Vietnam Journal of Mechanics, NCST of Vietnam Vol. **22**, 2000, No 2 (65 – 70)

THE BIFURCATION THEOREM ON THE PROBLEM OF THERMAL CONVECTION AND CONTAMINANT TRANSPORT IN UNDERGROUND WATER

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ABSTRACT. In the paper the bifurcation theorem on the problem of thermal convection and contaminant transport in underground water is proven.

1. Formulation of the problem

Equations of the problem on thermal convection and contaminant transport in underground water assume the dimensionless form [1]

$$\overline{v} = -\alpha\nabla p + R_T T\overline{\gamma} - R_C C\overline{\gamma} \tag{1.1}$$

$$\frac{\partial T}{\partial t} + \overline{v}\cdot\nabla T = \Delta T, \tag{1.2}$$

$$\varepsilon\frac{\partial C}{\partial t} + \overline{v}\cdot\nabla C = \frac{1}{Le}\Delta C, \tag{1.3}$$

$$\operatorname{div}\overline{v} = 0, \quad \text{in } \Omega, \tag{1.4}$$

with the boundary conditions:

$$v_n = 0, \quad T = T_1, \quad C = C_1 \quad \text{on } \partial\Omega, \tag{1.5}$$

and initial conditions

$$T_0 = T(x_0, y_0, z_0), \quad C_0 = C(x_0, y_0, z_0) \quad \text{at } t = t_0, \tag{1.6}$$

where the following notations are used: $\overline{v}$ denotes the velocity, p - pressure, T - temperature, C - concentration $\alpha = \dfrac{k}{H^2\nu}$, k - coefficient of permeability, H - a length scale, ν - coefficient of kinematical viscosity, $\varepsilon = \dfrac{\phi}{\sigma}$, ϕ - porosity, σ - heat capacity of porous media, $\overline{\gamma}$ - unit vector of the vertical upward axis Ox_3 in the

Cartesian coordinate system $Ox_1x_2x_3$, Le - Lewis number, R_T - thermal Rayleigh number, R_C - concentration Rayleigh number.

In [2] it is proved that there exists a mechanical equilibrium in the fluid:

$$\begin{aligned} \overline{v} &= 0, \\ C_0 &= -A_0^C x_3 + B_0^C, \\ T_0 &= -A_0^T x_3 + B_0^T, \end{aligned} \tag{1.7}$$

A_0^C, B_0^C, A_0^T, B_0^T the constants.

In [2] the existence theorem and the spectrum theorem of the linear problem are proved.

In this paper we prove the bifurcation theorem of the problem

$$\overline{v} = -\alpha \nabla p - R_C C \overline{\gamma} + R_T T \overline{\gamma}, \tag{1.8}$$

$$\overline{v} \cdot \nabla T - \overline{v} \cdot \overline{\gamma} = \Delta T, \tag{1.9}$$

$$\overline{v} \cdot \nabla C - \overline{v} \cdot \overline{\gamma} = \frac{1}{Le} \Delta C, \tag{1.10}$$

$$v_n = 0, \quad T = 0, \quad C = 0 \quad \text{on } \Omega. \tag{1.11}$$

2. The bifurcation theorem

From (1.1) - (1.4), (1.7) we can obtain

$$\overline{v} = -\alpha \nabla p - R_C C \overline{\gamma} + R_T T \overline{\gamma}, \tag{2.1}$$

$$\frac{\partial T}{\partial t} - \overline{v} \cdot \overline{\gamma} = \Delta T, \tag{2.2}$$

$$\varepsilon \frac{\partial C}{\partial t} - \overline{v} \cdot \overline{\gamma} = \frac{1}{Le} \Delta C, \tag{2.3}$$

$$\operatorname{div} \overline{v} = 0. \tag{2.4}$$

Suppose that

$$\begin{aligned} \overline{v}(x_1, x_2, x_3, t) &= \overline{v}(x_1, x_2, x_3) e^{-\lambda t}, \\ p(x_1, x_2, x_3, t) &= p(x_1, x_2, x_3) e^{-\lambda t}, \\ C(x_1, x_2, x_3, t) &= C(x_1, x_2, x_3) e^{-\lambda t}, \\ T(x_1, x_2, x_3, t) &= T(x_1, x_2, x_3) e^{-\lambda t}. \end{aligned}$$

From (2.1)-(2.4) we obtain

$$\overline{v} = -\alpha\nabla p - R_C C\overline{\gamma} + R_T T\overline{\gamma}, \tag{2.5}$$

$$-\lambda T - \overline{v}\cdot\overline{\gamma} = \Delta T, \tag{2.6}$$

$$-\lambda\varepsilon C - \overline{v}\cdot\overline{\gamma} = \frac{1}{Le}\Delta C, \tag{2.7}$$

$$\operatorname{div}\overline{v} = 0, \tag{2.8}$$

$$v_n = 0, \quad C = 0, \quad T = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

The problem (2.5)-(2.9) is equivalent to the following operator equations in the spaces $\tilde{L}_2(\Omega)$ and $H_2(\Omega)$ [3]

$$\overline{v} = -R_C B_{12} C + R_T B_{12} T, \tag{2.10}$$

$$\lambda T + B_{21}\overline{v} = AT, \tag{2.11}$$

$$\lambda\varepsilon C + B_{21}\overline{v} = \frac{1}{Le}AC, \tag{2.12}$$

where $L_2(\Omega)$ is the space of quadratically integrable vector functions in Ω, $H_2(\Omega)$ is the space of quadratically integrable scalar functions in Ω

$$\tilde{L}_2(\Omega) = \left\{\overline{v},\ \overline{v}\in L_2(\Omega),\ \operatorname{div}\overline{v} = 0 \text{ in } \Omega\ v_n = 0 \text{ on } \partial\Omega\right\},$$

$$B_{12}C \equiv \Pi C\overline{\gamma}, \quad B_{12}T \equiv \Pi T\overline{\gamma}, \quad B_{21}\overline{v} \equiv (\overline{v}\cdot\overline{\gamma}),$$

Π denotes an operator of orthogonal projection to $\tilde{L}_2(\Omega)$. The operator A is self-adjoint positive definite in $H_2(\Omega)$, its inverse operator is positive and compact [3].

In [2] it is proved that if $R_T > 0$, $R_C < 0$ or $R_T < 0$, $R_C > 0$ all normal disturbances vary monotonically with time-being either damped or amplified (monotonicity principle for disturbances) (see [4]).

Putting $\lambda = 0$ in (2.10) - (2.12) we obtain

$$\overline{v} = -R_C B_{12} C + R_T B_{12} T, \tag{2.13}$$

$$B_{21}\overline{v} = AT, \tag{2.14}$$

$$B_{21}\overline{v} = \frac{1}{Le}AC. \tag{2.15}$$

From (2.13)-(2.15) we have

$$\overline{v} = -R_C Le B_{12} A^{-1} B_{21}\overline{v} + R_T B_{12} A^{-1} B_{21}\overline{v}$$

or

$$\overline{v} = (R_T - LeR_c)\mathcal{A}\overline{v},$$
$$\mathcal{A} = B_{12}A^{-1}B_{21}. \tag{2.16}$$

Since the operator A^{-1} is positive and compact, operators B_{12}, B_{21} are adjoint and bounded, so the operator $\mathcal{A}$ is self-adjoint positive and compact and the eigenvalues $R = R_T - LeR_C$ are discrete and real.

The eigenvalues $R = R_T - LeR_C$ of the problem (2.16) are called the critical values of the problem (2.10)-(2.12). We have

Lemma. *The critical values* $R = R_T - LeR_C$ *of the problem* (2.10)-(2.12) *are discrete and real.*

We rewrite the problem (1.8)-(1.11) in the equivalent operator equations:

$$\overline{v} = -R_C B_{12} C + R_T B_{12} T,$$
$$(\overline{v} \cdot \nabla T) - B_{21}\overline{v} = -AT,$$
$$(\overline{v} \cdot \nabla C) - B_{21}\overline{v} = -\frac{1}{Le}AC$$

or

$$\overline{v} = -R_C B_{12} C + R_T B_{12} T, \tag{2.17}$$
$$T = -A^{-1}(\overline{v} \cdot \nabla T) + A^{-1}B_{21}\overline{v}, \tag{2.18}$$
$$C = -LeA^{-1}(\overline{v} \cdot \nabla C) + LeA^{-1}B_{21}\overline{v}. \tag{2.19}$$

From (2.18), (2.19) it follows

$$(E + A_{v1})T = A^{-1}B_{21}\overline{v}, \tag{2.20}$$
$$(E + A_{v2})C = LeA^{-1}B_{21}\overline{v}, \tag{2.21}$$

where

$$A_{v1}T \equiv A^{-1}(\overline{v} \cdot \nabla T),$$
$$A_{v2}C \equiv LeA^{-1}(\overline{v} \cdot \nabla C).$$

As in [5] it is easy to prove that the operators $(E + A_{v1})$ and $(E + A_{v2})$ get an inverse operator and A^{-1} is its differential Fresher at $\overline{v} = 0$:

$$(E + A_{v1})^{-1}A^{-1} = A^{-1} + G_1(\overline{v}), \tag{2.22}$$
$$(E + A_{v2})^{-1}A^{-1} = A^{-1} + G_2(\overline{v}), \tag{2.23}$$

where

$$\lim_{\|\overline{v}\|\to 0} \frac{\|G_1(\overline{v})\|_{H^1_{2,0}(\Omega)}}{\|\overline{v}\|_{\widetilde{W}^1_{2,0}(\Omega)}} = 0, \tag{2.24}$$

$$\lim_{\|\overline{v}\|\to 0} \frac{\|G_2(\overline{v})\|_{H^1_{2,0}(\Omega)}}{\|\overline{v}\|_{\widetilde{W}^1_{2,0}(\Omega)}} = 0, \tag{2.25}$$

$$H^1_{2,0}(\Omega) = \Big\{q \in H_2(\Omega),\ \nabla q \in H_2(\Omega),\ q = 0 \text{ on } \partial\Omega\Big\},$$
$$W^1_{2,0}(\Omega) = H^1_{2,0}(\Omega) \times H^1_{2,0}(\Omega) \times H^1_{2,0}(\Omega),$$
$$\widetilde{W}^1_{2,0}(\Omega) = \Big\{\overline{v} \in W^1_{2,0}(\Omega),\ \operatorname{div} \overline{v} = 0,\ \overline{v} = 0 \text{ on } \partial\Omega\Big\}.$$

From (2.20)-(2.23) it follows

$$T = (A^{-1} + G_1(\overline{v}))B_{21}\overline{v}, \tag{2.26}$$
$$C = Le(A^{-1} + G_2(\overline{v}))B_{21}\overline{v}. \tag{2.27}$$

Putting (2.26), (2.27) into (2.17) we get

$$\begin{aligned}\overline{v} &= -LeR_C B_{12}(A^{-1} + G_2(\overline{v}))B_{21}\overline{v} + R_T B_{12}(A^{-1} + G_1(\overline{v}))B_{21}\overline{v} \\ &= (R_T - LeR_C)B_{12}A^{-1}B_{21}\overline{v} - LeR_C B_{12}G_2(\overline{v})B_{21}\overline{v} + R_T B_{12}G_1(\overline{v})B_{21}\overline{v}\end{aligned}$$

or

$$\overline{v} = R\mathcal{A} + Q(\overline{v}), \tag{2.28}$$
$$Q(\overline{v}) \equiv -LeR_C B_{12}G_2(\overline{v})B_{21}\overline{v} + R_T B_{12}G_1(\overline{v})B_{21}\overline{v},$$

B_{12}, B_{21} are the bounded operators; using (2.24), (2.25) we get

$$\lim \frac{\|Q(\overline{v})\|_{H^1_{2,0}(\Omega)}}{\|\overline{v}\|_{\widetilde{W}^1_{2,0}(\Omega)}} = 0. \tag{2.29}$$

This implies that the operator $\mathcal{A}$ is a differential Fresher of the operator $R\mathcal{A}+Q$ at $\overline{v} = 0$.

Using the theorem of Krasnoselskii [6] we get

Theorem. *Let $R_i = R_{T_i} - LeR_{C_i}$ which are eigenvalues of problem* (2.13)-(2.15). *If R_T and R_C are such that $R = R_T - LeR_C < R_1$, then the problem* (1.8)-(1.11) *gets only a trivial solution. If $R_1 = R_{T_1} - LeR_{C_1}$ gets odd multiplicity then R_1 is a bifurcation point of the problem* (1.8)-(1.11).

This publication is completed with financial support from the Council for Natural Sciences of Vietnam.

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Received May 19, 2000

ĐỊNH LÝ PHÂN NHÁNH NGHIỆM CỦA BÀI TOÁN VỀ LAN TRUYỀN NHIỆT, CHẤT TRONG CÁC VỈA NƯỚC NGẦM

Bài báo đã chứng minh định lý phân nhánh nghiệm của bài toán về lan truyền nhiệt chất trong các vỉa nước ngầm.

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