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BODY PARAMETERS WITH APPLICATION OF OPTIMAL DIFFERENTIAL VARIATION METHOD IN SUPER CAVITY MODEL

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Abstract. When a slender body moves very fast through water at sufficient speed the cavity phenomena is happened. In a cavity model the bodys lengths and cavitator diameter are determined by the differential variation optimal method so that the velocity of body is maximum.

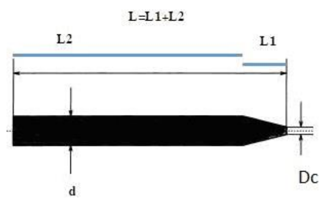
Keywords: Super cavity, Runge-Kutta methods, data assimilation.

1. INTRODUCTION

In hydrodynamics applications cavitation is the appearance of vapor bubbles and pockets inside homogeneous liquid medium. This phenomenon occurs because the pressure is reduced to the vapor pressure limit. We will study super cavity appearing by the very fast movement of slender body in water that makes uncontrolled gun-launched slender body. Except the body head called by cavitator is directly touching with water, the gas layer can be covered partial or full body depending on the design of body form. The body rotates about its nose. In this paper for simple calculation we choose cavitator formed by the plate disk with diameter (see Fig. 1). The body is consisted of two parts: the cone top and cylinder part with the diameter d.

In the super cavity model the following assumptions are (see [1]):

- The motion of the projectile is confined to a plane;
- The slender body rotates about its nose;
- The effect of gravity on the dynamics of this body is negligible;
- The motion of the slender body is not influenced by the presence of gas, water vapor or water drops in the cavity;



- L is the length of the slender body
- L_2 is the body's length of cylinder part
- L_1 is the body's length of cone top part
- d is the body's diameter
- D_c is the body's cavitator diameter

Fig. 1. Slender body geometery

The super cavity problems are studied in ([1–8]).

The Data Assimilation method using differential variation is based on the theory of optimal control for partial differential equation (Lions, 1971 see [9–11]). This method is applied to correct and find coefficients, solve the inverse problems, simulate the air and fluid pollution processes (see [12–18]). In this paper by applying the differential variation optimal method we will concentrate the study on the identification parameters L_1, L_2, D_c so that the distance way running by body is maximum.

2. GENERAL VARIATIONAL APPROACH

By the differential variation optimal method the length and cavitator diameter parameters L_1, L_2, D_c will be evaluated as the solution of an "Inverse Problem", basically as the solution of an optimization problem. The advantage is that there exist many efficient algorithms for solving this problem. Most of them require to compute the gradient of the function to be minimized. The cost function is done by solving an "Adjoin Model". The method is described in many papers together with the computational developments (see [9–18]). It can be summarized as follows:

Consider the mathematical equation of physic process that is described by the evolution problem: Let $X(t)$ the state vector describing the evolution of a system governed by the abstract equation

$$\frac{dX}{dt} = F\left(X, E_1, \ldots, E_n\right), \quad X\left(0\right) = X_0\,, \tag{1}$$

where $E_1, \ldots, E_n$ are the equation parameters; $X(t)$ is a unknown state vector belonging for any t to a Hilbert space $\Im$, $X_0 \in \Im$; F is a nonlinear operator mapping $Y \times Y_p$ to Y with $Y = L_2\left(0, T, \Im\right), \|.\|_Y = (.,.)_Y^{1/2}$, Y_p is Hilbert space (the space of model's parameters); Suppose that for given initial value $X(0) = X_0 \in \Im$ and $\left(E_1, \ldots, E_n\right) \in Y_p$ there exists a unique solution $X \in \Im$ to (1). To choose $E_1, \ldots, E_n$ so that the function $\int\limits_0^T H(X)dt$ is getting maximum in running process $(0, T)$ we introduce the functional called cost function

$$J\left(E_1, \ldots, E_n\right) = -\int\limits_0^T H(X)dt + \frac{1}{2}\sum_{i=1}^{n} \left\|E_i - E_{i,0}\right\|^2 . \tag{2}$$

Here $E_{1,0}, \ldots, E_{n,0}$ are priori evaluations of $E_1, \ldots, E_n$. The problem is to determine $E_1^*, \ldots, E_n^*$ by minimizing J. The second and the third terms in J are a regularization term, in the sense of Tykhonov, have a well posed problem (see [14, 18]).

The optimal solutions are characterized by $\vec{\nabla}.J(E_1^*, \ldots, E_n^*)$, where $\vec{\nabla}.J$ is the gradient of J. To compute this gradient we introduce $\bar{E}_i (i = 1, \ldots, n)$ the directions in the space Y_p. We will compute the Gateaux derivative of the cost function J by E_i in the directions of $\bar{E}_i$.

The Gateaux derivative of the cost function J in the directions of $\bar{E}_i (i = 1, \ldots, n)$ will be

$$\begin{aligned} \hat{J}(E_1, \ldots, E_n) &= -\sum_{i=1}^{n} \int_0^T \left(\frac{\partial H}{\partial X}, \hat{X}^{(i)} \right)_{\Im} dt + \sum_{i=1}^{n} \langle E_i - E_{i,0}, \bar{E}_i \rangle \\ &= -\sum_{i=1}^{n} \int_0^T \left(G(X), \hat{X}^{(i)} \right)_{\Im} dt + \sum_{i=1}^{n} \langle E_i - E_{i,0}, \bar{E}_i \rangle, \end{aligned} \tag{3}$$

where $G(X) = \dfrac{\partial H}{\partial X}$ and $\hat{X}^{(i)}$, $\hat{J}_{E_i}(E_1, \ldots, E_n)$ respectively are the Gateaux derivatives of X and J with respect to E_i in the directions $\bar{E}_i$. Here $<,>$ is the dot product associated with the norm operator $\|\|$. The optimal solution of problem is characterized by $\hat{J}(E_1, \ldots, E_n) = \vec{\nabla}.J.(\bar{E}_1, \ldots, \bar{E}_n)^T = 0$ where $\vec{\nabla}.J = \left(J'_{E_1}, \ldots, J'_{E_n} \right)$ is the gradient of J with respect to $E_1, \ldots, E_n$.

The superscript ^{T} indicates the transpose of the vector. To find $J'_{E_i}(E_1, \ldots, E_n)$ the same way as in ([9–13, 15–18]) we solve the adjoin equation of (1)

$$\frac{dP^{(i)}}{dt} + \left[\frac{dF}{dX} \right]^t \cdot P^{(i)} = G(X), \quad P^{(i)}(T) = 0. \tag{4}$$

Then by the similar way as in ([9–13, 15–18]) we have

$$J'_{E_i}(E_1, \ldots, E_n) = \int_0^T \left[\frac{\partial F}{\partial E_i} \right]^t P^{(i)} dt + E_i - E_{i,0} \tag{5}$$

Eqs. (1)-(5) are the Optimality System (O.S). The adjoin model will be run back word to get the gradient which are used to carry out an algorithm of optimization (see [9–13, 15–18]).

3. MATHEMATICAL MODEL FOR THE BODY MOTION

To describe the motion of body, a body fixed coordinate system as shown in Fig. 2 is chosen. (X_0, Y_0, Z_0) is the inertial reference frame with origin at 0 and (X_1, Y_1, Z_1) is the non-inertial reference frame with origin at A, the tip of the slender body. The X_1-axis coincides with the longitudinal axis of the slender body. The components of velocity of point A along X_0 and Z_0 direction are U_F and W_F respectively. The components of velocity of point A along X_1 and Z_1 direction are U and W respectively. The angular velocity about Y_0 axis is Q. The orientation angle of the body with respect to the Y_0 axis is ϑ.

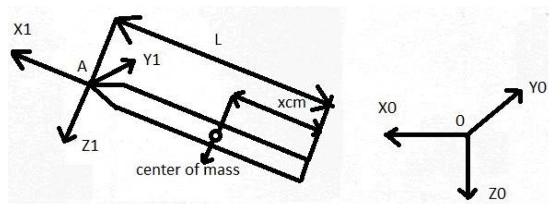


Fig. 2. Axes of body and inertial frames

The relationships between body and inertial fixed velocities are described by the following formula (see [19])

$$U_F = U\cos\vartheta + W\sin\theta, \quad W_F = -U\sin\vartheta + W\cos\vartheta, \quad \dot{\vartheta} = Q, \quad \vartheta(0) = \vartheta_0. \tag{6}$$

The mathematic cavity model (see [20]) is used to describe the motion of slender body under water in cavity. The motion of slender body in both phases is written by the following equations:

Phase 1: For $U^2 \gg W^2$ and $\rho A_c k\left(U,W,h\right)U^2 \gg 2mLQ^2$ the equation can be written as

$$\begin{aligned}
&\frac{\partial U}{\partial t} = -\frac{1}{2m}\rho k\left(U,W,h\right)A_c U^2, \quad \frac{\partial W}{\partial t} = QU, \quad \frac{\partial Q}{\partial t} = 0,\\
&\frac{\partial h}{\partial t} = -U\sin\vartheta + W\cos\vartheta, \quad \frac{\partial \vartheta}{\partial t} = Q,\\
&U\left(0\right) = U_0, \quad W\left(0\right) = W_0, \quad Q\left(0\right) = Q_0, \quad h(0) = h_0, \quad \vartheta(0) = \vartheta_0.
\end{aligned} \tag{7}$$

Phase 2: For $U^2 \gg W^2$ and $\rho A_c k\left(U,W,h\right)U^2 \gg 2mLQ^2$ the equation can be written as

$$\begin{aligned}
&\frac{\partial U}{\partial t} = -\frac{1}{2m}\rho k\left(U,W,h\right)F\left(A_c,r,l_k,\theta\right)U^2,\\
&\frac{\partial W}{\partial t} = KW^2\left[M_1 l_k + M_2 l_k x_{cm}\left(L - x_{cm}\right)\right] + 2KW\left[QM_2 L x_{cm} l_k\left(L - x_{cm}\right)\right] + QU,\\
&\frac{\partial Q}{\partial t} = -KM_2\left[W^2 l_k x_{cm} + 2WQLl_k x_{cm}\right], \quad \frac{\partial h}{\partial t} = -U\sin\vartheta + W\cos\vartheta, \quad \frac{\partial \vartheta}{\partial t} = Q,
\end{aligned} \tag{8}$$

where θ is the angle of slender body during impact with the cavity boundary,

$\tan\theta \approx \frac{W}{U}$ or $\theta \approx \arctan\frac{W}{U}$, $M_1 = -\frac{\rho d}{m}$, $M_2 = \frac{\rho d}{I}$,

$F\left(A_c,r,l_k,\theta\right) = A_c + r^2\cos^{-1}\left(\frac{r - l_k\tan\theta}{r}\right) - \left(r - l_k\tan\theta\right)\sqrt{dl_k\tan\theta}$

$k\left(U,W,h\right) = k_1 C_{D0}\left(1+\sigma\right)\cos^2\alpha, C_{D0} = 0.82,$

α is the angle between flow direction and body's direction in moving, $\cos\alpha \approx U/\sqrt{U^2+W^2}$, $p_\infty = \rho gh + P_{atm}$ ambient pressure, l_k is the wetted length of the body; k_1, K are parameters, h is the water depth between the body's position and water free surface, ρ is the mass density of water, x_{cm} is the distance between body's tail and it's center of mass,

m is the mass of the slender body, σ is the cavitation number $\sigma = 2\left(p_\infty - p_c\right)/\left(U^2 + W^2\right)$, I is the moment of inertia of the body about an axis parallel to the Y_1 axis and passing through its center of mass, $r = d/2$ is the radius of slender body, $A_c = \pi D_c^2/4$ is the area of the cavitator, $r_c = D_c/2$is the cavitator radius, $g = 9.81$ m/s is the gravity acceleration, p_c is the vapor pressure of water.

To get the above equations the following condition is needed: $l_k/L \ll 1$. The geometry of the cavity is given by (see [1,6,8]): $4\left(x - l/2\right)^2/l^2 + 4y^2/\left(D_k/2\right)^2 = 1$, where the maximum diameter D_k and length l of the cavity shape are given by the following formulas: $D_k = D_c\sqrt{k_1 C_{D0}\left(1+\sigma\right)/\sigma}$, $l = D_c\sqrt{\log\left(1/\sigma\right)}/\sigma$.

The depending on L and D_c of I_y, m and x_{cm} can be calculated by the following formulas:

$$m = \rho_{body} V, \quad x_{cm} = L_2 + \pi\left(0.5.R^2\left(L_1^2 - L_2^2\right) - L_1^2(R - 0.5.D_c)\left(5R + 1.5D_c\right)/12\right)/V,$$

$$I_y = \rho\pi\begin{bmatrix}\left(L_1^3R^2/3 + 0.2.L_1^3(R-0.5.D_c)^2 - 0.5.L_1^3R(R-0.5.D_c)\right) + \left(R^2L_2^3/3 + R^4L_2/4\right)\\ -Vx_G^2/\pi + 0.25\left(R^4L_1 + L_1(R-0.5.D_c)^4/5 + 2R^2L_1(R-0.5.D_c)^2\right.\\ \left.-2R^3L_1(R-0.5.D_c) - RL_1(R-0.5.D_c)^3\right)\end{bmatrix},$$

$$V = \pi\left[R^2(L_1+L_2) + \left[L_1(R-0.5D_c)^2/3 - RL_1(R-0.5D_c)\right]\right],$$

$$x_G = \pi\left(R^2\left(l_1^2 - l_2^2\right)/2 - l_1^2(R-0.5.D_c)\left(5R+1.5D_c\right)/12\right)/V,$$

Eqs. (7)-(8) can be rewritten as follows

$$\partial X/\partial t = A(X), \quad X(0) = X_0\,, \tag{9}$$

where $X = \left(U, W, Q, h, \vartheta\right)^T$ is an unknown state function vector of Eqs. (7)-(8), and

$$\begin{aligned}
&X_0 = \left(U_0, W_0, Q_0, h_0, \vartheta_0\right)^T,\\
&A(X) = \left[A_1(X), A_2(X), A_3(X), -U\sin\vartheta + W\cos\vartheta, Q\right]^T,\\
&A_1(X) = \begin{cases} -0.5\rho k\left(U,W,h\right)A_cU^2/m, & \text{in the firstphase}\\ -0.5\rho k\left(U,W,h\right)F\left(A_c, r, l_k, \theta\right)U^2/m, & \text{in the second phase}\end{cases}\\
&A_2(X) = \begin{cases} QU, & \text{in the first phase}\\ KC_1W^2 + KC_2W + QU, & \text{in the second phase}\end{cases}\\
&A_3(X) = \begin{cases} QU, & \text{in the first phase}\\ C_3W^2 + C_4WQ, & \text{in the second phase}\end{cases}\\
&C_1 = M_1l_k + M_2l_kx_{cm}\left(L - x_{cm}\right), \quad C_2 = 2M_2Lx_{cm}l_k\left(L - x_{cm}\right),\\
&C_3 = -M_2l_kx_{cm}\,, \quad C_4 = -M_2Ll_kx_{cm}\,,
\end{aligned} \tag{10}$$

4. DETERMINATION OF OPTIMAL GEOMETRIC PARAMETERS OF BODY

By the shape of slender body the relations of L_1, L_2, D_c, m, x_{cm} and I are built by the functions such as: $m = m\left(L_1, L_2, D_c\right)$, $x_{cm} = x_{cm}\left(L_1, L_2, D_c\right)$ and $I = I\left(L_1, L_2, D_c\right)$.

We will find L_1^*, L_2^*, D_c^* so that the way distance by X_0 direction running by the body described by the function $\int_0^T U_F(t)dt$ is maximum. Firstly, we have approximations

$L_{1,0}$, $L_{2,0}$ and $D_{c,0}$ of L_1, L_2 and D_c. Using the formula (1) we introduce the cost function

$$J\left(L_1, L_2, D_c\right) = -\int\limits_0^T U_F dt + \left(L_1 - L_{1,0}\right)^2/2 + \left(L_2 - L_{2,0}\right)^2/2 + \left(D_c - D_{c,0}\right)^2/2$$
$$= -\int\limits_0^T \left(U\cos\vartheta + W\sin\vartheta\right)dt + \left(L_1 - L_{1,0}\right)^2/2 + \left(L_2 - L_{2,0}\right)^2/2 + \left(D_c - D_{c,0}\right)^2/2. \tag{11}$$

Using the cost function from the formula (11) the continuous problem is to determine L_1^*, L_2^*, D_c^* minimizing $J\left(L_1, L_2, D_c\right)$. The optimal problem is written in the form

$$\begin{cases} \partial X/\partial t = A(X), \quad X(0) = X_0 \\ J\left(L_1^*, L_2^*, D_c^*\right) = \inf\limits_{L_1^*, L_2^*, D_c^*} J\left(L_1, L_2, D_c\right) \end{cases} \tag{12}$$

4.1. Computing the gradient $\vec{\nabla}.J(L_1, L_2, D_c)$ of the cost function J by L_1, L_2 and D_c parameters in the directions of $\overline{L}_1, \overline{L}_2, \bar{D}_c$

Let $\overline{L}_1, \overline{L}_2$ and $\bar{D}_c$ be some functions in the space of the control. The Gateaux derivative of the cost function J with respect to L and D_c in the directions of $\bar{L}$ and $\bar{D}_c$ will be

$$\hat{J}(L_1, L_2, D_c) = -\int\limits_0^T \left(G(X), \hat{X}_1\right)_{\Im} dt + \left(L_1 - L_{1,0}\right).\bar{L}_1 - \int\limits_0^T \left(G(X), \hat{X}_2\right)_{\Im} dt + \left(L_2 - L_{2,0}\right).\bar{L}_2$$
$$-\int\limits_0^T \left(G(X), \hat{X}_3\right)_{\Im} dt + \left(D_c - D_{c,0}\right)\bar{D}_c\,, \tag{13}$$

with

$$G\left(X\right) = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 & 0 & -U\sin\vartheta + W\cos\vartheta \end{bmatrix}^T, \tag{14}$$

and $\hat{X}^{(i)} = \left(\hat{U}^{(i)}, \hat{W}^{(i)}, \hat{Q}^{(i)}, \hat{h}^{(i)}, \hat{\vartheta}^{(i)}\right)^T$ $(i = 1, 3)$ respectively are the Gateaux derivatives of $X = (U, W, Q, h, \vartheta)^T$ by L_1, L_2 and D_c in the directions of $\overline{L}_1, \overline{L}_2, \bar{D}_c$.

Firstly, we will compute Gateaux derivatives $\hat{J}_{L_1}(L_1, L_2, D_c)$, $\hat{J}_{L_2}(L_1, L_2, D_c)$ and $\hat{J}_{D_c}(L_1, L_2, D_c)$ of the cost function J with respect to L_1, L_2 and D_c respectively in the directions of $\overline{L}_1$, $\overline{L}_2$ and $\bar{D}_c$

The Gateaux derivative equations of (9) with respect to body length parameter $L_i (i = 1, 2)$ in the directions of $\overline{L}_i$ are written as follows

$$\partial\hat{X}^{(i)}/\partial t = N(X)\hat{X}^{(i)} + B(X)\bar{L}_i, \quad \hat{X}^{(i)}(0) = 0, \tag{15}$$

where

$$N(X)=\left[\begin{array}{ccccc} N_{11}(X) & N_{12}(X) & 0 & N_{14}(X) & 0 \\ N_{21}(X) & N_{22}(X) & N_{23}(X) & 0 & 0 \\ 0 & N_{32}(X) & N_{33}(X) & 0 & 0 \\ -\sin\vartheta & \cos\vartheta & 0 & 0 & -U\cos\vartheta-W\sin\vartheta \\ 0 & 0 & 1 & 0 & 0 \end{array}\right], \tag{16}$$

$$N_{ij}=\begin{cases} N_{ij}^{(1)}, & \text{in the first phase} \\ N_{ij}^{(2)}, & \text{in the second phase} \end{cases} \quad (i=1,\ldots,3;\, j=1,\ldots,4)$$

$$N_{11}^{(1)}=-\frac{1}{2m}\rho k_1 C_{D0}\left(1+\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)}\right)\frac{\left(2U^4+3U^2W^2\right)}{\left(U^2+W^2\right)^{3/2}}A_c+\frac{1}{m}k_1\rho C_{D0}\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)^{5/2}}U^4A_c\,,$$

$$N_{12}^{(1)}=-\frac{1}{2m}\rho k_1 C_{D0}\left(1+\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)}\right)\frac{U^3W}{\left(U^2+W^2\right)^{3/2}}A_c+\frac{1}{m}k_1\rho C_{D0}\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)^{5/2}}WU^3A_c\,,$$

$$N_{14}^{(1)}=-\frac{\rho}{2m}k_1C_{D0}\frac{g}{0.5\left(U^2+W^2\right)^{3/2}}U^3A_cN_{21}^{(1)}=Q,\ N_{22}^{(1)}=0,\ N_{23}^{(1)}=U,\ N_{32}^{(1)}=0,\ N_{33}^{(1)}=0,$$

$$N_{11}^{(2)}=-\frac{1}{2m}k_1\rho C_{D0}\left(1+\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)}\right)\frac{\left[\left(2U^4+3U^2W^2\right)\right]F_c}{\left(U^2+W^2\right)^{3/2}}+\frac{1}{m}k_1\rho C_{D0}\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)^{5/2}}U^4F_c$$

$$+\frac{\rho}{2m}k_1C_{D0}\left(1+\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)}\right)\left(r^2\frac{\sin\left(\frac{r-l_k\tan\theta}{r}\right)\frac{l_k}{r}}{\cos^2\left(\frac{r-l_k\tan\theta}{r}\right)}-\frac{r\sqrt{\frac{l_kd}{\tan\theta}}}{2}+\frac{3}{2}l_k\sqrt{dl_k\tan\theta}\right)\frac{UW}{\sqrt{\left(U^2+W^2\right)}}\,,$$

$$N_{12}^{(2)}=-\frac{1}{2m}k_1\rho C_{D0}\left(1+\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)}\right)\frac{\left[-U^3W\hat{W}\right]F_c}{\left(U^2+W^2\right)^{3/2}}+\frac{1}{m}k_1\rho C_{D0}\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)^{5/2}}WU^3F_c$$

$$-\frac{\rho}{2m}k_1C_{D0}\left(1+\frac{p_\infty-p_c}{0.5\rho\left(U^2+W^2\right)}\right)\left(r^2\frac{\sin\left(\frac{r-l_k\tan\theta}{r}\right)\frac{l_k}{r}}{\cos^2\left(\frac{r-l_k\tan\theta}{r}\right)}-\frac{r\sqrt{\frac{l_kd}{\tan\theta}}}{2}+\frac{3}{2}l_k\sqrt{dl_k\tan\theta}\right)\frac{U^2}{\sqrt{\left(U^2+W^2\right)}}\,,$$

$$N_{14}^{(2)}=-\frac{\rho}{2m}k_1C_{D0}\frac{g}{0.5\left(U^2+W^2\right)^{3/2}}U^3F_c\,,\quad N_{33}^{(2)}=KC_4W,$$

$$N_{21}^{(2)}=Q,\quad N_{22}^{(2)}=2KC_1W+KC_2Q,\quad N_{23}^{(2)}=KC_2W+UN_{32}^{(2)}=2KC_3W+KC_4Q,$$

$$B^{(i)}=\left(B_1^{(i)},B_2^{(i)},B_3^{(i)},0,0\right)^T,$$

$$B_1^i=\begin{cases} \frac{1}{2m^2}k\left(U,W,h\right)U^2A_cm'_{L_1}, & \text{for the first phase} \\ \frac{1}{2m}k\left(U,W,h\right)U^2\left(\frac{F_cm'_{L_1}}{m}-F'_{c,l_k}l'_{k,L_1}\right), & \text{for the second phase} \end{cases}$$

$$F'_{c,l_k}=\left(r^2\frac{\sin\left(\frac{r-l_k\tan\theta}{r}\right)\frac{\tan\theta}{r}}{\cos^2\left(\frac{r-l_k\tan\theta}{r}\right)}-\frac{r\sqrt{\frac{d\tan\theta}{l_k}}}{2}+\frac{3}{2}\tan\theta\sqrt{dl_k\tan\theta}\right),$$

$$B_2^{(i)} = \begin{cases} 0, & \text{for the first phase} \\ C'_{1,L_i}W^2 + C'_{2,L_i}WQ, & \text{for the second phase} \end{cases}$$
$$B_3^{(i)} = \begin{cases} 0, & \text{for the first phase} \\ C'_{3,L_i}W^2 + C'_{4,L_i}WQ, & \text{for the second phase} \end{cases}$$

$C'_{1,L_i}, C'_{2,L_i}, C'_{3,L_i}, C'_{4,L_i}$ are the derivatives of those functions by length parameter L_i. Multiplying Eq. (15) by adjoin variable $P^{(i)} = \left(P_1^{(i)}, P_2^{(i)}, P_3^{(i)}, P_4^{(i)}, P_5^{(i)}\right)^T$ $(i = 1, 2)$ in the same space as X and then integrating by t between 0 and T and adding them we have

$$\begin{aligned} &\sum_{i=1}^{2} \left(\hat{X}^{(i)}(T), P^{(i)}(T)\right)_{\Im} - \left(\hat{X}^{(i)}(0), P^{(i)}(0)\right)_{\Im} = \\ &\qquad \sum_{i=1}^{2} \int_0^T \left(\hat{X}^{(i)}, \frac{dP^{(i)}}{dt} + F^{(i)}\left(X, P^{(i)}\right)\right)_{\Im} dt - \sum_{i=1}^{2} \bar{L}_i \int_0^T B^{(i)} \cdot P^{(i)T} dt, \end{aligned} \tag{17}$$

where $F^{(i)}\left(X, P^{(i)}\right) = N^t.P^{(i)}$ with $N(X)$ is defined by the formula (16).

By the similar way the Gateaux derivative equations of (7)-(8) with respect to cavitator diameter parameter D_c in the directions of $\bar{D}_c$ are written as follows

$$\partial\hat{X}^{(3)}/dt = N(X)\hat{X}^{(3)} + B^{(3)}(X)\bar{D}_c\,, \quad \hat{X}^{(3)}(0) = 0, \tag{18}$$

where

$$B^{(3)} = \left(B_1^{(3)}, B_2^{(3)}, B_3^{(3)}, 0, 0\right)^T,$$
$$B_1^{(3)} = \begin{cases} 0.5.k\,(U, W, h)\,U^2 A_c {m'}_{D_c}/m^2, & \text{for the first phase} \\ 0.5.k\,(U, W, h)\,U^2\left(F_c {m'}_{D_c}/m - {F'}_{c,l_k}{l'}_{k,D_c}\right)/m, & \text{for the second phase} \end{cases}$$
$$B_2^{(3)} = \begin{cases} 0, & \text{for the first phase} \\ C'_{1,D_c}W^2 + C'_{2,D_c}WQ, & \text{for the second phase} \end{cases}$$
$$B_3^{(3)} = \begin{cases} 0, & \text{for the first phase} \\ C'_{3,D_c}W^2 + C'_{4,D_c}WQ, & \text{for the second phase} \end{cases}$$

Here $C'_{1,D_c}, C'_{2,D_c}, C'_{3,D_c}, C'_{4,D_c}, m'_{D_c}$ are the derivatives of the functions C_1, C_2, C_3, C_4, m by the parameter D_c. Multiplying Eq. (18) by adjoin variable $P^{(3)} = \left(P_1^{(3)}, P_2^{(3)}, P_3^{(3)}, P_4^{(3)}, P_5^{(3)}\right)^T$ in the same space as X and then integrating by t between 0 and T we have

$$\begin{aligned} &\left(\hat{X}^{(3)}(T), P^{(3)}(T)\right)_{\Im} - \left(\hat{X}^{(3)}(0), P^{(3)}(0)\right)_{\Im} = \\ &\qquad \int_0^T \left(\hat{X}^{(3)}, \frac{dP^{(3)}}{dt} + F^{(3)}\left(X, P^{(3)}\right)\right)_{\Im} dt - \bar{D}_c \int_0^T P^{(3)T} B^{(3)} dt, \end{aligned} \tag{19}$$

where $F^{(3)}\left(X, P^{(3)}\right) = N^t.P^{(3)}$. Adding (17) and (19) we have

$$\sum_{i=1}^{3}\left(\hat{X}^{(i)}(T), P^{(i)}(T)\right)_{\Im} - \left(\hat{X}^{(i)}(0), P^{(i)}(0)\right)_{\Im} =$$
$$\sum_{i=1}^{3}\int_0^T \left(\hat{X}^{(i)}, dP^{(i)}/dt + F^{(i)}\left(X, P^{(i)}\right)\right)_{\Im} dt - \sum_{i=1}^{2}\bar{L}_i\int_0^T B^{(i)}\cdot {P^{(i)}}^T dt - \bar{D}_c\int_0^T B^{(3)}.{P^{(3)}}^T dt. \tag{20}$$

Then if $P^{(i)}(i = 1, 3)$ are the solutions of the following equations

$$dP^{(i)}/dt + F^{(i)}\left(X, P^{(i)}\right) = G(X), \quad P^{(i)}(T) = 0. \tag{21}$$

The Eq. (20) becomes

$$\sum_{i=1}^{3}\int_0^T \left(\hat{X}^{(i)}, G(X)\right)_{\Im} dt = \sum_{i=1}^{2}\bar{L}_i\int_0^T B^{(i)}\cdot P^{(i)T} dt + \bar{D}_c\int_0^T B^{(3)}.{P^{(3)}}^T dt. \tag{22}$$

Applying the Eq. (22) to the Eq. (13) we have

$$\hat{J}(L_1, L_2, D_c) = \vec{\nabla}.J.(\bar{L}_1, \bar{L}_2, \bar{D}_c)^T,$$

where

$$\vec{\nabla}.J\left(L_1, L_2, D_c\right) = \left(J'_{L_1}\left(L_1, L_2, D_c\right), J'_{L_2}\left(L_1, L_2, D_c\right), J'_{D_c}\left(L_1, L_2, D_c\right)\right), \tag{23}$$

$$J'_{L_i}\left(L_1, L_2, D_c\right) = -\bar{L}_i\int_0^T B^{(i)}\cdot P^{(i)T} dt + \left(L_i - L_{i,0}\right).\bar{L}_i, \quad (i = 1, 2), \tag{24}$$

$$J'_{D_c}\left(L_1, L_2, D_c\right) = -\bar{D}_c\int_0^T B^{(3)}.{P^{(3)}}^T dt + \left(D_c - D_{c,0}\right)\bar{D}_c. \tag{25}$$

4.2. Algorithm to solve the optimal problem

- Solve Eqs. (9) by Runge-Kutta methods
- Solve the adjoint Eq. (21)
- Get the function $J'_{L_i}(i = 1, 2)$ by the formula (24)
- Get the function J'_{D_c} by the formula (25)
- Calculate the gradient $\vec{\nabla}.J\left(L_1, L_2, D_c\right)$ of the cost function $J\left(L_1, L_2, D_c\right)$ by the formula (23)
- Solve the optimal problem (12) with the gradient $\vec{\nabla}.J\left(L_1, L_2, D_c\right)$

4.3. Simulation experiment on finding the parameters L_1, L_2 and D_c so that velocity U is maximum

We will consider the body with $d = 0.57$ cm, $\rho_{body} = 7680$ kg/m^3, $L_1 \in$ [0.5 cm, 20 cm], $L_2 \in$ [5 cm, 30 cm], $D_c \in$ [0.16 cm, 0.2 cm]. The function of distance way by X direction with changing L_1, L_2 and fixed $D_c = 0.16$ cm is presented in the left of Fig. 3. The function of distance way by X direction changing L, D_c and fixed $L_1 = 2.5$ cm is presented in the right of Fig. 3. The maximum area of the distance way by X_0 direction is shown by the red color area. The velocity U and way distance processes by X_0 direction with or without optimal values of couples (L_1, L_2, D_c) are shown in the Fig. 4. By this figure it is easy to see that with the optimal values $L_1^* = 0.5$ cm, $L_2^* = 22.5$ cm, $D_c^* = 0.16$ cm the way distance is maximum and reaches to 27.2 m. With the two other parameters (L_1, L_2, D_c) the way distances are less than 20 m but bigger than 17 m.

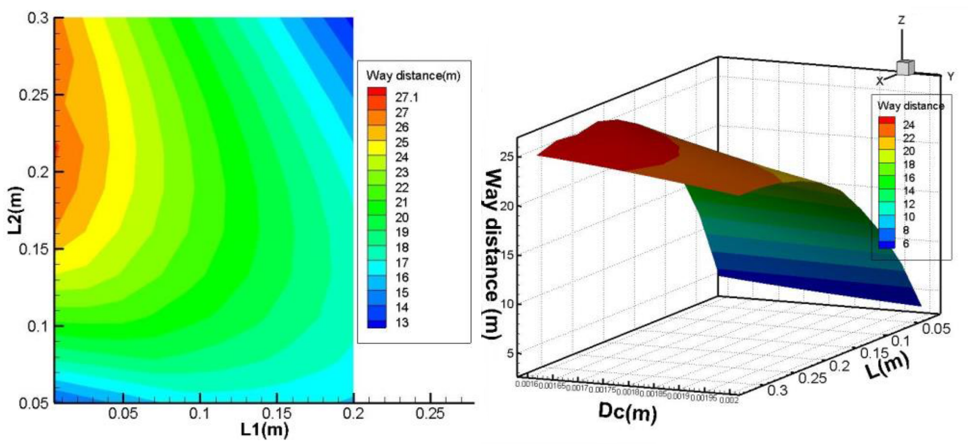


Fig. 3. The area of way distance by X direction depending on L_1 and L_2 in optimal process with fixed $D_c = 0.16$ cm (left); The area of way distance by X direction depending on L and D_c in optimal process with fixed $L_1 = 2.5$ cm (right)

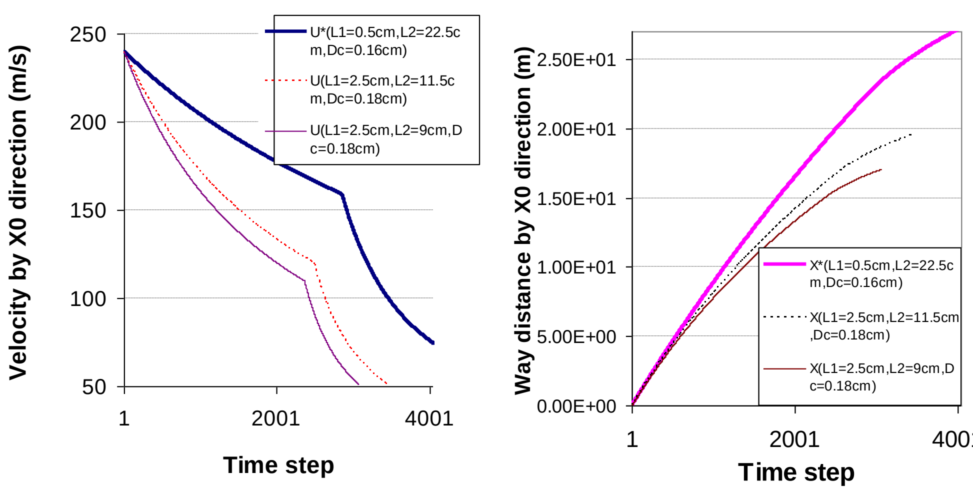


Fig. 4. Velocity by X direction with or without optimal (L_1, L_2, D_c) (left); Way distance by X_0 direction with or without optimal (L_1, L_2, D_c) (right)

5. CONCLUSION

In the paper the problem of determination of optimal geometric parameters of a slender body moving fast through water is considered by using the differential variation optimal method. The following observations are obtained:

- Fig. 3 shows that in the case $D_c = 0.16$ cm and $L_1 \in [0.5\text{ cm}, 2.5\text{ cm}]$, $L_2 \in [15\text{ cm}, 30\text{ cm}]$ the way distance is longer than 24 m.

- Fig. 4 shows that by data assimilation method the chosen values (L_1^*, L_2^*, D_c^*) make the way distance to be maximal. Thus, the optimal differential variation method can be used as a good tool to choose best geometric parameters in the problem of body running fast through water.

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