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APPROXIMATE ANALYSIS OF SOME TWO-DEGREE-OF-FREEDOM NONLINEAR RANDOM SYSTEMS BY AN EXTENSION OF GAUSSIAN EQUIVALENT LINEARIZATION

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ABSTRACT. The paper presents the analysis of some two-degree-of-freedom nonlinear systems under random excitation using Local Mean Square Error Criterion which is an extension of Gaussian Equivalent Linearization. The results obtained shows that the new technique can be very efficiently used not only for simple-degree-of-freedom systems as presented in the previous papers, but also for multi-degree-of-freedom ones. The solution's accuracy obtained by the proposed technique is much more improved than that using the traditional linearization. The conclusions in the paper point up the significance of this technique.

1. Introduction

Gaussian equivalent linearization (GEL) proposed by Caughey [1] is presently the simplest tool widely used for analysis of nonlinear stochastic problems. However, a major limitation of this method is perhaps that its accuracy decreases as the nonlinearity increases, and for many cases it can leads to unacceptable errors. Therefore, GEL has been developed by many authors [2-9] to obtain more improved solution accuracy.

N. D. Anh & M. Dipaola [8] proposed "Local Mean Square Error Criterion" (LOMSEC) which is an extension of GEL. The Authors gave initial tests based on Duffing and Vanderpol oscillators under a zero mean Gaussian white noise. Following the initial efforts of Anh & Dipaola, L. X. Hung investigated and developed the proposed technique through analysis of a series of diverse nonlinear random systems [10-11]. The obtained results show advance of LOMSEC, especially the solution accuracy is significantly improved.

However, so far the proposed technique has been just tested for nonlinear random simple-degree-of-freedom (SDOF) systems. So, the problem concerned by this paper is to develop the method LOMSEC for nonlinear random multi-degree-of-freedom (MDOF) systems.

Exact solutions to Fokker-Planck (FPK) equation are known only for special cases. Specifically, this equation can be solved for linear systems in any dimensions and can be solved for a limited class of nonlinear systems in two-dimensions. Some special stationary solutions are also known for systems in more than two-dimensions

[14], but these solutions typically require special relationships between the system and excitation parameters, which are unlikely to be met in practice. Numerical methods for solving the forward and backward FPK equation, available for two- and three-dimensional states are computationally very expensive, especially when investigating the effects of varying the system parameters on the probability density function (PDF), response moments, and system reliability. Some methods for reasonable approximate solutions of MDOF systems as well as its limitations are mentioned in [13]. In general, nonlinear random MDOF systems troubled many researchers in various areas for almost half a century was that it was generally difficult with any available method to obtain desirable approximate solutions of highly nonlinear random systems.

The paper analyzes two nonlinear random two-degree-of-freedom systems whose exact PDF solutions are known. The significantly improved solution accuracy as well as the advance by using LOMSEC instead of using GEL is shown based on a simultaneous comparison to the exact solution. Numerical analysis processes are verified by using the computerized program Mathematica 3.0 [17].

2. Local Mean Square Error Criterion

First of all, we recall some basic ideas of the method of GEL. Suppose that the mechanical structure discredited by a MDOF system is described by a set of nonlinear first order differential equations:

$$\dot{z} = g(z) + f(t) \tag{2.1}$$

where a dot denotes time differentiation, $z = (z_1, z_2, \dots, z_n)^T$ is a vector of state variables, n is a natural number, g is a nonlinear vector function of components of z, $f(t)$ is a stationary Gaussian random excitation vector, with zero mean. Suppose that a stationary solution to Eqn. 2.1 exists. Denote:

$$e(z) = \dot{z} - g(z) - f(t). \tag{2.2}$$

Eqn. 2.1 can be rewritten in the form:

$$e(z) = 0. \tag{2.3}$$

Following the GEL method, we introduce new linear terms in the expression of $e(z)$:

$$e(z) = \dot{z} - Az + Az - g(z) - f(t), \tag{2.4}$$

where $A = \{a_{ij}\}$ is a $n \times n$ constant matrix. Let vector y be a stationary solution of the linearized equation:

$$\dot{y} - Ay - f(t) = 0. \tag{2.5}$$

The vector y is Gaussian since the excitation vector $f(t)$ is Gaussian. Using Eqn. 2.5 one gets from (2.4):

$$e(y) = Ay - g(y). \tag{2.6}$$

Thus, if consider y as an approximation to the solution of the original nonlinear Eqn. 2.1 it is seen that $e(y)$ is an equation error which should be minimized from an optimal criterion. There are some criteria for determining the matrix of linearization, for example, Shocha and Soong [6], Naess [7], Anh and Schiehlen [9], etc. The most extensively used criterion is the mean square error criterion Caughey [1], which requires that the mean squares of error be minimum (here called as Caughey criterion):

$$\langle e_i^2(y)\rangle \to \min_{a_{ij}} \quad i,j = 1,\ldots,n, \tag{2.7}$$

where $e_i(y)$ are components of $e(y)$. The criterion (2.7) leads to the necessary condition:

$$A = \langle g(y)y^T\rangle\langle yy^T\rangle^{-1}. \tag{2.8}$$

From Eqn. 2.8 it is seen that the matrix of linearization A of the linearized Eqn. 2.5, in turn, depends on the statistics of the response. If in the matrix A higher order joint moments of the response appear, they can be expressed in terms of second order moments since y is a Gaussian random vector.

So, the classical version of GEL as described above, supposes that the minimization of the equation error may give a minimization of the solution error. It should be noted that up to now there is no theoretical proof of GEL; its accuracy has been investigated only by the comparison of the solutions obtained by GEL with their exact solutions if available or with simulation solutions. No mathematical link between the equation error and the solution error has been established. For the full information it should also be noted that there is another version of the mean square error criterion in which the linearized process y in (2.8) is replaced by the original nonlinear one z. In that version the mean square error criterion can give the exact solution, for example, when the excitation process is white noise one.

Denote by $p(y)$ the joint probability density function of the response vector y to the Eqn. 2.5. The criterion (2.7) can be rewritten in the explicit form:

$$\int_{-\infty}^{+\infty}(n)\int_{-\infty}^{+\infty} e_i^2(y)p(y)dy \longrightarrow \min_{a_{ij}}. \tag{2.9}$$

Since the integration is taken over all the co-ordinate space $y \in (-\infty; +\infty)$, the criterion (2.7) may be called "Global Mean Square Error Criterion". An extension of the concept, which supposes that the global mean square criterion (2.9) can leads to a large error for some nonlinear systems, especially as strong non-linearity. To increase the accuracy, the expected integration should be taken only in a domain where the response vector y is concentrated, yields the "local mean square error criterion" (LOMSEC) [8] :

$$\left[e_i^2(y)\right] \longrightarrow \min_{a_{ij}}, \tag{2.10}$$

where

$$\left[e_i^2(y)\right] = \int_{-y_1^0\sigma_{y1}}^{y_1^0\sigma_{y1}} (n) \int_{-y_n^0\sigma_{yn}}^{y_n^0\sigma_{yn}} (\cdot)p(y)dy, \tag{2.11}$$

and $y_1^0, y_2^0, \ldots, y_n^0$ are given positive values, $\sigma_{y1}, \ldots, \sigma_{yn}$ are square roots of variances of components $y_1, y_2, \ldots, y_n$. It is noted that as in GEL the values $\sigma_{y1}, \ldots, \sigma_{yn}$ are considered as independent parameters from a_{ij} when minimizing (2.10). Thus, the LOMSEC (2.10) yields the necessary conditions similar to (2.8):

$$A = \left[g(y)y^T\right]\left[yy^T\right]^{-1}. \tag{2.12}$$

The linear Eqn. 2.5 can be solved together with Eqn. 2.8 (Caughey) or Eqn. 2.12 (Lomsec) by any of the existing analytical measure, of which some measures should be summarized here:

1/ The exact PDF of Eqn. 2.5 can be found by solving FPK equation, using this PDF for determining higher-order moments which appeared in Eqn. 2.8/2.12. Then, the moment equations in combination with Eqn. 2.8/2.12 become a closure-set of equations. Caughey and Lomsec solution respectively come from this closure-set. However, the closure-set covers a series of nonlinear algebraic equations which is unlikely to be solved by using the existing mathematical software (so far, some available in Vietnam such as Mathematica 3.0, Maple 5.0, ...).

2/ Some cyclic procedures for numerical solutions for GEL may be used such as Naess [7], Assaf and Utku [12], take the procedure in [12] for example:

(a) Assign an initial value to the instantaneous correlation matrix $\langle yy^T\rangle$

(b) Use Eqn. 2.8 to construct the matrix A.

(c) Solve Eqn. 2.5 for the new instantaneous correlation matrix $\langle yy^T\rangle$.

(d) Repeat steps (b) and (c) until results from cycle to cycle are similar.

This procedure can be also applied for Lomsec by using Eqn. 2.12 and Eqn. 2.5, in addition at the step (a) positive values $y_1^0, y_2^0, \ldots, y_n^0$ are given.

3. Illustrative examples

Example 1. Consider the nonlinear random two-degree-of-freedom system, which was analyzed by Guo-Kang Er and Vai Pan Iu using an approximate PDF method [13]:

$$\ddot{y}_1 + \frac{1}{2}a_1(S_{11}\dot{y}_1 + 2a_2S_{12}\dot{y}_2) + 2a_3y_1 + 4a_4y_1^3 + 6a_5y_1^5 = \sqrt{S_{11}}\xi_1(t),$$
$$\ddot{y}_2 + \frac{1}{2}a_1[2(1-a_2)S_{12}\dot{y}_1 + S_{22}\dot{y}_2] + 2a_6y_2 + 4a_7y_2^3 + 6a_8y_2^5 = \sqrt{S_{22}}\xi_2(t), \tag{3.1}$$

where $a_1, a_2, \ldots, a_8$ are some constants; $\xi_i(t)$, $(i = 1, 2)$, is Gaussian white noise. Denoting $y_1 = x_1$, $\dot{y}_1 = x_2$, $y_2 = x_3$, $\dot{y}_2 = x_4$, we can express the system by the following four-dimensions nonlinear random system:

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -\frac{1}{2}a_1(S_{11}x_2 + 2a_2S_{12}x_4) - 2a_3x_1 - 4a_4x_1^3 - 6a_5x_1^5 + \sqrt{S_{11}}\xi_1(t),\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= -\frac{1}{2}a_1[2(1-a_1)S_{12}x_2 + S_{22}x_4] - 2a_6x_3 - 4a_7x_3^3 - 6a_8x_3^5 + \sqrt{S_{22}}\xi_2(t).
\end{aligned} \tag{3.2}$$

For this system, the exact stationary PDF solution found [14-15] does not depend on the parameters $S_{11}, S_{12}, S_{22}, a_2$:

$$p(x_1, x_2, x_3, x_4) = C\exp\Big\{ - a_1\big[\frac{1}{2}(x_2^2 + x_4^2) + a_3x_1^2 + a_4x_1^4 + a_5x_1^6 + a_6x_3^2 + a_7x_3^4 + a_8x_3^6\big]\Big\}, \tag{3.3}$$

where C is a constant determined from the normalization condition:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 = 1. \tag{3.4}$$

The exact solutions in the form of second order moments:

$$\langle x_i^2\rangle_e = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^2p(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 \quad i = 1 \text{ and } 3. \tag{3.5}$$

From (3.3), since the variables x_1, x_2, x_3, x_4 are independent, so (3.3) and (3.4) can be expressed as follows:

$$\begin{aligned}
p(x_1, x_2, x_3, x_4) &= p(x_1)p(x_2)p(x_3)p(x_4),\\
C &= C_1C_2C_3C_4,
\end{aligned} \tag{3.6}$$

where $\quad p(x_i) = C_i\exp\{q(x_i)\};\quad$ and $\quad C_i = \Big[\int_{-\infty}^{\infty}\exp\{q(x_i)\}dx_i\Big]^{-1}$

A three-dimensional graphic of the joint-PDF $p(x_1, x_3)$ with a set of the parameters value given $a_1 = a_3 = a_4 = a_6 = 1$; $a_5 = a_7 = a_8 = 100$; is shown in Fig. 1

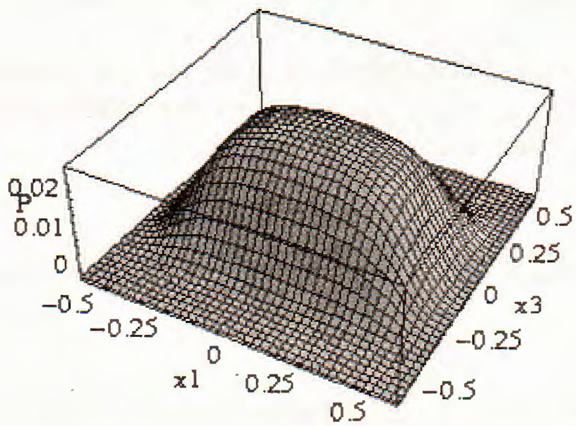

Fig. 1. Three-dimensional graphic of the joint-PDF $p(x_1, x_3)$

With (3.6) we get from (3.5) a more simplified expression:

$$\left\langle x_i^2 \right\rangle_e = \int_{-\infty}^{\infty} x_i^2 p(x_i) dx_i \qquad i = 1 \text{ and } 3. \tag{3.7}$$

We put linearization terms to substitute for the nonlinear terms in the system (3.2):

$$\begin{aligned} 4a_4 x_1^3 + 6a_5 x_1^5 &= \alpha x_1, \\ 4a_7 x_3^3 + 6a_8 x_3^5 &= \beta x_3, \end{aligned} \tag{3.8}$$

where α and β are linearization coefficients. The linearized system is:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{1}{2} a_1 (S_{11} x_2 + 2a_2 S_{12} x_4) - (2a_3 + \alpha) x_1 + \sqrt{S_{11}} \xi_1(t), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -\frac{1}{2} a_1 [2(1 - a_2) S_{12} x_2 + S_{22} x_4] - (2a_6 + \beta) x_3 + \sqrt{S_{22}} \xi_2(t). \end{aligned} \tag{3.9}$$

The assumption (3.8) leads to an equation error as follows:

$$\begin{aligned} e_1 &= \alpha x_1 - (4a_4 x_1^3 + 6a_5 x_1^5), \\ e_2 &= \beta x_3 - (4a_7 x_3^3 + 6a_8 x_3^5). \end{aligned} \tag{3.10}$$

The solution of the linearized system (3.9) is known:

$$\langle x_1^2\rangle = \frac{1}{a_1(2a_3+\alpha)}, \quad \text{and} \quad \langle x_3^2\rangle = \frac{1}{a_1(2a_6+\beta)}. \tag{3.11}$$

Caughey criterion yields the following condition:

$$\langle e_1^2\rangle \to \min_{\alpha}; \text{ and } \langle e_2^2\rangle \to \min_{\beta}; \Rightarrow \left\langle e_1\frac{\partial e_1}{\partial\alpha}\right\rangle = 0; \text{ and } \left\langle e_2\frac{\partial e_2}{\partial\beta}\right\rangle = 0. \tag{3.12}$$

Expanding the condition (3.12), one gets:

$$\begin{aligned}\alpha &= 90a_5\langle x_1^2\rangle^2 + 12a_4\langle x_1^2\rangle,\\ \beta &= 90a_8\langle x_3^2\rangle^2 + 12a_7\langle x_3^2\rangle.\end{aligned} \tag{3.13}$$

The closure-equation system is obtained by the combination of (3.11) and (3.13):

$$\begin{aligned}90a_5\langle x_1^2\rangle_G^3 + 12a_4\langle x_1^2\rangle_G^2 + 2a_3\langle x_1^2\rangle_G - \frac{1}{a_1} = 0,\\ 90a_8\langle x_3^2\rangle_G^3 + 12a_7\langle x_3^2\rangle_G^2 + 2a_6\langle x_3^2\rangle_G - \frac{1}{a_1} = 0.\end{aligned} \tag{3.14}$$

Solve the equation system (3.14), one gets Caughey solution with attention that only positive real solution of (3.14) to be taken.

LOMSEC criterion yields the following condition:

$$\left[e_1\frac{\partial e_1}{\partial\alpha}\right]_{-r\sigma_{x1}}^{+r\sigma_{x1}} = 0; \quad \text{and} \quad \left[e_2\frac{\partial e_2}{\partial\beta}\right]_{-r\sigma_{x3}}^{+r\sigma_{x3}} = 0, \tag{3.15}$$

where $(-r\sigma_{xi}, +r\sigma_{xi})$ are the expected integration domains, σ_{xi} are square roots of variances of components x_i $(i = 1, 3)$, r is a given positive value.

Some important formulas for LOMSEC process should be recalled here [10-11]:

$$\begin{aligned}&\left[x^{2n}\right]_{-r\sigma_x}^{r\sigma_x} = T_{n,r}\langle x^2\rangle^n \quad \text{with} : n = 1, 2, \ldots;\\ &T_{n,r} = \int_{-r}^{r} t^{2n}n(t)dt; \quad n(t) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{t^2}{2}\right\}.\end{aligned} \tag{3.16}$$

Expanding the condition (3.15) and applying (3.16), one gets:

$$\begin{aligned}\alpha &= 6a_5H_r\langle x_1^2\rangle^2 + 4a_4K_r\langle x_1^2\rangle,\\ \beta &= 6a_8H_r\langle x_3^2\rangle^2 + 4a_7K_r\langle x_3^2\rangle,\end{aligned} \tag{3.17}$$

where

$$K_r = \frac{\int_0^r t^4 n(t)dt}{\int_0^r t^2 n(t)dt}; \quad H_r = \frac{\int_0^r t^6 n(t)dt}{\int_0^r t^2 n(t)dt}. \tag{3.18}$$

The closure-equation system is obtained by the combination of (3.11) and (3.17):

$$\begin{aligned} &6a_5 H_r \langle x_1^2 \rangle_{LG}^3 + 4a_4 K_r \langle x_1^2 \rangle_{LG}^2 + 2a_3 \langle x_1^2 \rangle_{LG} - \frac{1}{a_1} = 0, \\ &6a_8 H_r \langle x_3^2 \rangle_{LG}^3 + 4a_7 K_r \langle x_3^2 \rangle_{LG}^2 + 2a_6 \langle x_3^2 \rangle_{LG} - \frac{1}{a_1} = 0. \end{aligned} \tag{3.19}$$

Ideal integration domain $(-r_e, +r_e)$ which gives $\langle x_i^2 \rangle_{LG} = \langle x_i^2 \rangle_e$ can be found by imputing to the equation (3.19) the exact solution $\langle x_i^2 \rangle_e$ obtained from (3.7). Based on the exact values of r_e which depend on the various nonlinearity, we select an expected (reasonable) integration domain $(-\bar{r}, +\bar{r})$ for LOMSEC solution $\langle x_i^2 \rangle_{LG}$.

Consider the system with the following values of the parameters:

$a_1 = a_3 = a_4 = a_6 = 1$; $a_5 = a_7 = a_8 = \varepsilon(0.1 - 100)$; S_{11}, S_{12}, S_{22}, $a_2 =$ arbitrary;

The numerical results of the solutions (the Exact $\langle x_i^2 \rangle_e$, Caughey $\langle x_i^2 \rangle_G$, Lomsec $\langle x_i^2 \rangle_{LG}$) as well as the error evaluations D_G (%) and D_{LG} (%) are given in tables 1 and 2. The dependence of Lomsec solution on the various integration domain as well as the correlation of the solutions (Exact, Caughey, Lomsec) at a given value of nonlinearity ε, for example, here $\varepsilon = 100$ are shown in figures 2.

Table 1. The numerical result of the response x_1

ε	$\langle x_1^2 \rangle_e$	$\langle x_1^2 \rangle_G$	$D_G(\%)$	r_e	$\langle x_1^2 \rangle_{LG}$	$D_{LG}(\%)$
0.1	0.22625	0.20601	-8.946	2.50065	0.23428	3.549
1	0.18974	0.16033	-15.500	2.37346	0.19318	1.813
10	0.12036	0.09250	-23.147	2.22877	0.11741	-2.451
100	0.06318	0.04611	-27.018	2.15897	0.05974	-5.445

Table 2. The numerical result of the response x_3

ε	$\langle x_3^2 \rangle_e$	$\langle x_3^2 \rangle_G$	$D_G(\%)$	r_e	$\langle x_3^2 \rangle_{LG}$	$D_{LG}(\%)$
0.1	0.33940	0.30936	-8.851	2.59215	0.35538	4.708
1	0.18974	0.16033	-15.500	2.37346	0.19318	1.813
10	0.08307	0.06884	-17.130	2.30391	0.08314	0.084
100	0.03073	0.02574	-16.238	2.29574	0.03068	-0.163

The expected integration domain is $\bar{r} = 2.3$; The probability $P(x_i)_{r=2.3} = 0.97855$

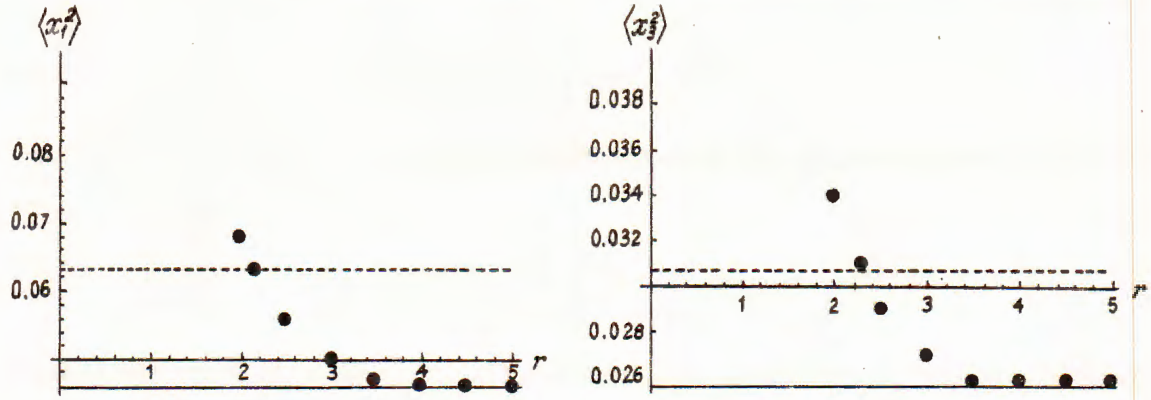

Fig. 2. Graph of the Exact, Caughey and Lomsec solutions with $\varepsilon = 100$.
Denote: ---- Exact, — Caughey, • • • Lomsec

Comments. Lomsec solution accuracy is much more improved than that of Caughey one, especially as strong nonlinearity. The exact solution is always bigger than the Caughey, meanwhile the Lomsec solution varies in accordance with value of the integration domain (r). In Fig. 2. We can see that the curve of the Lomsec solution crosses the exact at a point in accordance with a defined value of r, and approaches to the Caughey in the process of $r \to \infty$.

Example 2. Consider the following nonlinear random two-degree-of-freedom system, which was analyzed by Wen Yao Jia and Tong Fang using an another approximate PDF method [16]:

$$\ddot{x}_i + \beta_i \dot{x}_i + \frac{\partial}{\partial x_i} U(x_i) = w_i(t) \quad i = 1, 2, \tag{3.20}$$

where

$$U(x_i) = \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2 + \lambda_1 x_1^4 + \lambda_3 x_1^2 x_2^2 + \lambda_5 x_2^4. \tag{3.21}$$

Under the following assumptions:

$$\begin{aligned} &\langle w_i(t)\rangle = 0; \quad i = 1, 2, \\ &\langle w_i(t) w_j(t+\tau)\rangle = 2\pi k_i \delta_{ij} \delta(\tau); \quad i, j = 1, 2, \\ &\beta_i = R k_i; \quad i = 1, 2. \end{aligned} \tag{3.22}$$

The excitation in (3.20) can be rewritten using (3.22):

$$w_i(t) = \sqrt{2\pi k_i}\, \xi_i(t) = \sqrt{2\pi \frac{\beta_i}{R}}\, \xi_i(t). \tag{3.23}$$

The FPK equation corresponding to the system (3.20) has an exact solution for the stationary PDF:

$$f(x_1, x_2) = C \exp\Big\{ -\frac{R}{\pi} U(x_1, x_2) \Big\}, \tag{3.24}$$

where C is determined by the normalization condition, i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2) dx_1 dx_2 = 1; \Rightarrow C = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Big\{ -\frac{R}{\pi} U(x_1, x_2) \right]^{-1} \tag{3.25}$$

A three-dimensional graphic of the joint-PDF $f(x_1, x_2)$ with a set of the parameters value given $R = 0.5$; $\beta_1 = \beta_2 = 0.1$; $\omega_1 = 2$; $\omega_2 = 4$; $\lambda_1 = \lambda_3 = \lambda_5 = 100$; is shown in Fig. 3.

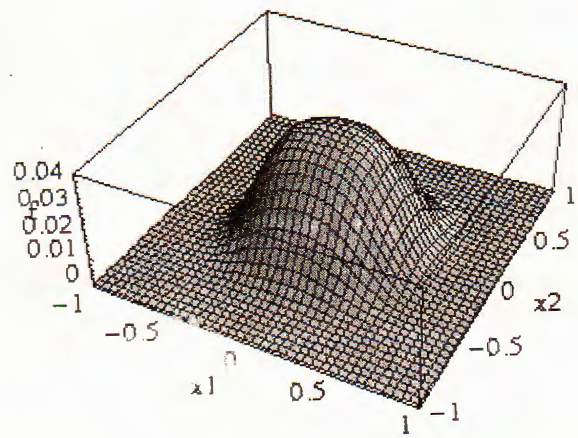

Fig. 3. Three-dimensional graphic of the joint-PDF $f(x_1, x_2)$

The exact solutions in the form of second order moments:

$$\langle x_i^2 \rangle_e = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^2 f(x_1, x_2) dx_1 dx_2 \quad i = 1, 2. \tag{3.26}$$

From (3.20), (3.21), (3.23), the original system can be rewritten:

$$\begin{aligned} \ddot{x}_1 + \beta_1 \dot{x}_1 + \omega_1^2 x_1 + 4\lambda_1 x_1^3 + 2\lambda_3 x_1 x_2^2 &= \sqrt{2\pi \frac{\beta_1}{R}}\, \xi_1(t), \\ \ddot{x}_2 + \beta_2 \dot{x}_2 + \omega_2^2 x_2 + 4\lambda_5 x_2^3 + 2\lambda_3 x_1^2 x_2 &= \sqrt{2\pi \frac{\beta_2}{R}}\, \xi_2(t). \end{aligned} \tag{3.27}$$

The simple linearization process can be applied for the system (3.27) by the following substitutes:

$$
\begin{aligned}
4\lambda_1 x_1^3 + 2\lambda_3 x_1 x_2^2 &= \rho_1 x_1, \\
4\lambda_5 x_2^3 + 2\lambda_3 x_1^2 x_2 &= \rho_2 x_2.
\end{aligned}
\tag{3.28}
$$

The linearized system is governed by the following two-equation system:

$$
\begin{aligned}
\ddot{x}_1 + \beta_1 \dot{x}_1 + (\omega_1^2 + \rho_1) x_1 &= \sqrt{2\pi \frac{\beta_1}{R}}\, \xi_1(t), \\
\ddot{x}_2 + \beta_2 \dot{x}_2 + (\omega_2^2 + \rho_2) x_2 &= \sqrt{2\pi \frac{\beta_2}{R}}\, \xi_2(t).
\end{aligned}
\tag{3.29}
$$

The assumption (3.28) leads to an equation error as follows:

$$
\begin{aligned}
e_1 &= 4\lambda_1 x_1^3 + 2\lambda_3 x_1 x_2^2 - \rho_1 x_1, \\
e_2 &= 4\lambda_5 x_2^3 + 2\lambda_3 x_1^2 x_2 - \rho_2 x_2.
\end{aligned}
\tag{3.30}
$$

The solution of the linearized system (3.29) is found:

$$
\langle x_1^2 \rangle = \frac{\pi}{R(\omega_1^2 + \rho_1)}; \quad \text{and} \quad \langle x_2^2 \rangle = \frac{\pi}{R(\omega_2^2 + \rho_2)}. \tag{3.31}
$$

Caughey criterion yields the following condition:

$$
\left\langle e_1 \frac{\partial e_1}{\partial \rho_1} \right\rangle = 0; \quad \text{and} \quad \left\langle e_2 \frac{\partial e_2}{\partial \rho_2} \right\rangle = 0. \tag{3.32}
$$

Expanding the condition (3.32) and in combination with (3.31), we get a closure-equation system so that gives Caughey solution:

$$
\begin{aligned}
12\lambda_1 \langle x_1^2 \rangle_G^2 + 2\lambda_3 \langle x_1^2 \rangle_G \langle x_2^2 \rangle_G + \omega_1^2 \langle x_1^2 \rangle_G - \frac{\pi}{R} &= 0, \\
12\lambda_5 \langle x_2^2 \rangle_G^2 + 2\lambda_3 \langle x_1^2 \rangle_G \langle x_2^2 \rangle_G + \omega_2^2 \langle x_2^2 \rangle_G - \frac{\pi}{R} &= 0.
\end{aligned}
\tag{3.33}
$$

LOMSEC criterion yields the following condition:

$$
\left[e_1 \frac{\partial e_1}{\partial \rho_1} \right]_{-r\sigma_{x_1}}^{+r\sigma_{x_1}} = 0; \quad \text{and} \quad \left[e_2 \frac{\partial e_2}{\partial \rho_2} \right]_{-r\sigma_{x_2}}^{+r\sigma_{x_2}} = 0 \tag{3.34}
$$

where $(-r\sigma_{xi}, +r\sigma_{xi})$ are the expected integration domains, σ_{xi} are square roots of variances of components x_i $(i = 1, 2)$, r is a given positive value.

Expanding the condition (3.34) in combination with (3.31) and applying (3.16), we get a closure-equation system so that gives LOMSEC solution:

$$\begin{aligned}
&4\lambda_1 K_r \langle x_1^2\rangle_{LG}^2 + 2\lambda_3 H_r \langle x_1^2\rangle_{LG}\langle x_2^2\rangle_{LG} + \omega_1^2\langle x_1^2\rangle_{LG} - \frac{\pi}{R} = 0,\\
&4\lambda_5 K_r \langle x_2^2\rangle_{LG}^2 + 2\lambda_3 H_r \langle x_1^2\rangle_{LG}\langle x_2^2\rangle_{LG} + \omega_2^2\langle x_2^2\rangle_{LG} - \frac{\pi}{R} = 0,
\end{aligned} \tag{3.35}$$

where

$$K_r = \frac{\int_0^r t^4 n(t)dt}{\int_0^r t^2 n(t)dt}; \quad H_r = \frac{\int_0^r t^2 n(t)dt}{\int_0^r n(t)dt}. \tag{3.36}$$

The ideal integration domain $(-r_e, +r_e)$ which gives $\langle x_i^2\rangle_{LG} = \langle x_i^2\rangle_e$, and the expected integration domain $(-\bar{r}, +\bar{r})$ for LOMSEC solution are determined by the way as presented in example 1.

Consider the system with the following values of the parameters:
$R = 0.5$; $\beta_1 = \beta_2 = 0.1$; $\omega_1 = 2$; $\omega_2 = 4$; $\lambda_1 = \lambda_3 = \lambda_5 = \mu(0.1 - 100)$.

The numerical results of the solutions as well as the error evaluations are given in tables 3 and 4. The dependence of Lomsec solution on the various integration domain and the correlation of the solutions (Exact, Caughey, Lomsec) at a given value of nonlinearity μ, for example, here $\mu = 100$ are shown in figures 4.

Table 3. The numerical result of the response x_1

μ	$\langle x_1^2\rangle_e$	$\langle x_1^2\rangle_G$	$D_G(\%)$	r_e	$\langle x_1^2\rangle_{LG}$	$D_{LG}(\%)$
0.1	1.17821	1.15140	-2.275	2.86613	1.20360	2.155
1	0.60378	0.55671	-7.796	2.53319	0.60607	0.379
10	0.22519	0.20077	-10.844	2.42279	0.22169	-1.554
100	0.07462	0.06591	-11.673	2.39088	0.07300	-2.171

Table 4. The numerical result of the response x_2

μ	$\langle x_2^2\rangle_e$	$\langle x_2^2\rangle_G$	$D_G(\%)$	r_e	$\langle x_2^2\rangle_{LG}$	$D_{LG}(\%)$
0.1	0.37680	0.37664	-0.042	3.45000	0.37892	0.563
1	0.30640	0.30284	-1.162	2.94430	0.31267	2.046
10	0.16987	0.16015	-5.722	2.59630	0.17212	1.325
100	0.06766	0.06122	-9.518	2.44672	0.06716	-0.739

The expected integration domain $\bar{r} = 2.5$; The probability $P(x_i)_{r=2.5} = 0.98758$

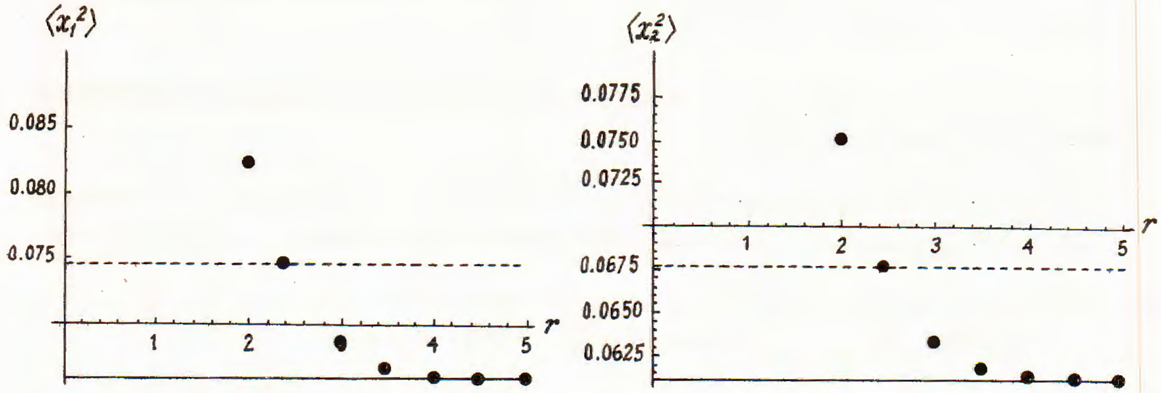

Fig. 4. Graphic of the Exact, Caughey and Lomsec solutions with $\mu = 100$.
Denote: ---- Exact, — Caughey, ● ● ● Lomsec

Comments. The numerical analysis obtained from example 2 leads to the similar comments to example 1.

4. Conclusions

Through the illustrative examples, the obtained results show that Lomsec technique can be efficiently used not only for nonlinear random systems with SDOF [10-11] but also for two-DOF. The most significant advantage of Lomsec technique is to obtain much more improved solutions compared with using Caughey criterion.

A defined value (integration domain) exists, that leads to the exact solution by using Lomsec technique. It means that in principle, it is possible for Lomsec criterion to find exact solution, meanwhile this is impossible for Caughey criterion.

By the way of changing the limitation of integration domain, the Lomsec provides with a lot of different approximate solutions, and as $r = \infty$ the Lomsec gives Caughey solution.

The investigation result leads out a suggestion of fact that it is possible to use an expected value (for example $\bar{r} = 2.5$) for the similar systems. This makes the application more convenient to solve the practical technical problems.

The proposed technique may be extended to other two-DOF systems as well as to MDOF (more two-DOF) systems.

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PHÂN TÍCH GẦN ĐÚNG MỘT SỐ HỆ NGẪU NHIÊN PHI TUYẾN HAI BẬC TỰ DO BẰNG MỘT PHÁT TRIỂN CỦA TUYẾN TÍNH HÓA TƯƠNG ĐƯƠNG GAUSS

Bài báo trình bày việc phân tích một vài hệ phi tuyến hai bậc tự do chịu kích động ngẫu nhiên dùng tiêu chuẩn sai số bình phương trung bình khu vực - một phát triển của phương pháp tuyến tính hóa tương đương Gauss. Các kết quả nhận được chỉ ra rằng kỹ thuật mới có thể được sử dụng rất hiệu quả không chỉ đối với các hệ một bậc tự do như đã trình bày trong các bài báo trước, mà còn đối với các hệ nhiều bậc tự do. Độ chính xác của lời giải nhận được bởi phương pháp dự kiến được cải thiện đáng kể hơn so với dùng tuyến tính hóa truyền thống. Những kết luận trong bài báo cho thấy rõ ý nghĩa của kỹ thuật mới này.