Vietnam Journal of Mechanics, NCST of Vietnam Vol. 22, 2000, No 1 (19 – 28)

A NON-LINEAR ELEMENT FOR ANALYSING ELASTIC FRAME STRUCTURES AT LARGE DEFLECTIONS

NGUYEN DINH KIEN
Institute of Mechanics, 264 Doi Can, Hanoi

ABSTRACT. A non-linear finite element for analysing elastic frame structures at large deflections is presented. A co-rotational technique combined with classical beam theory with the inclusion of the effect of axial forces is adopted. The element nodal force vector is derived from the strain energy of the element. The element tangent stiffness matrix is obtained by differentiating the nodal force vector, with respect to the degree of freedom (d.o.f.). The obtained formulations have a simple and compact mathematical forms which are easy to implement into a computer program. An incremental interactive technique based on Newton-Raphson method is adopted to solve the non-linear equation and to trace the equilibrium paths of the structures. Numerical examples are presented to show the accuracy and efficiency of the proposed formulations.

1. Introduction

The use of finite element method in engineering practice is increasing rapidly. This trend is observed because the integrated use of finite element softwares in CAD can have a significant benefit on the productivity of a design team, increase the functionality of the design and decrease its cost [1].

Lately, the finite element method has proven to be very effective in linear analysis [2], and the solutions have also been obtained for some non-linear problems [3]. However, much research is now underway to solve the problems of the continuum mechanics and to improve the finite element formulations as well as numerical integration procedures and computer program implementation.

In the field of structural mechanics, the analyses of beam and frame structures are of great importance. With the fast development of computational technology, and the importance of the beam and frame structures in practical applications, the non-linear analyses on such kinds of the structures have become a central topic in the structural mechanics. Various formulations for non-linear beams and frames using different strategies have been proposed [4, 5, 6]. However, the proposed formulations are more or less complicated, or restricted to small rotations during the deformation process of the element. More effort on the development of efficient finite element formulations in the topic is still needed.

Recently, Kuo and Fang [7] proposed a finite element formulation for elastic frames by using the body attached co-ordinate method which is somehow similar to the co-rotational technique. The proposed formulation can remove the restriction of small rotations, but the obtained formulation for nodal force vector does not show explicitly the role of the attached co-ordinate. In addition, the element stiffness matrix presented in [7] is superimposed of the bending and geometrical stiffness matrices of a basic beam element and the stiffness matrix of the linear bar, which does not directly reflect the deformation characteristics of the element.

The co-rotational technique has been used in a combination with the principle of virtual work for deriving the geometrically non-linear beam element [8]. The formulations obtained by using the technique are simple, but the derived procedure is not. Moreover, no example was given in [8] to show the accuracy and efficiency of the formulations.

In this paper, a simple procedure for deriving geometrically non-linear frame element which can be used for analysing $2D$ frame structures at large deflections is presented. The proposal method based on the energy approach and the co-rotational technique results in mathematical compact forms of the element formulations which are easily implemented into a computer program. Some numerical examples are also presented to demonstrate the accuracy and efficiency of the proposed element.

2. Beams and frames at large deflections

Assume a plane beam element in a fixed co-ordinates (x, z) with the x and z axes are directed along and perpendicular to the beam axis is considered. For small deflections, linear theory with the definition of engineering strain is adequate to model the behaviour of the beam. For large deflections, the problem is a geometrically non-linearity, and an approximate measure such as Green strain [3, 9] must be employed

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{2.1}$$

where, u_0 and w are the axial and lateral displacements of material particles on the neutral axis of the element, respectively; z - the distance from the considering point to the neutral axis.

For solving a non-linear problem in general, and the geometrically non-linearity in particular by finite element method, the tangent stiffness matrix and nodal force vector are needed to construct the equilibrium equation [8]. The standard finite element procedure presented in [3], for instance, produces the element stiffness matrix which is a combination of the small displacement stiffness matrix, initial stiffness matrix and large displacement stiffness matrix. The procedure to derive the stiffness matrix and nodal force vector from this standard procedure is, more or less complicated, which partly caused by the non-linear part $\frac{1}{2}\left(\frac{\partial w}{\partial w}\right)^2$ in the definition of the Green strain (2.1). In addition, the obtained stiffness matrix and nodal force vector are not easy to implement into a computer program because of their complex forms. These difficulties can be remarkably reduced by using the energy method and adopting the co-rotational technique which allows to use the definition of engineering strain.

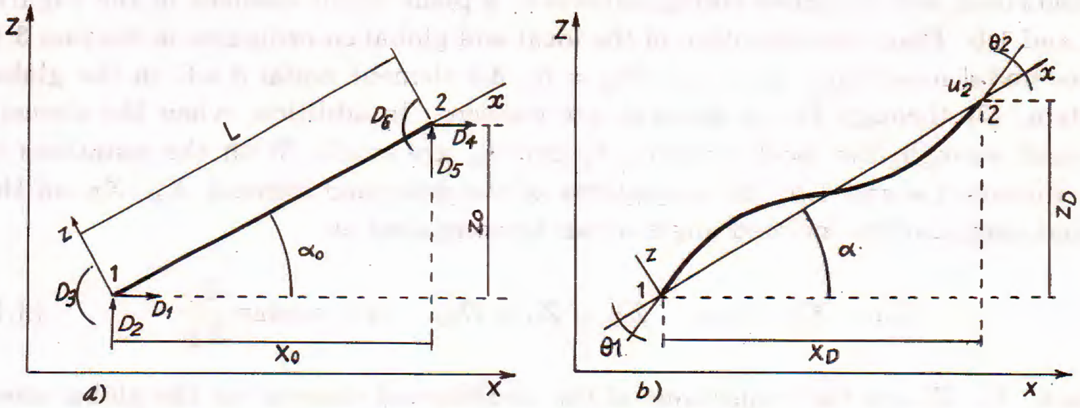


Fig. 1. Plane frame element in local and global co-ordinates
a) undeformed state; b) deformed state

3. Co-rotational technique

3.1. Local and global co-ordinates

The co-rotational technique was initially introduced by Belytschko and co-workers for analysing dynamic problems [10, 11]. The central idea of this technique is to introduce local co-ordinate which continuously rotates and translates with the element. The technique for a typical $2D$ frame element is illustrated in the Figures 1.a and 1.b, where undeformed and deformed configurations of the element are respectively shown. The global co-ordinate (X, Z) is fixed in space, while the local one (x, z) is attached to the element, moves and rotates with it. The original of the local co-ordinate, $x = z = 0$, is always attached to the node 1 of the element, and the axis x is directed through the node 2. By using this co-rotational co-ordinate, the finite formulations of the element can be formed in the local co-ordinate by using the standard, small strain, small deflection relationship, and therefore the difficulties caused by the non-linear definition of strain (2.1) to the formation of the finite element formulations can be avoided. The obtained finite element formulations such as the element stiffness matrix, nodal force vector in the local co-ordinates are then transferred into the global co-ordinate by the use of transformation relationships which can be obtained from the rigid rotation of the local co-ordinate, respects to the global one. The global formulations are then assembled in a standard way of finite element method in order to obtain the structural stiffness matrix and nodal force vector, and construct a system of equilibrium equations for the structure.

3.2. Relationship between d.o.f. in the local and global co-ordinates

The first thing we need for constructing the finite element formulations based on the co-rotational technique is the relationship between the components of local displacement vector $\{\mathbf{d}\} = \{d_1\, d_2 \ldots d_6\}^T$ and those of global displacement vector $\{\mathbf{D}\} = \{D_1\ D_2\ \ldots D_6\}^T$. This relationship can be formed by considering the undeformed and deformed configurations of a plane frame element in the Figures 1.a and 1.b. From the definition of the local and global co-ordinates in Section 3.1, three nodal local d.o.f. $u_1 = v_1 = v_2 = 0$. All element nodal d.o.f. in the global system, D_1 through D_6, in general, are nonzero. In addition, when the element is small enough, the local rotation, θ_1 and θ_2, are small. With the notations in the Figures 1.a and 1.b, the projections of the deformed element X_D, Z_D on the global axes, and its inclined angle α can be computed as

$$X_D = X_0 + D_{41}, \quad Z_D = Z_0 + D_{52}, \quad \alpha = \arctan\frac{Z_D}{X_D} \tag{3.1}$$

where, X_0, Z_0 are the projections of the undeformed element on the global axes, and

$$D_{41} = D_4 - D_1, \quad D_{52} = D_5 - D_2.$$

Therefore, the nonzero local d.o.f. $d_3 = \theta_1$, $d_4 = u_2$, $d_6 = \theta_2$ can be expressed through the global d.o.f. $D_1,\ D_2 \ldots D_6$ as

$$\begin{aligned} &\theta_1 = D_3 - (\alpha - \alpha_0) = D_3 - \theta_r, \quad \theta_2 = D_6 - (\alpha - \alpha_0) = D_6 - \theta_r; \\ &u_2 = \frac{1}{L + L_D}[(2X_0 + D_{41})D_{41} + (2Z_0 + D_{52})D_{52}] \end{aligned} \tag{3.2}$$

where, $\theta_r = (\alpha - \alpha_0)$ is rigid rotation of the local co-ordinate during deforming process of the element; L_D - the length of the element in the deformed state, which can be computed as

$$L_D = \sqrt{X_D^2 + Z_D^2} = \sqrt{(X_0 + D_{41})^2 + (Z_0 + D_{52})^2} \tag{3.3}$$

4. Finite element formulation

4.1. Kinematic relations and interpolations

Thanks to the definition of the local co-ordinate, the membrane strain and curvature of the element in the local co-ordinate can be assumed to be linear, and they are expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \kappa = \frac{\partial^2 w}{\partial x^2} \tag{4.1}$$

where, u, w respectively are axial and lateral displacements of the element in the local co-ordinate. In the finite element context, the displacements u and

w can be interpolated through the element nodal displacements u_2, θ_1 and θ_2 ($u_1 = v_1 = v_2 = 0$) by using the standard interpolation scheme for the beam element, that is linear for u and Hermitian for w

$$u = \sum_{i=1}^{2} N_{u_i} d_{u_i} = N_2 u_2 \quad , \qquad w = \sum_{i=1}^{4} N_{w_i} d_{w_i} = N_{w_2}\theta_1 + N_{w_4}\theta_2 \tag{4.2}$$

where, N_{u_i}, N_{w_i} are the shape functions which have the forms [2]:

$$N_{u_2} = \frac{x}{L}, \quad N_{w_2} = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}, \quad N_{w_4} = -\frac{x^2}{L} + \frac{x^3}{L^2}. \tag{4.3}$$

From (4.2) and (4.3), the strain and curvature defined in (4.1) can be written as

$$\varepsilon_x = \frac{u_2}{L}, \quad \kappa = C_{w_2}\theta_1 + C_{w_4}\theta_2 \tag{4.4}$$

where,

$$C_{w_i} = \frac{\partial^2 N_{w_i}}{\partial x^2}, \quad i = 2, 4. \tag{4.5}$$

4.2. Nodal force vector and tangent stiffness matrix

The element nodal force vector and tangent stiffness matrix are derived in this Section by using the expression of element strain energy. For a frame element with length of L as shown in Figure 1, the strain energy can be expressed through the local strain and curvature as follows:

$$U = \frac{1}{2}\int_0^L (EA\varepsilon_x^2 + EI\kappa^2)dx \tag{4.6}$$

where, EA and EI represent axial and flexural rigidities, respectively. From (4.4) and with the aid of symbolic computational software MAPLE [12], the expression for strain energy (4.6) can easily be obtained as

$$\begin{aligned} U &= \frac{1}{2}\int_0^L \Big[EA\Big(\frac{u_2}{L}\Big)^2 + EI(C_{w_2}\theta_1 + C_{w_4}\theta_2)^2\Big]dx \\ &= \frac{1}{2}\frac{EA}{L}u_2^2 + 2\frac{EI}{L}(\theta_1^2 + \theta_1\theta_2 + \theta_2^2). \end{aligned} \tag{4.7}$$

The element nodal vector $\mathbf{F}_{EL}$ with only three components: N_{L_2} - the axial force at node 2; M_{L_1}, M_{L_2} - the moments at nodes 1 and 2, and the tangent stiffness matrix $\mathbf{K}_{EL}$ in the local co-ordinate can be obtained as the first and second order derivatives of the strain energy U with respect to the local d.o.f. [3]:

$$\begin{aligned}\mathbf{F}_{EL} = \{N_{L_2}\ M_{L_1}\ M_{L_2}\}^T &= \left\{\frac{\partial U}{\partial u_2}\ \frac{\partial U}{\partial \theta_1}\ \frac{\partial U}{\partial \theta_2}\right\}^T \\ &= \left\{\frac{EA}{L}u_2\ 2\frac{EI}{L}(2\theta_1+\theta_2)\ 2\frac{EI}{L}(\theta_1+2\theta_2)\right\}^T\end{aligned} \tag{4.8}$$

$$\mathbf{K}_{EL} = \left[\frac{\partial^2 U}{\partial d_i \partial d_j}\right] = \begin{bmatrix} \frac{\partial^2 U}{\partial u_2^2} & \frac{\partial^2 U}{\partial u_2 \partial \theta_1} & \frac{\partial^2 U}{\partial u_2 \partial \theta_2} \\ \frac{\partial^2 U}{\partial \theta_1 \partial u_2} & \frac{\partial^2 U}{\partial \theta_1^2} & \frac{\partial^2 U}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 U}{\partial \theta_2 \partial u_2} & \frac{\partial^2 U}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 U}{\partial \theta_2^2} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & 4\frac{EI}{L} & 2\frac{EI}{L} \\ 0 & 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \tag{4.9}$$

As it is seen from (4.8) and (4.9), the obtained formulations for the element nodal force vector and stiffness matrix in the local system have quite simple forms. Since the strain energy of the element is invariant with respect to the co-ordinate, the global nodal force vector $\mathbf{F}_{EG}$ with the aid of (3.2), can be written as

$$\mathbf{F}_{EG} = \left\{\frac{\partial U}{\partial D_i}\right\} = \left\{\frac{\partial U}{\partial d_j}\right\}\left\{\frac{\partial d_j}{\partial D_i}\right\} = \mathbf{B}_1^T\,\mathbf{F}_{EL} \tag{4.10}$$

where,

$$\mathbf{B}_1 = \left[\frac{\partial d_j}{\partial D_i}\right] \tag{4.11}$$

is (6×3) transformation matrix defined by the relationship between the local and global d.o.f.

The element global tangent stiffness matrix for the element is differentiation of the global nodal force vector (4.10) with respect to the global d.o.f.

$$\begin{aligned}\mathbf{K}_{EG} &= \left[\frac{\partial \mathbf{F}_{EG_i}}{\partial D_j}\right] \\ &= \mathbf{B}_1^T\,\mathbf{K}_{EL}\,\mathbf{B}_1 + N_{L_2}\left[\frac{\partial^2 u_2}{\partial D_i \partial D_j}\right] + M_{L_1}\left[\frac{\partial^2 \theta_1}{\partial D_i \partial D_j}\right] + M_{L_2}\left[\frac{\partial^2 \theta_2}{\partial D_i \partial D_j}\right]\end{aligned} \tag{4.12}$$

where, $\mathbf{K}_{EL} = \left[\frac{\partial \mathbf{F}_{EL_i}}{\partial d_j}\right]$ is the ordinary linear element stiffness matrix. From (3.2), it is easy to show that

$$\left[\frac{\partial^2 \theta_1}{\partial D_i \partial D_j}\right] = \left[\frac{\partial^2 \theta_2}{\partial D_i \partial D_j}\right] = -\left[\frac{\partial^2 \alpha_r}{\partial D_i \partial D_j}\right]. \tag{4.13}$$

This gives the global tangent stiffness matrix a mathematical compact form

$$\mathbf{K}_{EG} = \mathbf{B}_1^T \mathbf{K}_{EL} \mathbf{B}_1 + N_{L_2} \mathbf{B}_2 + (M_{L_1} + M_{L_2}) \mathbf{B}_3 \tag{4.14}$$

with the (6×6) transformation matrices $\mathbf{B}_2$ and $\mathbf{B}_3$

$$\mathbf{B}_2 = \left[\frac{\partial^2 u_2}{\partial D_i \partial D_j} \right], \quad \mathbf{B}_3 = - \left[\frac{\partial^2 \alpha_r}{\partial D_i \partial D_j} \right]. \tag{4.15}$$

The obtained formulations for the global nodal force vector and element tangent stiffness matrix (4.14), (4.10) are enough for constructing the equilibrium equations of the geometrically non-linear problem. The procedure for computing the element nodal force vector and tangent stiffness matrix in the global co-ordinate is as follows: From the global d.o.f., the local d.o.f. are computed by using the equation (3.2); the transformation matrices $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ are computed from (4.11) and (4.15); the local element nodal force vector and stiffness matrix are then computed from equations (4.8) and (4.9), respectively; at last, the global element nodal force vector and tangent stiffness matrix are then computed from the equations (4.10) and (4.14). The standard procedure of the finite element method is then employed to construct the structural stiffness matrix and internal force vector.

4.3. Equilibrium equation

The non-linear equation for the structure may be expressed by [3]

$$\{\boldsymbol{\Psi}\} = \{\mathbf{F}\} - \lambda\{\mathbf{P}\} \tag{4.16}$$

where, $\{\mathbf{F}\}$ is the structural internal force vector obtained by assembling the element nodal force vectors $\mathbf{F}_{EG}$ of (4.10); $\{\boldsymbol{\Psi}\}$ is vector of unbalance force between the internal force vector $\{\mathbf{F}\}$ and the external nodal force $\lambda\{\mathbf{P}\}$; λ represents a loading parameter and $\mathbf{P}$ is a normalized loading vector. The equation (4.16) can be solved by the combined incremental and iterative technique based on Newton-Raphson method [8].

5. Numerical examples

The obtained formulations were implemented into a computer program using MATLAB [13]. With the aid of MAPLE software, the computer code was easily developed. Some tests are then performed to demonstrate the accuracy and effectiveness of the developed formulations.

5.1. Simply supported column under axial compressive force

The large deflection behaviour of the simply supported column under axial compressive force P at the end is used as the first test for the developed formulations. The column and its data are shown on Figure 2.a. A small transverse load

$q = 0.0001$ at the middle of the column is used as an imperfection. The problem is well known as buckling of Euler column. An equilibrium path before and after critical state is sought here by using 16 elements. Figure 2.b shows the relationship between the two quantities: P/P_{cr} - the ratio of the external load to the critical load, and d/L - the ratio between midpoint deflection and the column length. As it is seen in the figure, a very good agreement between the present numerical result and the analytical solution [15]. The proposed element is not only well predicted the critical load but the post-critical behaviour of the column also.

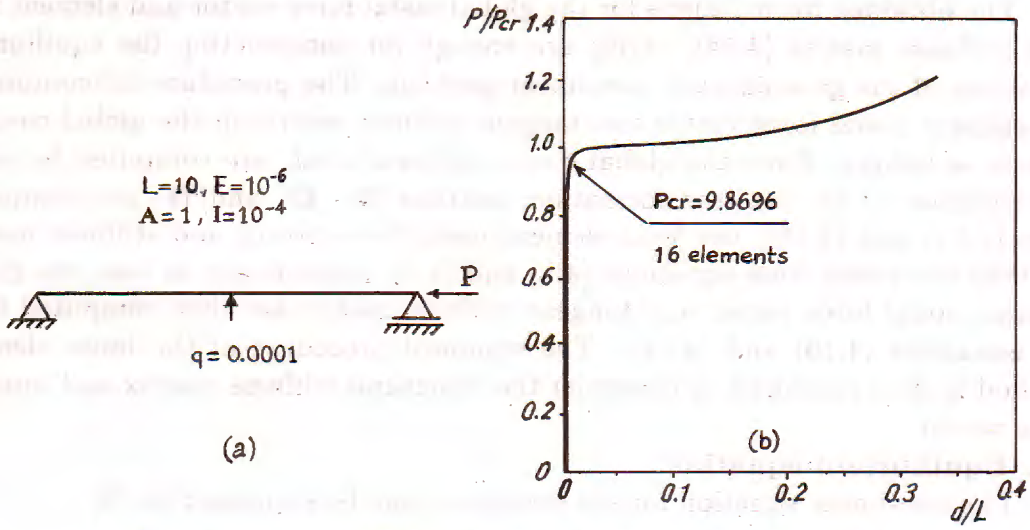


Fig. 2. Buckling of Euler column under axial compressive load
a) geometry and material data; b) external load versus midpoint deflection

5.2. Cantilever beam under end moment

The cantilever beam under an end moment with the data shown in the Figure 3 is used as the second test. The calculated axial and vertical displacements $u_{cal.}, w_{cal.}$ at the free end by using 16 elements are given in the Table 1. The exact solutions [14] u_{exact}, w_{exact} are also presented in the Table for the purpose of comparison. The result shows the accuracy of the proposed element.

Table 1. The axial and vertical displacements at the free and of the cantileve beam

M	$u_{cal.}$	u_{exact}	$w_{cal.}$	w_{exact}
1	-0.0166	-0.0166	0.4966	0.4966
2	-0.0665	-0.0665	0.9967	0.9967
4	-0.2643	-0.2643	1.9735	1.9735
8	-1.0321	-1.0330	3.7916	3.7912
10	-1.5839	-1.5853	4.5977	4.5970

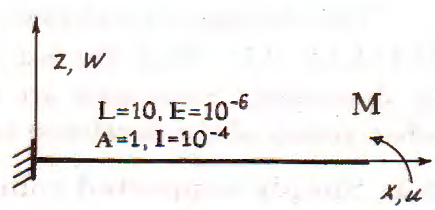
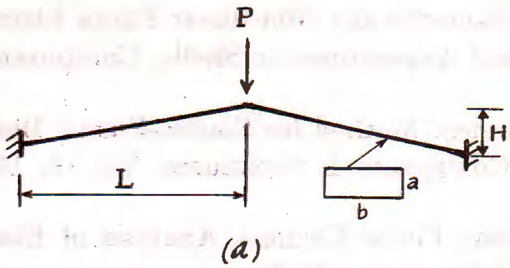


Fig. 3. Cantilever beam under the end moment

5.3. Toggle frame under concentrate load



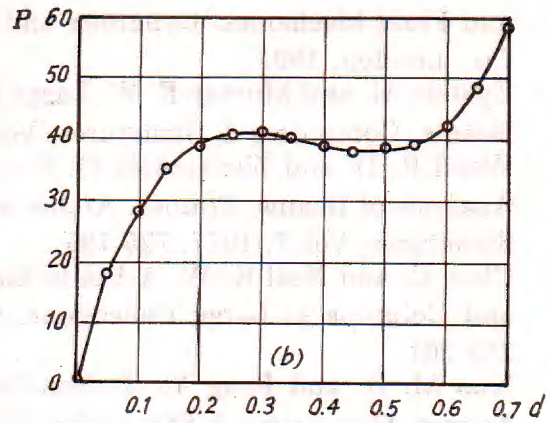


Fig. 4. The toggle frame under concentrate load
a) geometry and material data; b) external load versus loaded point displacement

A toggle frame previous studied analytically by Williams [15] and numerically by Kuo and Fang [7] is used here to verify the proposed formulations. The geometry and material data of the frame are given in the Figure 4.a. Only 4 elements were used to model the behaviour of the frame. The displacement control technique was adopted to overcome the difficulties due to the limit points in this case. The relationship between the external load P and displacement d of the loaded point is shown in the Figure 4.b. The calculated result is very good in agreement to the analytical one. The maximum increment used in this example is 3.

6. Conclusions

The complication of the frame element under large deflection can remarkably be reduced by using the co-rotational technique. The technique used in a combination with the energy method resulted a mathematical compact forms of the finite formulations which benefit the procedure of implementation of the formulations into a computer program. The numerical examples have shown the accuracy and efficiency of the developed formulations.

Acknowledgement. A part of the work was done during the author's stay at the Department of Structural Engineering, Royal Institute of Technology, Sweden. The financial support from the Department to the project "*Large Deflection Analysis of Structures*" is acknowledged. The manuscript has been read by Dr. Pham Duc Chinh with useful comments and corrections.

REFERENCES

1. Bathe K. J., Walczak J. and Zhang H. Some Recent Advances for Practical Finite Element Analysis, Computers & Structures, Vol. 47, 1993, 511-521.
2. Zienkiewicz O. C. and Taylor R. L. The Finite Element Method, Vol. 1 Basic Formulation and Linear Problems, 4^{th} edition, McGraw-Hill Co., London, 1991.
3. Zienkiewicz O. C. and Taylor R. L. The Finite Element Method, Vol. 2 Solid

and Fluid Mechanics, Dynamics and Non-linearity, 4^{th} edition, McGraw-Hill Co., London, 1991.
4. Epstein M. and Murray E. W. Large Deformation In-plane Analysis of Elastic Beams, Computers & Structures, Vol. 6, 1976, 1-9.
5. Wood R. D. and Zienkiewick O. C. Geometrically Non-linear Finite Element Analysis of Beams, Frames, Arches and Axissymmetric Shells, Computers & Structures, Vol. 7, 1977, 725-735.
6. Cheb C. and Neal K. W. A Finite Element Method for Elastic-Plastic Beams and Columns at Large Deflections, Computers & Structures, Vol. 18, 1984, 255-261.
7. Kuo M. H. and Fang Y. H. Non-linear Finite Element Analysis of Elastic Frames, Computers & Structures, Vol. 26, 1987, 693-701.
8. Crisfield M. A. Non-linear Finite Element Analysis of Solids and Structures, Vol. 1: Essentials, John Willey & Sons, Chichester, 1991.
9. Géradin M. and Rixen D. Mechanical Vibrations; Theory and Application to Structural Dynamics, 2^{nd} edition, John Willey & Sons, Chichester, 1996.
10. Belytchko T. and Hseih B. J. Non-linear Transient Finite Element Analysis with Converted Co-ordinates, International Journal for Numerical Methods in Engineering, Vol. 7, 1973, 255-271.
11. Belytchko T. and Glaun L. W. Applications of Higher Order Co-rotational Stretch Theories to Non-linear Finite Element Analysis, Computers & Structures, Vol. 10, 1979, 175-182.
12. MAPLE V, Language Reference Manual, Spring-Verlag, New York, 1991
13. MATLAB, High-Performance Numeric Computation and Visualization Software, Reference Guide, The Math Works Inc., 1992
14. Timoshenko S. P. and Gere J. M. Theory of Elastic Stability, Mc.Graw-Hill, New York, 1961.
15. Williams F. W. An Approach to the Nonlinear Behavior of Members of a Rigid Jointed Plane Framework with Finite Deflections, Quaterly Applied Mathematics, Vol. 17, 1964, 451-469.

Received June 30, 1999

PHẦN TỬ PHI TUYẾN TRONG PHÂN TÍCH KHUNG ĐÀN HỒI
CÓ BIẾN DẠNG LỚN

Bài báo trình bày phần tử phi tuyến trong phân tích kết cấu khung đàn hồi. Công thức được đưa ra dựa trên kỹ thuật tọa độ cùng quay kết hợp với lý thuyết dầm cổ điển có tính tới ảnh hưởng của lực dọc trục. Véc tơ lực nút của phần tử được tính từ biểu thức năng lượng biến dạng. Ma trận độ cứng phần tử thu nhận được bằng cách vi phân vectơ lực nút theo các bậc tự do của phần tử. Các công thức nhận được có dạng đơn giản và là các biểu thức toán học gọn ghẽ, dễ dàng cho việc lập chương trình tính toán. Kỹ thuật lặp tăng dần trên cơ sở phương pháp Niutơn-Raphson được sử dụng để giải bài toán phi tuyến và thu nhận đường cân bằng của kết cấu. Các ví dụ số chứng tỏ tính chính xác và hữu hiệu của các công thức đề nghị.