one.

The investigation result leads out a new suggestion for choosing the integration interval $\bar{r}$ (for example it is quite possible to choose $\bar{r} = 2.5$) for general application to the similar non-linear stochastic systems. This makes the application more convenient to solve the practical technical problems.

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TUYẾN TÍNH HÓA CÁC HỆ PHI TUYẾN CHỊU KÍCH ĐỘNG NGẪU NHIÊN BẰNG TIÊU CHUẨN SAI SỐ BÌNH PHƯƠNG TRUNG BÌNH KHU VỰC

Bài báo trình bày phân tích ba hệ phi tuyến chịu kích động ngẫu nhiên bằng tiêu chuẩn sai số bình phương trung bình khu vực - một phát triển của phương pháp tuyến tính hóa tương đương Gauss do Caughey đề xuất. Các kết quả nhận được chỉ ra rằng kỹ thuật mới cho phép nhận được lời giải chính xác hơn so với dùng tiêu chuẩn Caughey. Bài báo đưa ra một số kết luận mới mà chưa được phát hiện bởi các tác giả nghiên cứu trước đây.