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A NOTE ON THE METHOD OF HARMONIC BALANCE

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ABSTRACT. Free oscillation period of the Duffing oscillator with cubic non-linearity was examined. A comparison between the exact period and those obtained by the asymptotic and the harmonic balance methods was done. It was shown that the results given by the harmonic balance are acceptable even for large oscillations whereas the asymptotic method can only be applied for small oscillations.

Introduction

The perturbation methods (the Poincaré, the asymptotic and the multiple scale methods) can only be applied to weakly non-linear systems. This well-known remark has been verified in [1] through a comparison between the exact free oscillation period of the Duffing oscillator (1.1) and that given by the Poincaré method.

The present article deals with the same question but for the asymptotic and the harmonic balance methods; the latter is often used for studying strong non-linear systems although its accuracy needs to be carefully examined. It will be shown that, for large oscillations, the results obtained from the asymptotic method are to be rejected whereas those of the harmonic balance method are acceptable.

§1. The system under consideration and the exact period

Consider the Duffing oscillator described by the differential equation:

$$\ddot{x} + x + \varepsilon x^3 = 0, \tag{1.1}$$

where x is an oscillatory variable, overdots denote the derivatives with respect to time t; 1 is the linear frequency; ε is a positive parameter.

The free oscillation with the initial conditions

$$x(t)\big|_{t=0} = x_0, \qquad \dot{x}(t)\big|_{t=0} = 0 \tag{1.2}$$

has the period [2]:

$$T = \frac{4}{\sqrt{1 + A_0}} \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \tag{1.3}$$

where $A = \varepsilon x_0^2$, $m = \dfrac{A_0}{2(1+A_0)}\cdot$

§2. Free oscillation period from the asymptotic solution

As well-known, the asymptotic method [3] can only be applied if ε is small enough. Here, to make necessary comparison, the asymtotic period is given for arbitrary ε.

The expansion of the asymptotic solution is of the form:

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \dots \tag{2.1}$$

$$\begin{cases} \dot{a} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots \\ \omega &= \dot{\psi} = 1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots \end{cases} \tag{2.2}$$

where: a is the amplitude of the first harmonic; ψ is the phase angle; ω is the frequency to be calculated; $u_i(a,\psi)$ $(i = 1, 2, \dots)$ are functions of (a,ψ), 2π-periodic in ψ but do not contain the first harmonics; $A_(a)$ and $B_i(a)$ $(i = 1, 2, \dots)$ are functions of a.

Substituting (2.1), (2.2) into (1.1) and vanishing the terms of like powers of ε, then those of like harmonics yield:

$$A_1 = 0, \qquad B_1 = \frac{3}{8}a^2, \quad u_1 = \frac{a^3}{32}\cos 3\psi, \tag{2.3}$$

$$A_2 = 0. \qquad B_2 = -\frac{15a^4}{256}\cdot \tag{2.4}$$

Thus

- To the first approximation:

$$\begin{aligned} &\dot{a} = 0 \quad \text{i.e.} \quad a = \text{const}, \\ &\omega = \dot{\psi} = 1 + \frac{3}{8}\varepsilon a^2 \quad \text{i.e.} \quad \psi = \Big(1 + \frac{3}{8}\varepsilon a^2\Big)t, \\ &x_1 = a\cos\psi \quad \text{is the first approximation solution,} \\ &x_{1*} = a\cos\psi + \frac{\varepsilon a^3}{32}\cos 3\psi \quad \text{is the refinement of the} \\ &\qquad\qquad \text{first approximation solution.} \end{aligned} \tag{2.5}$$

- To the second approximation

$$\begin{aligned} &\dot{a} = 0 \quad \text{i.e.} \quad a = \text{const}, \\ &\omega = \dot{\psi} = 1 + \frac{3}{8}\varepsilon a^2 - \frac{15}{256}\varepsilon^2 a^4 \quad \text{i.e.} \quad \psi = \Big(1 + \frac{3}{8}\varepsilon a^2 - \frac{15}{256}\varepsilon^2 a^4\Big)t, \\ &x_2 = a\cos\psi + \frac{\varepsilon a^3}{32}\cos 3\psi \quad \text{is the second approximation solution.} \end{aligned} \tag{2.6}$$

Note that, corresponding to the solutions x_1, x_{1*}, x_2, the initial abscissa x_0 can be expressed as:

$$x_0 = x_1(0) = a, \quad x_0 = x_{1*}(0) = a + \frac{\varepsilon a^3}{32}, \quad x_0 = x_2(0) = a + \frac{\varepsilon a^3}{32}. \tag{2.7}$$

Hence, the formulas of the periods are:

- for the first approximation solution:

$$T_1 = \frac{2\pi}{\omega} = \frac{2\pi}{1 + \frac{3}{8}A} \quad \text{with} \quad A = \varepsilon a^2 = \varepsilon x_0^2 = A_0, \tag{2.8}$$

- for the refinement of the first approximation solution

$$T_{1*} = \frac{2\pi}{1 + \frac{3}{8}A} \quad \text{with } A \text{ satisfying} \quad A_0 = A\Big(1 + \frac{A}{32}\Big)^2, \tag{2.9}$$

- for the second approximation solution:

$$T_2 = \frac{2\pi}{1 + \frac{3}{8}A - \frac{15}{256}A^2} \quad \text{with } A \text{ satisfying} \quad A_0 = A\Big(1 + \frac{A}{32}\Big)^2. \tag{2.10}$$

§3. Free oscillation period from the harmonic balance solution

We apply now the harmonic balance method to solve the question stated. The one-component solution takes the form:

$$x = a\cos\omega t, \quad a = \text{const}, \quad \omega = \text{const to be calculated.} \tag{3.1}$$

Substituting (3.1) into (1.1) and vanishing the harmonic $\cos\omega t$ we obtain:

$$\omega^2 = 1 + \frac{3}{4}\varepsilon a^2. \tag{3.2}$$

Since $x_0 = a$, the period of the one-component solution is given by:

$$\widetilde{T}_1 = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 + \frac{3}{4}A}}, \tag{3.3}$$

where $A = \varepsilon a^2 = \varepsilon x_0^2 = A_0$.

The two-component solution is:

$$x = a\cos\omega t + b\cos 3\omega t, \tag{3.4}$$

where: $a = \text{const}$, $b = \text{const}$, $\omega = \text{const}$ to be calculated.

Again, substituting (3.4) into (1.1) and vanishing the terms $\cos\omega t$, $\cos 3\omega t$, we obtain respectively

$$\begin{cases} (1-\omega^2)a + \frac{3\varepsilon}{4}(a^3 + a^2 b + 2ab^2) &= 0, \\ (1-9\omega^2)b + \frac{\varepsilon}{4}(a^3 + 6a^2 b + 3b^3) &= 0. \end{cases} \tag{3.5}$$

Letting $b = \beta a$, we rewrite (3.5) as:

$$\omega = \sqrt{1 + \frac{3}{4}A(1 + \beta + 2\beta^2)}\,, \tag{3.6}$$

$$51A\beta^3 + 27A\beta^2 + (21A + 32)\beta - A = 0. \tag{3.7}$$

Since $x_0 = a + b = a(1+\beta)$, the relation between $A = \varepsilon a^2$ and $A_0 = \varepsilon x_0^2$ is:

$$A_0 = \varepsilon x_0^2 = \varepsilon a^2(1+\beta)^2 = A(1+\beta)^2. \tag{3.8}$$

Hence, the equation (3.7) can be rewritten as:

$$(51A_0 + 32)\beta^3 + (27A_0 + 64)\beta^2 + (21A_0 + 32)\beta - A_0 = 0. \tag{3.9}$$

Finally, the period of the two-component solution can be expressed as:

$$\widetilde{T}_2 = \frac{2\pi}{\sqrt{1 + \frac{3}{4}A(1 + \beta + 2\beta^2)}}\,, \tag{3.10}$$

where β, A - as functions of A_0 - must be determined from (3.8), (3.9).

§4. Concluding remarks

The periods obtained from the presented methods are given below in the following table

A_0	T	T_1	T_{1*}	T_2	$\widetilde{T}_1$	$\widetilde{T}_2$
0	6.28319	6.28319	6.28319	6.28319	6.28319	6.28319
0.01	6.25796	6.25971	6.25973	6.25976	6.25976	6.25976
0.1	6.06066	6.05608	6.05744	6.06082	6.06004	6.06065
1.0	4.76802	4.56959	4.64106	4.82704	4.74964	4.76736
4	3.17971	2.51327	2.81326	3.92813	3.14159	3.17724
10	2.19179	1.32278	1.76963	7.47998	2.15511	2.18897
40	1.15182	0.39270	0.85044	-0.65345	1.12849	1.14986
100	0.73626	0.16320	0.54232	-0.17898	0.72073	0.73491
1000	0.23434	0.01671	0.20106	-0.01795	0.22928	0,23390

It can be seen that even for large oscillations, the harmonic balance solution is acceptable: for $A_0 = 1000$, the relative error of $\widetilde{T}_1$ is of order 2% and that of $\widetilde{T}_2$ is smaller than $2^\circ/_{\circ\circ}$.

Another remark: for small oscillations, the asymptotic solution is acceptable and the second approximation T_2 is more accurate than the first ones (T_1 and T_{1*}); for large oscillations, as well-known the conclusion is the reverse and the second approximation becomes irrational ($T_2 \to \infty$ as $A_0 \to A_{0*} \approx 13.44659$ and $T_2 < 0$ as $A_0 > A_{0*}$).

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GHI CHÚ VỀ PHƯƠNG PHÁP CÂN BẰNG ĐIỀU HÒA

Xét chu kỳ dao động tự do của chấn tử Duffing với yếu tố phi tuyến bậc ba. Chu kỳ tính theo phương pháp tiệm cận và phương pháp cân bằng điều hòa được so sánh với chu kỳ chính xác. Nhận thấy kết quả thu được từ phương pháp cân bằng điều hòa được chấp nhận trong khi phương pháp tiệm cận - như đã biết - chỉ có thể áp dụng cho các dao động nhỏ.