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THE INTERACTION BETWEEN TWO PARAMETRIC EXCITATIONS OF THE SECOND AND THIRD DEGREES

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ABSTRACT. The interaction between nonlinear oscillations is an interesting problem which has attracted many researches. The interaction between forced and parametric excitations and between two parametric excitations of first and second degrees and first and third degrees, is informed in [1]. In the present paper, we study interaction between two parametric excitations of the second and third degrees. The asymptotic method of nonlinear mechanics in combination with a computer is used.

1. Stationary Oscillations

Let us consider a dynamic system governed by the differential equation

$$\ddot{x} + \omega^2 x = \varepsilon\{\Delta x - h\dot{x} - \gamma x^3 + 2px^2 \cos\omega t + 2qx^3 \cos(2\omega t + 2\sigma)\}, \tag{1.1}$$

where $2p > 0$, $2q > 0$ are intensities of parametric excitations of second and third degrees, respectively, and 2σ $(0 \leq 2\sigma < 2\pi)$ is the phase shift between two excitations, $h \geq 0$ is the friction coefficient, γ is the coefficient of restoration of the third degree, $\varepsilon > 0$ is a small parameter, $\varepsilon\Delta = \omega^2 - 1$ is the detuning parameter and 1 is the own frequency of the system under consideration; overdot denotes the derivative relative to time.

The solution of equation (1.1) will be found in the form

$$x = a\cos\psi, \quad \dot{x} = -a\omega\sin\psi, \quad \psi = \omega t + \theta, \tag{1.2}$$

where a and θ are unknown functions of time, which satisfy the relationship

$$\dot{a}\cos\psi - a\dot{\theta}\sin\psi = 0. \tag{1.3}$$

By substituting (1.2) into (1.1) and combining it with (1.3) we obtain the following equations for new variables a and θ:

$$\begin{cases} \dot{a} = -\dfrac{\varepsilon}{\omega} F \sin\psi, \\ a\dot{\theta} = -\dfrac{\varepsilon}{\omega} F \cos\psi, \end{cases} \tag{1.4}$$

where εF denotes the right hand side of equation (1.1). In the first approximation we can replace (1.4) by averaged equations

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{\omega}\langle F\sin\psi\rangle, \\ a\dot{\theta} &= -\frac{\varepsilon}{\omega}\langle F\cos\psi\rangle, \end{aligned} \tag{1.5}$$

where $\langle \mathcal{F}\rangle$ is an averaged value in time of the function $\mathcal{F}$. It is easy to verify the following form of equations (1.5):

$$\begin{cases} \dot{a} = -\dfrac{\varepsilon}{4\omega}\big[2h\omega a + pa^2\sin\theta + qa^3\sin\theta + qa^3\sin(2\theta-2\sigma)\big], \\ a\dot{\theta} = -\dfrac{\varepsilon}{2\omega}\Big[\Big(\Delta - \dfrac{3}{4}\gamma a^2\Big)a + \dfrac{3}{2}pa^2\cos\theta + qa^3\cos(2\theta-2\sigma)\Big]. \end{cases} \tag{1.6}$$

The stationary amplitude $a = a_0 = \text{const} \neq 0$ and phase $\theta = \theta_0 = \text{const}$ satisfy the equations

$$\begin{cases} f_0 = 2h\omega + (pa_0\cos\sigma)\sin\xi + (pa_0\sin\sigma)\cos\xi + qa_0^2\sin 2\xi = 0, \\ g_0 = \Delta - \dfrac{3}{4}\gamma a_0^2 - \dfrac{3}{2}pa_0\sin\sigma\sin\xi + \dfrac{3}{2}pa_0\cos\sigma\cos\xi + qa_0^2\cos 2\xi = 0, \\ \xi = \theta_0 - \sigma. \end{cases} \tag{1.7}$$

The elimination of the phase θ_0 between two equations (1.7) can be realized through two steps [1]. The first step is to transform (1.7) into a system of two equations with trigonometric functions of only one argument ξ by means of the transformation

$$\begin{aligned} f &= -\Big(\frac{1}{4}p\cos\sigma + qa_0\cos\xi\Big)a_0 f_0 + \Big(-\frac{1}{4}p\sin\sigma + qa_0\sin\xi\Big)a_0 g_0 \\ &= A\sin\xi + B\cos\xi - E = 0, \\ g &= -\Big(\frac{1}{4}p\sin\sigma + qa_0\sin\xi\Big)a_0 f_0 + \Big(\frac{1}{4}p\cos\sigma - qa_0\cos\xi\Big)a_0 g_0 \\ &= G\sin\xi + H\cos\xi - K = 0, \end{aligned} \tag{1.8}$$

with determinant

$$T = a_0^2\Big(q^2a_0^2 - \frac{1}{16}p^2\Big), \tag{1.9}$$

where

$$
\begin{aligned}
A &= \frac{3}{8}p^2a_0^2 - \frac{5}{8}p^2a_0^2\cos^2\sigma + qa_0^2\Big[\Delta - \Big(q + \frac{3}{4}\gamma\Big)a_0^2\Big] = X - T,\\
X &= \frac{5}{16}p^2a_0^2 - \frac{5}{8}p^2a_0^2\cos^2\sigma + qa_0^2\Big(\Delta - \frac{3}{4}\gamma a_0^2\Big),\\
B &= G = -\frac{5}{16}p^2a_0^2\sin 2\sigma - 2qh\omega a_0^2, \qquad (1.10)\\
H &= -\frac{1}{4}p^2a_0^2 + \frac{5}{8}p^2a_0^2\cos^2\sigma - qa_0^2\Big[\Delta + \Big(q + \frac{3}{4}\gamma\Big)a_0^2\Big] = -(X + T),\\
E &= \frac{1}{2}ph\omega a_0\cos\sigma + \frac{p}{4}a_0\Big[\Big(5q - \frac{3}{4}\gamma\Big)a_0^2 + \Delta\Big]\sin\sigma,\\
K &= \frac{1}{2}ph\omega a_0\sin\sigma + \frac{pa_0}{4}\Big[\Big(5q + \frac{3}{4}\gamma\Big)a_0^2 - \Delta\Big]\cos\sigma.
\end{aligned}
$$

Solving equations (1.8) relative to $\sin\xi$, $\cos\xi$ we obtain

$$D\sin\xi = D_1, \quad D\cos\xi = D_2,$$

here

$$
\begin{aligned}
D &= \begin{vmatrix} A & B \\ G & H \end{vmatrix} = T^2 - (X^2 + B^2),\\
D_1 &= \begin{vmatrix} E & B \\ K & H \end{vmatrix} = -ET - (EX + BK),\\
D_2 &= \begin{vmatrix} A & E \\ G & K \end{vmatrix} = -KT + KX - BE.
\end{aligned}
$$

The amplitude - frequency relationship will be

$$W(\Delta, a_0^2) = D_1^2 + D_2^2 - D^2 = 0. \qquad (1.11)$$

As shown in [1] one can present (1.11) in the form

$$W(\Delta, a_0^2) = TW_0(\Delta, a_0^2), \qquad (1.12)$$

where

$$
\begin{aligned}
W_0(\Delta, a_0^2) = &-T^3 + [2(X^2 + B^2) + E^2 + K^2]T \qquad (1.13)\\
&+ (X^2 + B^2)(P_1^2 + Q_1^2 - P_0^2 - Q_0^2) + 2X(E^2 - K^2) + 4EBK,
\end{aligned}
$$

$$
\begin{aligned}
P_0 &= 2h\omega, & Q_0 &= \Delta - \frac{3}{4}\gamma a_0^2,\\
P_1 &= \frac{5}{4}pa_0\sin\sigma, & Q_1 &= \frac{5}{4}pa_0\cos\sigma.
\end{aligned}
$$

and the expressions T, X, B, E, K are given by formulae (1.9) and (1.10). Using the equation

$$W_0(\Delta, a_0^2) = 0, \tag{1.14}$$

we obtain the resonance curves depending on the parameters of the system under consideration. Fig. 1 shows the resonance curves for the parameters $q = 0.08$, $p = 0.04$, $\gamma = 0.2$, $h = 0.01$. The resonance curves consist of two quasi-parallel "parabola" branches, 1 and 2. The function $W_0(\Delta, a_0^2)$ vanishes on these branches, it is positive between these branches and is negative in the other parts of the plane (a_0^2, Δ). In figure 1, curve 3 represents the equation $D = 0$, which divides the plane (a_0^2, Δ) into the regions of $D > 0$ and $D < 0$.

Increasing the friction coefficient h (see Fig. 2 for $q = 0.08$, $p = 0.04$, $\gamma = 0.2$ and $h = 0.02$) the resonance curves move up.

Increasing the intensity of the parametric excitation of the third degree (parameter q), the resonance curves spread wide (see Fig. 3 for $q = 0.11$, $p = 0.04$, $\gamma = 0.2$, $h = 0.01$ and Fig. 4 for $q = 0.2$, $p = 0.04$, $\gamma = 0.2$, $h = 0.01$). The stable branches of the resonance curves disappear when q is large enough (see Fig. 4).

Increasing the intensity of the parametric excitation of the second degree (parameter p), the upper branch of the resonance curve moves up and the lower branch moves down. (see Fig. 5 for $q = 0.08$, $p = 0.05$, $\gamma = 0.2$, $h = 0.01$ and Fig. 6 for $q = 0.08$, $p = 0.07$, $\gamma = 0.2$, $h = 0.01$).

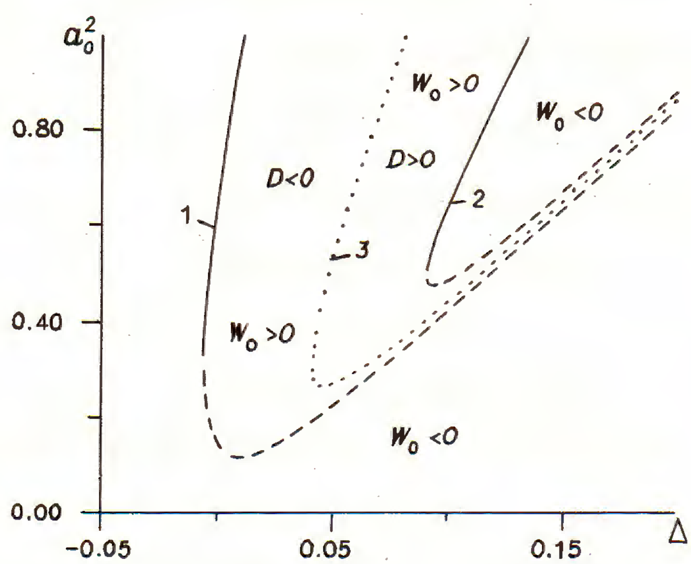

Fig. 1. Resonance curves for $q = 0.08$, $p = 0.04$, $\gamma = 0.2$, $h = 0.01$

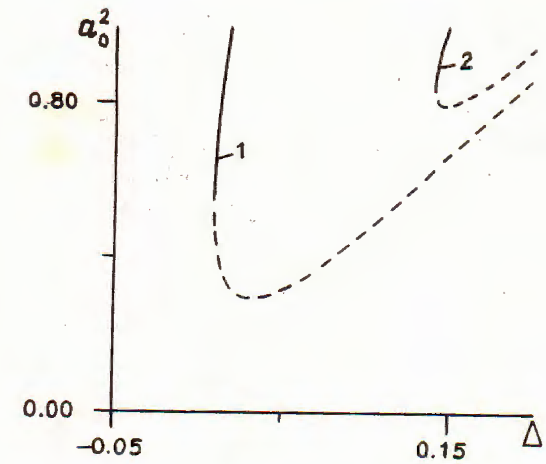

Fig. 2. Resonance curves for $q = 0.08$, $p = 0.04$, $\gamma = 0.2$, $h = 0.02$

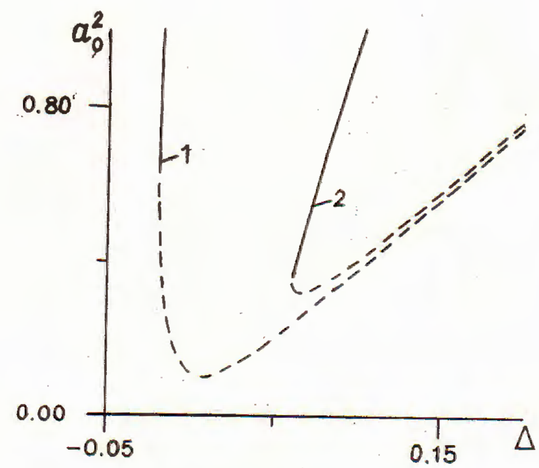

Fig. 3. Resonance curves for $q = 0.11$, $p = 0.04$, $\gamma = 0.2$, $h = 0.01$

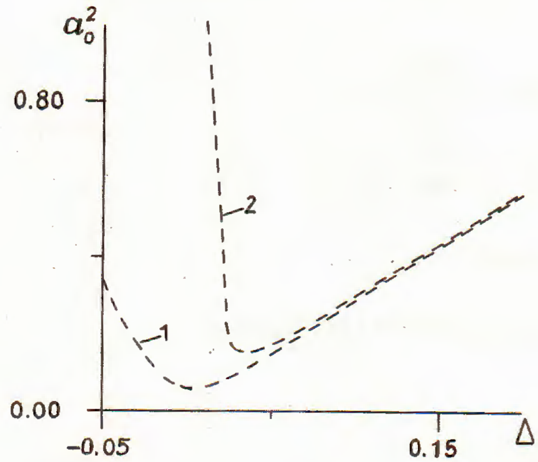

Fig. 4. Resonance curves for $q = 0.2$, $p = 0.04$, $\gamma = 0.2$, $h = 0.01$

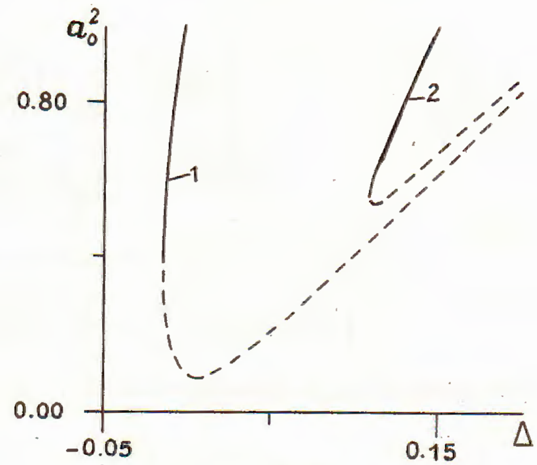

Fig. 5. Resonance curves for $q = 0.08$, $p = 0.05$, $\gamma = 0.2$, $h = 0.01$

Strongly increasing both intensities p and q, we have the resonance curves shown in Fig. 7 (for $q = 0.7$, $p = 0.4$, $\gamma = 0.2$, $h = 0.1$).

2. Stability of stationary Oscillations

The variational equations for the stationary solution of equations (1.6) are of the form

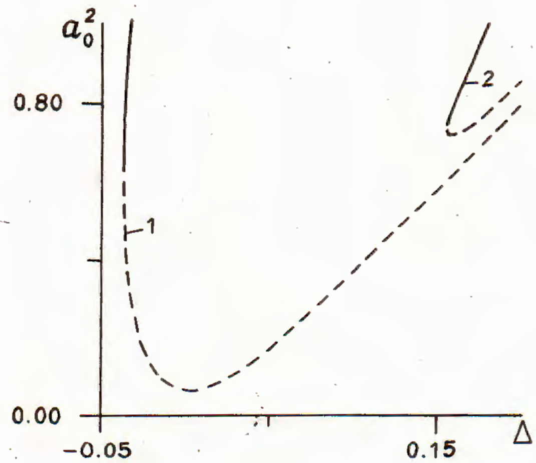

Fig. 6. Resonance curves for
$q = 0.08$, $p = 0.07$, $\gamma = 0.2$, $h = 0.01$

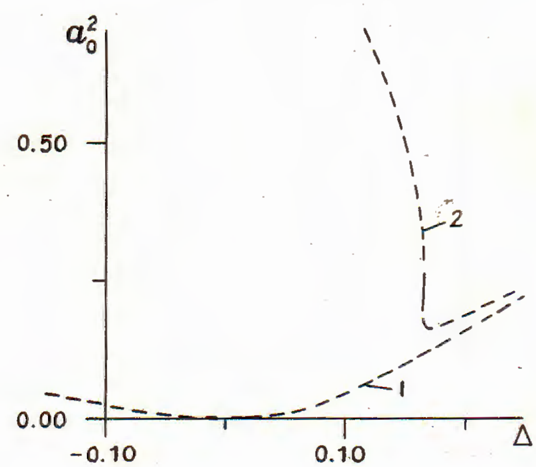

Fig. 7. Resonance curves for large intensities of parametric excitations
$q = 0.7$, $p = 0.4$, $\gamma = 0.2$, $h = 0.1$

$$\begin{aligned} \frac{d}{dt}\delta a &= -\frac{\varepsilon}{4\omega}\Big\{\Big(\frac{\partial f^*}{\partial a}\Big)_0\delta a + \Big(\frac{\partial f^*}{\partial \theta}\Big)_0\delta\theta\Big\}, \\ a_0\frac{d}{dt}\delta\theta &= -\frac{\varepsilon}{2\omega}\Big\{\Big(\frac{\partial g^*}{\partial a}\Big)_0\delta a + \Big(\frac{\partial g^*}{\partial \theta}\Big)_0\delta\theta\Big\}, \end{aligned} \tag{2.1}$$

where

$$\begin{cases} f^* = 2h\omega a + pa^2\sin\theta + qa^3\sin(2\theta - 2\sigma), \\ g^* = \Big(\Delta - \frac{3}{4}\gamma a^2\Big)a + \frac{3}{2}pa^2\cos\theta + qa^3\cos(2\theta - 2\sigma). \end{cases} \tag{2.2}$$

The characteristic equation for (2.1) is

$$a_0\lambda^2 + \frac{\varepsilon}{2\omega}\Big[\Big(\frac{\partial g^*}{\partial \theta}\Big)_0 + \frac{a_0}{2}\Big(\frac{\partial f^*}{\partial a}\Big)_0\Big]\lambda + \frac{\varepsilon^2}{8\omega^2}L = 0, \tag{2.3}$$

$$L = \Big(\frac{\partial f^*}{\partial a}\Big)_0\Big(\frac{\partial g^*}{\partial \theta}\Big)_0 - \Big(\frac{\partial f^*}{\partial \theta}\Big)_0\Big(\frac{\partial g^*}{\partial a}\Big)_0. \tag{2.4}$$

Since $N = \Big(\frac{\partial g^*}{\partial \theta}\Big)_0 + \frac{a_0}{2}\Big(\frac{\partial f^*}{\partial a}\Big)_0 = 2h\omega a_0$, then the first stability condition $N > 0$ is fulfilled for $h > 0$. The second stability condition will be

$$L > 0. \tag{2.5}$$

Since $(f^*)_0 = a_0 f_0 = 0$, $(g^*)_0 = a_0 g_0 = 0$ on the resonance curve (see (1.7) and (2.2)), then from equations (1.8), on this curve we have

$$L_2 = \Big(\frac{\partial f}{\partial a}\Big)_0\Big(\frac{\partial g}{\partial \theta}\Big)_0 - \Big(\frac{\partial f}{\partial \theta}\Big)_0\Big(\frac{\partial g}{\partial a}\Big)_0 = \frac{1}{a_0^2}TL. \tag{2.6}$$

In [1], it was proved that

$$L_2 = \frac{1}{2D}\frac{\partial W}{\partial a}.$$

Hence, the stability condition (2.5) leads to

$$\frac{a_0^2}{2DT}\frac{\partial W}{\partial a} = \frac{a_0^2}{2D}\frac{\partial W_0}{\partial a_0} > 0.$$

or

$$\frac{1}{D}\frac{\partial W}{\partial a_0^2} > 0. \tag{2.7}$$

With this inequality we identify the stable and unstable branches of the resonance curve $W_0 = 0$. The curves $W_0 = 0$ (curves 1 and 2) and curve $D = 0$ (curve 3 in Fig. 1) divide the plane (a_0^2, Δ) into regions where the functions W_0 and D have either a positive or a negative sign. The curve 1 lies in the regions of negative function D. Following the rule shown in [2], when moving upwards along a straight line parallel to the ordinate axis a_0^2 which cuts resonance curve, if one moves from the zone $W_0 > 0$ ($W_0 < 0$) to the zone $W_0 < 0$ ($W_0 > 0$) then the intersection point of the straight line with the resonance curve corresponds to the stable (unstable) state of oscillations. Similarly, we can identify the stability of the branches of curve 2, which lies in the zone $D > 0$. In figures 1-6 the stable (unstable) branches are represented by heavy (dashed) lines, where the inequality (2.7) is fulfilled (not fulfilled). It is easy to show that the zero solution $a = 0$ of equations (1.6) is stable.

3. Concluding remark

In this paper the asymptotic stationary solutions of equation (1.1) and their stability are of special interest. The problem under consideration becomes complicated due to the appearance of trigonometrical functions with different arguments. By suitable transformation the phase elimination is realized and the equation for amplitude and frequency (resonance curve) is obtained. The resonance curves sonsist of two "parabola" branches. Their location and shape depend on the parameters of parametric excitations and on the coefficient of friction. The stability of stationary solution is investigated by using the inequality (2.7).

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TƯƠNG TÁC GIỮA HAI KÍCH ĐỘNG THÔNG SỐ BẬC HAI VÀ BẬC BA

Bài báo trình bày kết quả nghiên cứu sự tương tác giữa hai kích động thông số bậc hai và bậc ba, được mô tả bởi phương trình (1.1). Ở đây đã sử dụng phương pháp tiệm cận của cơ học phi tuyến kết hợp với máy tính. Việc khử pha của dao động dừng đã được thực hiện bằng những phép biến đổi đặc biệt. Các đường cộng hưởng gồm hai nhánh dạng pa-ra-bôn ngửa với kích thước và dạng phụ thuộc vào các thông số của kích động thông số và lực cản. Vấn đề ổn định của dao động dừng cũng đã được xét chi tiết.

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