Vietnam Journal of Mechanics, NCST of Vietnam Vol. 23, 2001, No 2 (110 - 115)

ON THE RELATIONS BETWEEN KINETIC ENERGY AND LINEAR MOMENTUM, ANGULAR MOMENTUM OF THE PARTICLE AND OF THE RIGID BODY

NGUYEN VAN KHANG
Hanoi University of Technology

ABSTRACT. Using the definition for the partial derivative of a scalar in respect to the vector, this paper presents the relations between kinetic energy and linear momentum, angular momentum of the particle and of the rigid body. The obtained results are useful for the investigation of the dynamics of multibody systems.

1. Introduction

The linear and angular momentum of the particle are the basic dynamic quantities of Newton's mechanics. The kinetic energy of the particle is the basic dynamic quantity of Lagrange's mechanics [1, 2]. In the present paper we use the definition for the partial derivative of a scalar α in respect to the vector $\mathbf{x}$ [4, 5]

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \partial \alpha \cdot \frac{1}{\partial \mathbf{x}^T} = \left[\frac{\partial \alpha}{\partial x_1}, \frac{\partial \alpha}{\partial x_2}, \dots, \frac{\partial \alpha}{\partial x_n}\right] \tag{1.1}$$

in order to study the relation between the kinetic energy of the particle and the rigid body and their linear and angular momentum.

2. Relation between kinetic energy and linear momentum of the particle

The expression for the linear momentum and the kinetic energy of the particle has the following form [1, 2]

$$\mathbf{P} = m\,\mathbf{v}, \qquad T = \frac{1}{2} m\,\mathbf{v}^2, \tag{2.1}$$

where m is the mass and $\mathbf{v}$ is the velocity vector of the particle.

In the reference system $Oxyz$ (Fig. 1), the linear momentum and kinetic energy of the particle can be written as

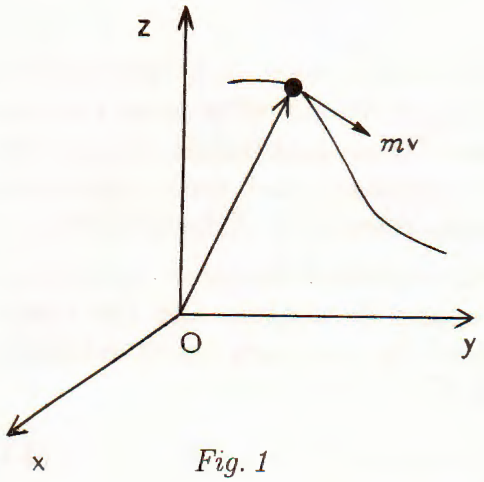

Fig. 1

$$\mathbf{P} = \left[m\dot{x}, m\dot{y}, m\dot{z}\right]^T, \tag{2.2}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{2.3}$$

Theorem 1. *Partial derivative of kinetic energy of the particle in respect to its velocity vector is equal to the transposed vector of the linear momentum of the particle*

$$\frac{\partial T}{\partial \mathbf{v}} = \mathbf{P}^T. \tag{2.4}$$

Proof. According to definition (1.1) we find

$$\frac{\partial T}{\partial \mathbf{v}} = \partial T \cdot \frac{1}{\partial \mathbf{v}^T} = \left[\frac{\partial T}{\partial \dot{x}}, \frac{\partial T}{\partial \dot{y}}, \frac{\partial T}{\partial \dot{z}}\right].$$

From the expression for the kinetic energy of the particle (2.3) it follows that

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z},$$

With it we have

$$\frac{\partial T}{\partial \mathbf{v}} = \left[m\dot{x}, m\dot{y}, m\dot{z}\right] = \mathbf{P}^T.$$

Theorem 2. *Partial derivative of kinetic energy of the particle in respect to its linear momentum vector is equal to the transposed vector of the velocity vector of the particle*

$$\frac{\partial T}{\partial \mathbf{P}} = \mathbf{v}^T. \tag{2.5}$$

Proof. According to the definition (1.1) and the expression (2.2) we have

$$\frac{\partial T}{\partial \mathbf{P}} = \partial T \cdot \frac{1}{\partial \mathbf{P}^T} = \left[\frac{\partial T}{m\partial \dot{x}}, \frac{\partial T}{m\partial \dot{y}}, \frac{\partial T}{m\partial \dot{z}}\right].$$

From the expression for the kinetic energy of the particle (2.3) we can calculate

$$\frac{\partial T}{m\partial \dot{x}} = \dot{x}, \quad \frac{\partial T}{m\partial \dot{y}} = \dot{y}, \quad \frac{\partial T}{m\partial \dot{z}} = \dot{z}.$$

Substituting this result into the above equation we obtain

$$\frac{\partial T}{\partial \mathbf{P}} = [\dot{x},\ \dot{y},\ \dot{z}] = \mathbf{v}^T.$$

3. Relation between kinetic energy of the rigid body and its linear and angular momentum

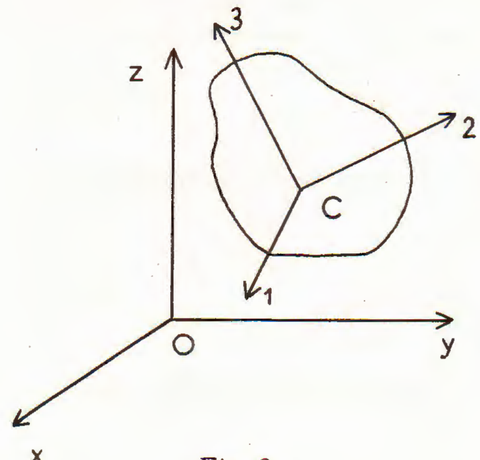

Fig. 2

The configuration of a rigid body in space can be identified by using six coordinates. Three coordinates describe the body translation, and three coordinates define the orientation of the body (Fig. 2). The expressions of the linear momentum, the angular momentum and the kinetic energy of the rigid body have the form [1, 3, 4, 5, 6]

$$\mathbf{P} = m\,\mathbf{v}_C, \tag{3.1}$$

$$\mathbf{L}_C = \vec{\mathbf{J}}_C\,\boldsymbol{\omega}, \tag{3.2}$$

$$T = \frac{1}{2}m\,\mathbf{v}_C^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\vec{\mathbf{J}}_C\cdot\boldsymbol{\omega}, \tag{3.3}$$

where m is the mass of the body, $\mathbf{v}_C$ is the velocity vector of mass center, $\boldsymbol{\omega}$ is the angular velocity of the rigid body and $\vec{\mathbf{J}}_C$ is the inertia tensor of the rigid body relative to its mass center.

The expressions (3.1), (3.2) and (3.3) can be written in a matrix form as [1, 3, 4, 5]

$$\mathbf{P} = m\,\mathbf{v}_C, \quad \mathbf{L}_C = \mathbf{J}_C\cdot\boldsymbol{\omega},$$
$$T = \frac{1}{2}m\,\mathbf{v}_C^T\mathbf{v}_C + \frac{1}{2}\boldsymbol{\omega}^T\mathbf{J}_C\boldsymbol{\omega},$$

where $\mathbf{J}_C$ is the matrix of the inertia tensor $\vec{\mathbf{J}}_C$.

In the reference system $Oxyz$ (Fig. 2), the vectors $\mathbf{P}$ and $\mathbf{v}_C$ therefore have the form

$$\mathbf{P} = \left[m\dot{x},\ m\dot{y},\ m\dot{z}\right]^T, \tag{3.4}$$

$$\mathbf{v}_C = \left[\dot{x}_C,\ \dot{y}_C,\ \dot{z}_C\right]^T. \tag{3.5}$$

It is known that if the origin of the body reference is attached to the mass center of the rigid body and the body reference C123 is the principal inertial axes system, the inertia matrix $\mathbf{J}_C$ and angular velocity $\boldsymbol{\omega}$ of the rigid body can be written as [1]

$$\mathbf{J}_C = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix}, \tag{3.6}$$

$$\boldsymbol{\omega} = \left[\omega_1,\ \omega_2,\ \omega_3\right]^T, \tag{3.7}$$

in which the moments of inertia J_{11}, J_{22}, J_{33} are constant.

Using the expressions (3.6), (3.7) the angular momentum $\mathbf{L}_C$ and the kinetic energy T of the rigid body can be written as

$$\mathbf{L}_C = \left[J_{11}\omega_1,\quad J_{22}\omega_2,\quad J_{33}\omega_3\right]^T, \tag{3.8}$$

$$T = \frac{1}{2}m(\dot{x}_C^2 + \dot{y}_C^2 + \dot{z}_C^2) + \frac{1}{2}(J_{11}\omega_1^2 + J_{22}\omega_2^2 + J_{33}\omega_3^2). \tag{3.9}$$

Now we shall prove the theorems of the relations between the kinetic energy and the linear and angular momentum of the rigid body.

Theorem 3. *Partial derivative of the kinetic energy of the rigid body in respect to its velocity vector of mass center is equal to the transposed vector of the linear momentum of the rigid body*

$$\frac{\partial T}{\partial \mathbf{v}_C} = \mathbf{P}^T. \tag{3.10}$$

Proof. According to the definition (1.1) we have

$$\frac{\partial T}{\partial \mathbf{v}_C} = \partial T \cdot \frac{1}{\partial \mathbf{v}_C^T} = \left[\frac{\partial T}{\partial \dot{x}_C}, \frac{\partial T}{\partial \dot{y}_C}, \frac{\partial T}{\partial \dot{z}_C}\right]$$

From the expression for the kinetic energy of the rigid body (3.9) gets

$$\frac{\partial T}{\partial \dot{x}_C} = m\dot{x}_C, \quad \frac{\partial T}{\partial \dot{y}_C} = m\dot{y}_C, \quad \frac{\partial T}{\partial \dot{z}_C} = m\dot{z}_C.$$

Therefore it is

$$\frac{\partial T}{\partial \mathbf{v}_C} = \left[m\dot{x}_C,\ m\dot{y}_C,\ m\dot{z}_C\right] = \mathbf{P}^T.$$

Theorem 4. *Partial derivative of the kinetic energy of the rigid body in respect to its linear momentum vector is equal to the transposed vector of the velocity vector of the mass center of the rigid body*

$$\frac{\partial T}{\partial \mathbf{P}} = \mathbf{v}_C^T. \tag{3.11}$$

Proof. According to the definition (1.1) we have

$$\frac{\partial T}{\partial \mathbf{P}} = \partial T \cdot \frac{1}{\partial \mathbf{P}^T} = \left[\frac{\partial T}{m\partial \dot{x}_C}, \frac{\partial T}{m\partial \dot{y}_C}, \frac{\partial T}{m\partial \dot{z}_C}\right].$$

From the expression for the kinetic energy of the rigid body (3.9) we obtain

$$\frac{\partial T}{m\partial \dot{x}_C} = \dot{x}_C, \quad \frac{\partial T}{m\partial \dot{y}_C} = \dot{y}_C, \quad \frac{\partial T}{m\partial \dot{z}_C} = \dot{z}_C.$$

Thus, we have

$$\frac{\partial T}{\partial \mathbf{P}} = \left[\dot{x}_C,\ \dot{y}_C,\ \dot{z}_C\right] = \mathbf{v}_C^T.$$

Theorem 5. *Partial derivative of the kinetic energy of the rigid body in respect to its angular vector is equal to the transposed vector of the angular momentum vector relative to its mass center*

$$\frac{\partial T}{\partial \boldsymbol{\omega}} = \mathbf{L}_C^T. \tag{3.12}$$

Proof. According to the definition (1.1) we have

$$\frac{\partial T}{\partial \boldsymbol{\omega}} = \partial T \cdot \frac{1}{\partial \boldsymbol{\omega}^T} = \left[\frac{\partial T}{\partial \omega_1}, \frac{\partial T}{\partial \omega_2}, \frac{\partial T}{\partial \omega_3}\right].$$

From the expression for the kinetic energy of the rigid body (3.9) yields

$$\frac{\partial T}{\partial \omega_1} = J_{11}\omega_1, \quad \frac{\partial T}{\partial \omega_2} = J_{22}\omega_2, \quad \frac{\partial T}{\partial \omega_3} = J_{33}\omega_3.$$

Therefore it is

$$\frac{\partial T}{\partial \boldsymbol{\omega}} = \left[J_{11}\omega_1, \quad J_{22}\omega_2, \quad J_{33}\omega_3\right] = \mathbf{L}_C^T.$$

Theorem 6. *Partial derivative of the kinetic energy of the rigid body in respect to its angular momentum vector relative to the mass center of rigid body is equal to the transposed vector of the angular vector of the rigid body*

$$\frac{\partial T}{\partial \mathbf{L}_C} = \boldsymbol{\omega}^T. \tag{3.13}$$

Proof. According to the definition (1.1) and correcting the expression (3.8) we have

$$\frac{\partial T}{\partial \mathbf{L}_C} = \partial T \cdot \frac{1}{\partial \mathbf{L}_C^T} = \left[\frac{1}{J_{11}}\frac{\partial T}{\partial \omega_1},\ \frac{1}{J_{22}}\frac{\partial T}{\partial \omega_2},\ \frac{1}{J_{33}}\frac{\partial T}{\partial \omega_2}\right].$$

The expression for the kinetic energy of the rigid body yields

$$\frac{1}{J_{11}}\frac{\partial T}{\partial \omega_1} = \omega_1, \quad \frac{1}{J_{22}}\frac{\partial T}{\partial \omega_2} = \omega_2, \quad \frac{1}{J_{33}}\frac{\partial T}{\partial \omega_3} = \omega_3.$$

With them we have

$$\frac{\partial T}{\partial \mathbf{L}_C} = [\omega_1, \quad \omega_2, \quad \omega_3] = \boldsymbol{\omega}^T.$$

4. Conclusions

The paper presents two theorems of the relation between kinetic energy and linear momentum of the particle, two theorems of the relation between kinetic energy of the rigid body and its linear momentum, two theorems of the relation between kinetic energy and angular momentum of the rigid body relative to its mass center. These theorems are simple and useful.

This publication is completed with the financial support of the Council for Natural Sciences of Vietnam.

REFERENCES

1. Fischer U., Stephan W. Prinzipien und Methoden der Dynamik, Fachbuchverlag, Leipzig 1972.
2. Magnus K., Müller H. H. Grundlagen der Technischen Mechanik, 6. Auflage. B. G. Teubner, Stuttgart 1990.
3. Kane T. R., Levinson D. A. Dynamics: Theory and Applications, McGraw-Hill, New York 1985.
4. Witenburg J. Dynamics of Systems of Rigid Bodies, B. G. Teubner, Stuttgart 1977.
5. Schiehlen W. Technische Dynamik, B. G. Teubner, Stuttgart 1986.
6. Shabana A. A. Dynamics of Multibody Systems, John Wiley & Sons, New York 1989.

Received August 9, 2000

VỀ QUAN HỆ GIỮA ĐỘNG NĂNG VỚI ĐỘNG LƯỢNG,
MÔMEN ĐỘNG LƯỢNG CỦA CHẤT ĐIỂM VÀ VẬT RẮN

Trong bài báo này sử dụng định nghĩa về đạo hàm của hàm vô hướng theo biến vectơ, tác giả đã thiết lập một số hệ thức quan hệ giữa động năng với động lượng, mômen động lượng của chất điểm và vật rắn. Các kết quả thu được có thể sử dụng trong nghiên cứu động lực học hệ nhiều vật.