CALCULATION OF THE MATTER PROPAGATION IN THE RIVER OR OPEN CHANNEL SYSTEM

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ABSTRACT. In this paper we consider the following problems: The existence of solution, the stability of finite difference scheme and the non-negative property of numerical solution.

Introduction

The differential equation describing diffusion process of matter S in a river or open channel is as follows:

$$\frac{\partial S}{\partial t} + v\frac{\partial S}{\partial x} - \frac{1}{\omega}\frac{\partial}{\partial x}\left(\omega P\frac{\partial S}{\partial x}\right) + bS = \phi, \tag{0.1}$$

where

x is the coordinate along the streamed,

t - the time, v - the average velocity,

ω - the area of cross section,

$P \geq \delta_0 > 0$ - the general diffusion coefficient,

$b \geq 0$ - the decay coefficient.

Equation (0.1) is of parabolic type. In order to solve this boundary problem, beside the initial condition at the time $t = 0 : S(x,0) = S^0(x)$, one more boundary condition is needed at every boundary $S(L_i, t) = S_{L_i}(t)$, $(L_1 = x_1 = 0, L_2 = x_N = L)$.

At the inflow boundary L_i the boundary condition will be given by $S(L_i, t) = f_i(t) \geq 0$. At the outflow boundary we will consider that $S(L_i, t)$ is resulted from the transport process or $\left.\dfrac{\partial S}{\partial x}\right|_{L_i} = 0$.

In the river or open channel system (see fig. 1) beside the boundary conditions at the boundary nodes A, B, C, it is necessary to give the adjoint conditions at the internal nodes D, E, F.

These adjoint conditions are resulted from the law of matter conservation and on the supposition that there are no source, no creation, no decay of matter S at the internal nodes (see [5]).

$$\frac{\partial}{\partial t}\int_{\widetilde{V}} S_i d\widetilde{V} + \int_{\widetilde{S}} S_i v_n d\widetilde{S} + \int_{\widetilde{S}} P_i\left(\frac{\partial S}{\partial x}\right)_i d\widetilde{S} = 0,$$

$$\frac{\partial}{\partial t}\int\limits_{\widetilde{V}} S_i d\widetilde{V} + \sum_{i=1}^{I^D} \alpha_i S_i Q_i + \sum_{i=1}^{I^D} \alpha_i \omega_i P_i \left(\frac{\partial S}{\partial x}\right)_i = 0,$$

where

I^D is the set of branches having common internal node D,

Q_i - the discharge at the node D of the river branch i,

v_n - the projection of the velocity vector on the external normal vector of the boundary $\widetilde{S}$,

$$\alpha_i = \begin{cases} 1 & \text{if } D \text{ is right boundary of river branch } i, \\ -1 & \text{if } D \text{ is left boundary of river branch. } i. \end{cases}$$

Suppose that the component $\dfrac{\partial}{\partial t}\displaystyle\int\limits_{\widetilde{V}} S_i d\widetilde{V}$ is very small in comparison with the other ones, and the value S_i^D of the all branches i $(i \in I^D)$ are equal $(S_i^D = S^D)$ then we obtain the following adjoint condition

$$\sum_{i=1}^{I^D} \alpha_i \omega_i P_i \left(\frac{\partial S}{\partial x}\right)_i = 0. \tag{0.2}$$
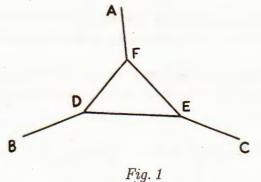
If the diffusion coefficient P does not depend on S, the equation (0.1) is linear. In the opposite cases (0.1) is nonlinear equation and we can linearise (0.1), considering S on the coefficient P as known, using the values S^k at the previous time step t_k.

Equation (0.1) can be written in the following form

$$\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial x} - \frac{\partial}{\partial x}\left(P\frac{\partial S}{\partial x}\right) + bS = \phi, \tag{0.3}$$

where

$$a = v - \frac{P}{\omega}\frac{\partial \omega}{\partial x}, \quad P \geq \delta_0 > 0.$$

Fig. 1

1. Existence of the solution

The linear equation (0.3) with the initial condition $S(x,0) = S^0(x)$ and the boundary conditions $S(L_i, t) = S_{L_i}(t)$ $(i = 1,2)$ has a unique solution, if these conditions and the coefficients of the equation (0.3) are sufficiently smooth.

By the transformation

$$S = Ce^{g(x,t)+Mt}, \quad M = \text{const},$$

we have

$$\frac{\partial S}{\partial t} = e^{g+Mt}\left[\frac{\partial C}{\partial t} + (g_t' + M)C\right],$$
$$\frac{\partial S}{\partial x} = e^{g+Mt}\left(\frac{\partial C}{\partial x} + g_x'C\right), \tag{1.1}$$
$$\frac{\partial}{\partial x}\left(P\frac{\partial S}{\partial x}\right) = e^{g+Mt}\left[\frac{\partial}{\partial x}\left(P\frac{\partial C}{\partial x}\right) + 2Pg_x'\frac{\partial C}{\partial x} + C\frac{\partial}{\partial x}(Pg_x') + PC(g_x')^2\right].$$

Putting (1.1) into (0.3) yields:

$$\frac{\partial C}{\partial t} + (a - 2Pg_x')\frac{\partial C}{\partial x} - \frac{\partial}{\partial x}\left(P\frac{\partial C}{\partial x}\right) + \alpha C = \tilde{\phi}, \tag{1.2}$$

where

$$\begin{cases} \alpha = b + M + g_t' + ag_x' - \dfrac{\partial}{\partial x}(Pg_x') - P(g_x')^2, \\ \tilde{\phi} = \phi e^{-g(x,t)-Mt}. \end{cases}$$

The function $g(x,t)$ will be chosen so that $a - 2Pg_x' = 0$. For this $g_x' = \dfrac{a}{2P}$ or $g(x,t) = \displaystyle\int \frac{a}{2P}dx$ and M will be chosen so that $\alpha \geq 0$. Then equation (1.2) becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(P\frac{\partial C}{\partial x}\right) - \alpha C + \tilde{\phi} \tag{1.3}$$

with the following initial and boundary conditions

$$C(x,0) = C^0(x) = S^0(x)e^{-g(x,0)},$$
$$C(L_i,t) = C_{L_i}(t) = S_{L_i}(t)e^{-g(L_i,t)-Mt}.$$

We use new transformation

$$\overline{S} = C - \frac{L_2 - x}{L_2 - L_1}C_{L_1}(t) - \frac{x - L_1}{L_2 - L_1}C_{L_2}(t). \tag{1.4}$$

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Putting (1.4) into (1.3) we get:

$$\frac{\partial \overline{S}}{\partial t} = \frac{\partial}{\partial x}\left(P\frac{\partial \overline{S}}{\partial x}\right) - \alpha \overline{S} + \overline{\phi}, \qquad (1.5)$$

where

$$\overline{\phi} = \tilde{\phi} - \alpha\left[\frac{L_2 - x}{L_2 - L_1}C_{L_1}(t) + \frac{x - L_1}{L_2 - L_1}C_{L_2}(t)\right]$$
$$+ \frac{C_{L_2}(t) - C_{L_1}(t)}{L_2 - L_1}\frac{\partial P}{\partial x} - \frac{L_2 - x}{L_2 - L_1}C'_{L_1}(t) - \frac{x - L_1}{L_2 - L_1}C'_{L_2}(t).$$

The corresponding initial and boundary conditions of the equation (1.5) are:

$$\overline{S}(x,0) = e^{-g(x,0)}\left(S^0(x) - \frac{L_2 - x}{L_2 - L_1}S_{L_1}(0) - \frac{x - L_1}{L_2 - L_1}S_{L_2}(0)\right) \qquad (1.6)$$
$$\overline{S}(L_i, t) = 0; \quad (i = 1, 2).$$

The boundary problem (1.5), (1.6) with the coefficient $P \geq \delta_0 > 0$ and $\alpha \geq 0$ has a unique solution (see [4]). Therefore the linear equation (0.3) has also a unique solution.

2. Stability of the finite difference scheme

Differencing the equation (0.1) we get:

$$\frac{S_n^{k+1} - S_n^k}{\tau} + \frac{(v + |v|)_n}{2}\frac{S_n^{k+1} - S_{n-1}^{k+1}}{\Delta} + \frac{(v - |v|)_n}{2}\frac{S_{n+1}^{k+1} - S_n^{k+1}}{\Delta} -$$
$$\frac{1}{\omega_n \Delta}\left[(\omega P)_{n+1/2}\frac{S_{n+1}^{k+1} - S_n^{k+1}}{\Delta} - (\omega P)_{n-1/2}\frac{S_n^{k+1} - S_{n-1}^{k+1}}{\Delta}\right] + b_n S_n^{k+1} = \phi_n^{k+1}, \quad (2.1)$$

where $\Delta = x_{n+1} - x_n = \text{const}$, $\tau = t_{k+1} - t_k = \text{const}$.

The equation (2.1) can be rewritten under the form

$$\alpha_n S_{n+1}^{k+1} + \beta_n S_n^{k+1} + \gamma_n S_{n-1}^{k+1} = \delta_n, \qquad (2.2)$$

where

$$\alpha_n = -\frac{\tau(|v| - v)_n}{2\Delta} - \frac{\tau(\omega P)_{n+1/2}}{\omega_n \Delta^2},$$
$$\beta_n = 1 + \tau b_n + \frac{\tau|v|_n}{\Delta} + \frac{\tau(\omega P)_{n+1/2}}{\omega_n \Delta^2} + \frac{\tau(\omega P)_{n-1/2}}{\omega_n \Delta^2},$$
$$\gamma_n = -\frac{\tau(|v| + v)_n}{2\Delta} - \frac{\tau(\omega P)_{n-1/2}}{\omega_n \Delta^2},$$
$$\delta_n = S_n^k + \tau \phi_n^{k+1}.$$

Obviously, the coefficients α_n, β_n and γ_n satisfy the following conditions

$$\alpha_n < 0, \quad \beta_n > 0, \quad \gamma_n < 0,$$
$$\beta_n = |\alpha_n| + |\gamma_n| + \sigma_n, \quad \sigma_n = 1 + \tau b_n > 0. \tag{2.3}$$

In the simple case when $v = \text{const}$, $\omega = \text{const}$, $P = \text{const}$, $b = \text{const}$, (i.e. $\alpha_n = \alpha = \text{const}$, $\beta_n = \beta = \text{const}$ and $\gamma_n = \gamma = \text{const}$) we can prove the stability of this finite difference scheme (2.1).

Indeed, substituting the solution under the form $S_n^k = \lambda^k S^0 e^{in\varphi}$, where $i = \sqrt{-1}$ and $0 \le \varphi < 2\pi$ into the equation (2.2) with $\phi = 0$ we get

$$\lambda^{k+1}\left(\alpha e^{i(n+1)\varphi} + \beta e^{in\varphi} + \gamma e^{i(n-1)\varphi}\right)S^0 = \lambda^k S^0 e^{in\varphi}.$$

Dividing both sides of this expression by $S^0 e^{in\varphi}$ one deduces

$$\lambda^{k+1}(\alpha e^{i\varphi} + \beta + \gamma e^{-i\varphi}) = \lambda^k,$$
$$\lambda^{k+1}\left[\beta + (\alpha + \gamma)\cos\varphi + i(\alpha - \gamma)\sin\varphi\right] = \lambda^k.$$

From inequality $|(\alpha + \gamma)\cos\varphi| \le |\alpha + \gamma| \le |\alpha| + |\gamma|$ it is easy to verify that $\beta + (\alpha + \gamma)\cos\varphi = 1 + \tau b + |\gamma| + |\alpha| + (\alpha + \gamma)\cos\varphi > 1$, therefore

$$|\lambda| = \left|\frac{\lambda^{k+1}}{\lambda^k}\right| = \frac{1}{\sqrt{[\beta + (\alpha + \gamma)\cos\varphi]^2 + (\alpha - \gamma)^2 \sin^2\varphi}} < 1.$$

Let $|S_{n_0}^{k+1}| = \sup_n |S_n^{k+1}|$, from (2.2), (2.3) we have

$$\left|\beta S_{n_0}^{k+1} - |\alpha|S_{n_0+1}^{k+1} - |\gamma|S_{n_0-1}^{k+1}\right| = \left|S_{n_0}^k + \tau \phi_{n_0}^{k+1}\right|,$$
$$\left|S_{n_0}^{k+1}\right|(\beta - |\alpha| - |\gamma|) \le \sup_{2\le n\le N-1} \left|S_n^k\right| + \tau \sup_k \sup_{2\le n\le N-1} |\phi_n^k|,$$
$$\sup_{2\le n\le N-1} \left|S_n^{k+1}\right| \le \frac{1}{1+\tau b}\left[\sup_{2\le n\le N-1} \left|S_n^k\right| + \tau \sup_k \sup_{2\le n\le N-1} |\phi_n^k|\right]$$
$$< \sup_{2\le n\le N-1} \left|S_n^k\right| + \tau \sup_k \sup_{2\le n\le N-1} |\phi_n^k| \le \cdots$$
$$\cdots \le \sup_{2\le n\le N-1} \left|S_n^0\right| + (k+1)\tau \sup_k \sup_{2\le n\le N-1} |\phi_n^k|. \tag{2.4}$$

Denoting $\left\|S_h\right\|_h = \sup_k \sup_{1\le n\le N} |S_n^k|$, $\left\|\phi_h\right\|_h = \sup_k \sup_{2\le n\le N-1} |\phi_n^k|$,
$\left\|S_h^0\right\|_h = \sup_{1\le n\le N} |S_n^0|$, $\left\|S_{L_i}\right\|_h = \sup_k |S_{L_i}^k|$, from (2.4) one deduces

$$\left\|S_h\right\|_h \le \left\|S_h^0\right\|_h + T \cdot \left\|\phi_h\right\|_h + \left\|S_{L_i}\right\|_h.$$

It mean that the finite difference scheme (2.1) is stable unconditionally with respect to τ/Δ.

3. Calculation of the matter propagation on a river and the non-negative property of the numerical solution

Let the function $\phi(x, t)$, boundary condition and initial condition $S^0(x)$ be non-negative. We shall prove that the matter concentration S_n^k on a river is non-negative. For this we suppose that $S_n^k \geq 0$ and prove $S_n^{k+1} \geq 0$.

1) At the left boundary L_1:

i) We have the given boundary condition $S_{L_i}(t) = f_1(t) \geq 0$ if the flow is inflow (i.e. $v_1 \geq 0$), therefore

$$S_1^{k+1} = f_1(t_{k+1}) \geq 0.$$

ii) If the flow is outflow ($v_1 < 0$) we determine the value S as follows: Suppose that there are no source, no process of matter diffusion, creation, decay at the boundary, then we obtain the equation

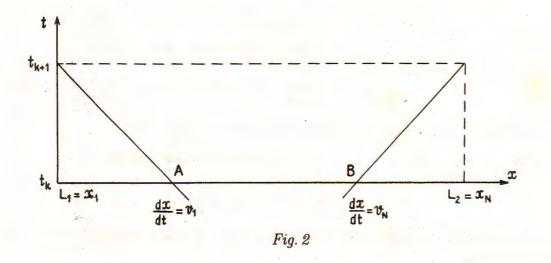$$\frac{\partial S}{\partial t} + v_1 \frac{\partial S}{\partial s} = 0. \tag{3.1}$$

Equation (3.1) can be solved by the following two methods:

- Method of characteristics (see fig. 2)
    * Determine the coordinate of the point A: $x_A = x_1 - v_1 \tau$
    * Determine the index n so that: $x_{n-1} < x_A < x_n$
    * Calculate value S_A^k by linear interpolation

$$S_A^k = S_{n-1}^k + \frac{x_A - x_{n-1}}{x_n - x_{n-1}}(S_n^k - S_{n-1}^k).$$

   * Take $S_1^{k+1} = S_A^k$ \hfill (3.2)

Fig. 2

- Difference method:
  * Determine the least index n so that

$$x_n - x_1 > \tau |v_1|,$$

$$\frac{S_1^{k+1} - S_1^k}{\tau} + v_1 \frac{S_n^k - S_1^k}{x_n - x_1} = 0,$$

$$S_1^{k+1} = \left(1 - \frac{\tau |v_1|}{x_n - x_1}\right) S_1^k + \frac{\tau |v_1|}{x_n - x_1} S_n^k. \tag{3.3}$$

Obviously by both methods we get $S_1^{k+1} \geq 0$ if $S_n^k \geq 0$.

2) At the right boundary L_2:

i) We have also the given boundary condition $S_{L_2}(t) = f_2(t) \geq 0$ if the flow is inflow ($v_N \leq 0$). Therefore

$$S_N^{k+1} = f_2(t_{k+1}) \geq 0.$$

ii) If the flow is outflow ($v_N > 0$) the value S_N^{k+1} may be calculated also from the equation

$$\frac{\partial S}{\partial t} + v_N \frac{\partial S}{\partial x} = 0$$

by two methods:

- Method of characteristics (see Fig. 2)

$$S_N^{k+1} = S_B^k. \tag{3.4}$$

- Difference method

$$S_N^{k+1} = \left(1 - \frac{\tau v_N}{x_N - x_n}\right) S_N^k + \frac{\tau v_N}{x_N - x_n} S_n^k, \tag{3.5}$$

where the maximum index n is chosen so that $\tau v_N < x_N - x_n$. We get also $S_N^{k+1} \geq 0$ if $S_n^k \geq 0$.

3) At the interior points x_n ($n = 2, 3, \ldots, N-1$)

Equation system (2.2) with the coefficients satisfying the condition (2.3) has the unique solution (see [1]).

System (2.2) is solved by the double sweep method

$$S_{n+1}^{k+1} = \mathcal{L}_{n+1} S_n^{k+1} + r_{n+1}, \tag{3.6}$$

where

$$\mathcal{L}_{n+1} = \frac{-\gamma_{n+1}}{\beta_{n+1} + \alpha_{n+1}\mathcal{L}_{n+2}}; \quad \mathcal{L}_N = 0,$$

$$r_{n+1} = \frac{\delta_{n+1} - \alpha_{n+1} r_{n+2}}{\beta_{n+1} + \alpha_{n+1}\mathcal{L}_{n+2}}; \quad r_N = S_N^{k+1}$$

and the computational error is not accumulated (see [2], [3]). From (3.6) we have

$$\mathcal{L}_N = 0 \quad \text{and} \quad r_N = S_N^{k+1} \geq 0.$$

Using inductive method we can prove that

$$0 < \mathcal{L}_n < 1 \quad \text{and} \quad r_n \geq 0 \quad (n = 2, \ldots, N-1).$$

Indeed, suppose that

$$0 \leq \mathcal{L}_{n+1} < 1 \quad \text{and} \quad r_{n+1} \geq 0.$$

From (2.2), (2.3) and $\phi(x,t) \geq 0$ it yields $\delta_n \geq 0$ and

$$\beta_n + \alpha_n \mathcal{L}_{n+1} = \beta_n - |\gamma_n| - |\alpha_n| + |\gamma_n| + |\alpha_n| - |\alpha_n|\mathcal{L}_{n+1}$$
$$= \sigma_n + |\gamma_n| + |\alpha_n|(1 - \mathcal{L}_{n+1}) > |\gamma_n|$$

therefore

$$0 \leq \mathcal{L}_n = \frac{-\gamma_n}{\beta_n + \alpha_n \mathcal{L}_{n+1}} = \frac{|\gamma_n|}{\beta_n + \alpha_n \mathcal{L}_{n+1}} < \frac{|\gamma_n|}{|\gamma_n|} = 1,$$
$$r_n = \frac{\delta_n - \alpha_n r_{n+1}}{\beta_n + \alpha_n \mathcal{L}_{n+1}} = \frac{\delta_n + |\alpha_n| r_{n+1}}{\beta_n + \alpha_n \mathcal{L}_{n+1}} \geq 0. \tag{3.7}$$

From (3.7) and (3.6) we get

$$S_{n+1}^{k+1} = \mathcal{L}_{n+1} S_n^{k+1} + r_{n+1} \geq 0.$$

If the boundary condition at the outflow boundary L_i is $\dfrac{\partial S}{\partial x}\bigg|_{L_i} = 0$, then from this boundary condition and (3.6) one deduces:

a) For the case L_2 is outflow and L_1 is inflow boundaries:

$$\mathcal{L}_N = 1, \quad r_N = 0 \quad \text{and} \quad S_1^{k+1} = f_1(t_{k+1}) \geq 0.$$

b) For the case L_1 is outflow and L_2 is inflow boundaries:

$$\mathcal{L}_N = 0, \quad r_N = f_2(t_{k+1}) \quad \text{and} \quad S_1^{k+1} = \frac{r_2}{1 - \mathcal{L}_2} \geq 0.$$

Conclusion. Any finite difference scheme with the coefficients α_n, β_n, γ_n of the equation (2.2) satisfying the condition (2.3) is unconditional stable and the numerical solution is not negative.

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4. Calculation of the matter propagation on a river system

We divide the nodes of a river system into two kinds: the boundary nodes and the internal nodes.

1) At the boundary node A, B, C (see Fig. 1)

The value $S_{1_i}^{k+1}$ or $S_{N_i}^{k+1}$ is calculated by the formulas (3.2), (3.3) or (3.4), (3.5), where $i \in I$, I is the set of all boundary nodes.

2) At the internal nodes D, E, F

We shall calculate simultaneously the values S_j^{k+1} of the all internal nodes $j \in J$, J is the set of internal nodes.

Differencing the adjoint condition (0.2) at the node D, we get

$$\sum_{i=1}^{I^D} \left(\frac{\omega P}{\Delta} \right)_i^D (S^D - S_i^*) = 0, \tag{4.1}$$

where $S_i^* = S_{m,i}^{k+1}$

$$m = \begin{cases} 2 & \text{if } D \text{ is left boundary of branch } i \\ N-1 & \text{if } D \text{ is right boundary of branch } i. \end{cases}$$

3) At the interior points of the river branch i the values $S_{n_i}^{k+1}$ are calculated by the following double sweep method

$$S_{n+1}^{k+1} = \widetilde{\mathcal{L}}_{n+1} S_n^{k+1} + \widetilde{r}_{n+1} S_N^{k+1} + R_{n+1}, \tag{4.2}$$

where

$$\widetilde{\mathcal{L}}_{n+1} = \frac{-\gamma_{n+1}}{\beta_{n+1} + \alpha_{n+1} \widetilde{\mathcal{L}}_{n+2}}; \quad \widetilde{\mathcal{L}}_N = 0,$$

$$\widetilde{r}_{n+1} = \frac{-\alpha_{n+1} \widetilde{r}_{n+2}}{\beta_{n+1} + \alpha_{n+1} \widetilde{\mathcal{L}}_{n+2}}; \quad \widetilde{r}_N = 1,$$

$$R_{n+1} = \frac{\delta_{n+1} - \alpha_{n+1} R_{n+2}}{\beta_{n+1} + \alpha_{n+1} \widetilde{\mathcal{L}}_{n+2}}; \quad R_N = 0.$$

From (4.2) with $n = 1$ it yields

$$S_2^{k+1} = \widetilde{\mathcal{L}}_2 S_1^{k+1} + \widetilde{r}_2 S_N^{k+1} + R_2 = \nu_2 S_1^{k+1} + \mu_2 S_N^{k+1} + \theta_2,$$

where $\nu_2 = \widetilde{\mathcal{L}}_2$, $\mu_2 = \widetilde{r}_2$, $\theta_2 = R_2$. We have in general

$$S_n^{k+1} = \nu_n S_1^{k+1} + \mu_n S_N^{k+1} + \theta_n, \tag{4.3}$$

where

$$\nu_n = \widetilde{\mathcal{L}}_n \nu_{n-1}; \qquad\qquad \nu_1 = 1,$$
$$\mu_n = \widetilde{r}_n + \widetilde{\mathcal{L}}_n \mu_{n-1}; \qquad\qquad \mu_1 = 0,$$
$$\theta_n = R_n + \widetilde{\mathcal{L}}_n \theta_{n-1}; \qquad\qquad \theta_1 = 0.$$

From (4.3) one deduces

$$S_{m,i}^{k+1} = \nu_{m,i} S_{1,i}^{k+1} + \mu_{m,i} S_{N,i}^{k+1} + \theta_{m,i}. \tag{4.4}$$

Putting (4.4) into (4.1) we get

$$\sum_{i=1}^{I^D} \left(\frac{\omega P}{\Delta}\right)_i^D \left(S^D - \nu_{m,i} S_{1,i}^{k+1} - \mu_{m,i} S_{N,i}^{k+1} - \theta_{m,i}\right) = 0. \tag{4.5}$$

One of the values $S_{1,i}^{k+1}$ or $S_{N,i}^{k+1}$ will be matter concentration S at the node D (it depends on that D is left or right boundary of river branch i). The other one will be concentration S at the second boundary of river branch i (for example at boundary node A, or internal node E, or internal node F).

Equation (4.5) is a linear equation describing the correlation of the matter concentrations S at $(I^D + 1)$ nodes of the river system.

At each internal node there is always one linear equation like (4.5). Therefore we get the linear algebraic equation system:

$$AS = c \tag{4.6}$$

having the order equal to the number of internal modes and the unknowns are the matter concentrations S at the mentioned nodes. Solving the equation system (4.6) we get the matter concentrations S at all of the internal nodes of the river system.

4) At the interior point x_n of each river branch i

We shall calculate the values $S_{n,i}^{k+1}$ $(n = 2, 3, \ldots, N-1)$ by the formula (4.2) or (4.3) or (3.6).

Remark. At each internal node (for example, node D), the values $S_{D_i}^{k+1}$ may be calculated separately. We have two methods for calculating these values.

1. From the adjoint condition (0.2) one deduces

$$S_{D_i}^{k+1} = S_D^{k+1} = \frac{\displaystyle\sum_{i=1}^{I^D} \left(\frac{\omega P}{\Delta}\right)_i S_i^*}{\displaystyle\sum_{i=1}^{I^D} \left(\frac{\omega P}{\Delta}\right)_i},$$

where

$$S_i^* = \begin{cases} S_{2,i}^{k+1}, & \text{if } D \text{ is left boundary of river branch } i \\ S_{N-1,i}^{k+1}, & \text{if } D \text{ is right boundary of river branch } i \end{cases}$$

and $S_{2,i}^{k+1}$, $S_{N-1,i}^{k+1}$ may be determine by the method of characteristic or explicit finite difference method.

2) Using other adjoint condition (see [7], [8]), we have

$$S_{D_j}^{k+1} = \frac{\sum\limits_{i \in I_v^D} |Q_{D_i}| S_{D_i}^{k+1}}{\sum\limits_{j \in I_r^D} |Q_{D_j}|}, \quad j \in I_r^D,$$

where I_v^D is the set of river branches having inflow at the node D, I_r^D - the set of branch having outflow at D, $I^D = I_v^D + I_r^D$, Q_{D_i} is the water discharge of branch i at D.

The values $S_{D_i}^{k+1}$, $i \in I_v^D$ are calculated by the method of characteristic.

The above algorithm was used for calculating the matter propagation in the DONGNAI-SAIGON river system (see [5]), where v is computed from the one dimensional Saint-Venant equation system (see [6])
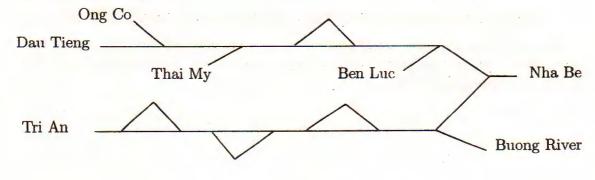
$$B\frac{\partial Z}{\partial t} + \frac{\partial Q}{\partial x} = q,$$

$$\frac{\partial Q}{\partial t} + 2v\frac{\partial Q}{\partial x} + B(c^2 - v^2)\frac{\partial Z}{\partial x} = \left[iB + \left(\frac{\partial \omega}{\partial x}\right)_h\right]v^2 - \frac{g\omega Q|Q|}{K^2}$$

by using the difference scheme

$$\frac{\partial f}{\partial t}\bigg|_{n+1/2}^{k+1} = \frac{f_{n+1}^{k+1} + f_n^{k+1} - f_{n+1}^k - f_n^k}{2\tau},$$

$$\frac{\partial f}{\partial x}\bigg|_{n+1/2}^{k+1} = \frac{f_{n+1}^{k+1} - f_n^{k+1}}{\Delta}.$$

And the DONGNAI-SAIGON river system was modelled as follows

Ong Co

Dau Tieng

Thai My

Ben Luc

Nha Be

Tri An

Buong River

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TÍNH LAN TRUYỀN VẬT CHẤT TRÊN HỆ THỐNG SÔNG HOẶC KÊNH HỞ

Bài báo trình bày các vấn đề sau: Sự tồn tại nghiệm của bài toán, sự ổn định của sơ đồ sai phân, tính chất không âm của nghiệm bằng số khi các điều kiện đầu và điều kiện biên không âm.