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VIBRATION ANALYSIS OF BEAMS SUBJECTED TO RANDOM EXCITATION BY THE DUAL CRITERION OF EQUIVALENT LINEARIZATION

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Abstract. In this paper responses of beams subjected to random loading are analyzed by the dual approach of the equivalent linearization method. The external random loading is assumed to be a space-wise and time-wise white noise in which the exact solutions of the modal equations can be found. A system of nonlinear algebraic equations for linearization coefficients of the modal linearized system is obtained in a closed form and is solved by the fixed-point iteration method. Results obtained from the proposed dual criterion are compared with the exact solution and those obtained from other approaches including energy method, and conventional linearization method. It is observed that the solution obtained by the dual criterion is in good agreement with the exact solution, especially, in the case of strong nonlinearity of beam.

Keywords: Random vibration, equivalent linearization, dual criterion, modal response, nonlinear beam.

1. INTRODUCTION

Over decades, the equivalent linearization (EQL) is one of the most extensively used methods in investigating mechanical systems. The earliest researches on the EQL method were carried out by Booton [1], Kazakov [2] and Caughey [3,4]. The fundamental idea of the method lies on replacing the original nonlinear system under a random external excitation by a linearized system under the same excitation in which linearization coefficients are found from a specified optimal criterion, for example, the mean-square error criterion [4], spectral criterion [5]. This method has developed and been applied successfully to find approximate responses of nonlinear system subjected random excitation. For discrete systems with single- and multi-degree-of-freedom, studies using the equivalent linearization can be found in some works [6–11], review articles [12–16], and in monographs by Crandall and Mark [17], Lin [18], Roberts and Spanos [19], Socha [5] with references therein. For continuous systems, the method of EQL is also applied [14, 20–26]. In vibration analysis of beam structures under random excitations, the EQL method is studied by several authors. In Ref. [21], Herberts considered the effect of the membrane force on the stresses in a simply supported Bernoulli-Euler beam by the method of EQL. Seide [22] investigated nonlinear mean-square multimode responses of beams subjected to uniform pressure uncorrelated in time. Using EQL method, he obtained mean-square stresses and displacements of beams with arbitrary end conditions. Iwan and Whirley [23] developed a version of the EQL that can be applied to continuous systems under non-stationary random excitations. Their technique allows the replacement of the original nonlinear system with a time-varying linear continuous system. In [24,25], a new technique of equivalent linearization method is proposed based on the energy approach. Unlike the traditional replacement of EQL method [3, 4], the energy method requires that the mean-square error between the potential energy of the original nonlinear system and that of corresponding linearized system must be minimum. In [26], Anh et al. extended the approach of regulated equivalent linearization (RGEL) in studying single-degree-of-freedom system to random vibration analysis of beams under random loadings. The effective of the RGEL method is recorded by its excellent performance in calculating approximate modal responses of the beam.

Recently, Anh et al. [27] have proposed a dual criterion of the EQL method for nonlinear single-degree-of-freedom systems under random excitations. The authors showed that the accuracy of the mean-square response obtained by the dual criterion is significantly improved when the nonlinearity is increasing. This dual approach is then extended to cases of multi-degree-of-freedom systems [28].

Naturally, the dual approach may be extended to random vibration analysis of many continuous systems. In this research, a version of dual criterion of the EQL method is developed for analyzing modal responses of beams subjected to random loading. A nonlinear algebraic system of linearization coefficients obtained in a closed form is solved by an iteration method. To elucidate the dual approach, the obtained results are compared with those of the conventional linearization and energy methods.

2. THE GOVERNING EQUATION OF BEAM

Consider the governing equation of a beam on elastic foundation, restrained at its ends and subjected to a space-wise distributed time-dependent loading $p(x,t)$ [24,25]

$$EI\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \mu A\frac{\partial^2 w}{\partial t^2} + \beta\frac{\partial w}{\partial t} + K_f w = p(x,t), \tag{1}$$

where the axial force N is given by

$$N = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{2}$$

Here, A and I are the area and inertia moment of the cross-section, respectively; E is the elastic modulus, μ the mass density, β the viscous damping coefficient, L the length of the beam, K_f the stiffness of the elastic foundation; $w(x,t)$ is the deflection of the beam. In this paper, a space-wise and time-wise white noise loading $p(x,t)$ is considered.

In order to solve (1) one expands $p(x,t)$in series [24]

$$p(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x), \tag{3}$$

where $q_n(t)$ $(n = 1, 2, \ldots)$ are zero-mean Gaussian white noise stationary random processes with corresponding correlations

$$E\left[q_m(t)\,q_n(t+\tau)\right] = 2\pi S_m \delta_{mn} \delta(\tau)\,, (m, n = 1, 2, \ldots) \tag{4}$$

in which δ_{mn} is the Kronecker-delta notation, $\delta(\tau)$ is the Dirac-delta; the quantities S_n $(n = 1, 2, \ldots)$ are constant spectral density values of random processes $q_n(t)$. The functions $\phi_n(x)$ $(n = 1, 2, \ldots)$ in the series (3) are modal shapes satisfying the following relationships

$$\frac{d^4\phi_n}{dx^4} = \frac{\mu A}{EI}\omega_n^2 \phi_n, \tag{5}$$

$$\int_0^1 \phi_m \phi_n d\xi = \delta_{mn}, \quad \xi = \frac{x}{L} \tag{6}$$

where ω_n $(n = 1, 2, \ldots)$ are natural frequencies associated with free vibration of the system (1) without the axial force, viscous damping, elastic foundation and external excitation for a simply supported boundary condition

$$\omega_n^2 = \frac{n^4 EI \pi^4}{\mu A L^4}. \tag{7}$$

Let the deflection function $w(x,t)$ of the beam be expended in terms of an appropriate orthogonal set of modal shapes $\phi_n(x)$ as follows

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\,\phi_n(x), \tag{8}$$

where $w_n(t)$ is the modal contribution corresponding to n^{th}-mode. Substituting Eq. (8) into the expression of the axial force N in Eq. (2) yields

$$N = \frac{EA}{2L^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K_{nm} w_n w_m, \tag{9}$$

where it is denoted

$$K_{nm} = \int_0^1 \frac{d\phi_n}{d\xi} \frac{d\phi_m}{d\xi} d\xi = K_{mn}. \tag{10}$$

In view of Eqs. (5) and (6), the governing equation (1) takes the form

$$\sum_{n=1}^{\infty}\left(\mu A\ddot{w}_n+\beta\dot{w}_n+K_f w_n+\mu A\omega_n^2 w_n\right)\phi_n-\frac{EA}{2L^2}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}w_i w_j w_n\frac{d^2\phi_n}{dx^2}=p\left(x,t\right). \tag{11}$$

Multiplication of Eq. (11) by ϕ_m, integration over the length of the beam, and then use of orthogonality conditions given in Eq. (6) yield a set of coupled nonlinear differential equations for modal amplitudes $w_m\left(t\right)$

$$\ddot{w}_m+\frac{\beta}{\mu A}\dot{w}_m+\omega_m^2 w_m+\frac{K_f}{\mu A}w_m+\frac{E}{2\mu L^4}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij}K_{nm}w_i w_j w_n=b_m\left(t\right), \tag{12}$$

where the random function $b_m\left(t\right)$ is given by

$$b_m\left(t\right)=\frac{1}{\mu AL}\int_0^L p\left(x,t\right)\phi_m\left(x\right)dx. \tag{13}$$

Further in the expansion (8), it is assumed that only the first M modes of the beam significantly contribute to formulate responses. Crandall and Yildiz [29] shown that if the infinite series that are representing quantities such as displacement, mean-square stresses, etc., converge, then the results can be made as accurate as desired by taking sufficiently large M. For this reason, Eq. (12) can be taken by the following finite form with the first M modes

$$\ddot{w}_m+\frac{\beta}{\mu A}\dot{w}_m+\omega_m^2 w_m+\frac{K_f}{\mu A}w_m+G_m\left(w_1,w_2,\ldots,w_M\right)=b_m\left(t\right), \tag{14}$$

in which nonlinear components $G_m\left(m=1,2,\ldots,M\right)$ are functions of M variables $w_1,\ldots,w_M$

$$G_m=G_m\left(w_1,w_2,\ldots,w_M\right)=\frac{E}{2\mu L^4}\sum_{n=1}^{M}\sum_{i=1}^{M}\sum_{j=1}^{M}K_{ij}K_{nm}w_i w_j w_n. \tag{15}$$

Our objective is to find approximate mean-square modal responses of the beam from the modal system (14). In the next section, the dual criterion of stochastic linearization method will be applied to this nonlinear system.

3. A DUAL CRITERION FOR MODAL EQUATIONS OF BEAM

The dual criterion of stochastic linearization method appears from an idea that the original nonlinear system can be replaced by an equivalent linearization system, and then this equivalent system is replaced by another nonlinear system that belongs to the same class of the original nonlinear system. Some obtained results using the dual criterion are presented in works of Anh et al. [27,28] for single- and multi-degree-of-freedom systems subjected to random excitations. Naturally, the dual criterion of stochastic linearization needs to be developed in investigating continuous systems under random excitation. For the governing equation of the beam in the modal form (14), one can make a linearization version as follows

$$\ddot{w}_m + \frac{\beta}{\mu A}\dot{w}_m + \omega_m^2 w_m + \frac{K_f}{\mu A} w_m + \omega_m^2 k_{eq,m} w_m = b_m(t), \tag{16}$$

where $k_{eq,m}$ $(m = 1, 2, \ldots, M)$ are non-dimensional linearization coefficients determined from a specified criterion of stochastic linearization. We here utilize the dual criterion [27, 28] for determining coefficients $k_{eq,m}$ $(m = 1, 2, \ldots, M)$. In the first step, the original nonlinear term G_m is replaced by a linearized one $\omega_m^2 k_{eq,m} w_m$, and then the linear term $\omega_m^2 k_{eq,m} w_m$ is replaced by another nonlinear quantity, $\lambda_m G_m$, that can be considered as a term belonging to the same class of the original function G_m, where the coefficients $k_{eq,m}$and λ_m are determined from the following proposed criterion for beam vibration

$$e_1 = E\left[\left(G_m - \omega_m^2 k_{eq,m} w_m\right)^2\right] + \rho E\left[\left(\omega_m^2 k_{eq,m} w_m - \lambda_m G_m\right)^2\right] \to \min_{k_{eq,m},\lambda_m}, \tag{17}$$

with the detuning parameter ρ taking two values 0 or 1. When the parameter ρ is equal to zero, the criterion (17) becomes the conventional mean-square error criterion which can be found in the literature. On the other hand, as the parameter ρ is taken to be 1, the criterion (19) is so-called dual one. In this criterion, the first expectation can be understood as a component of the conventional replacement, whereas the second one describes a dual replacement of the linearization problem. Similar to the conventional linearization (see [6]), the criterion (17) leads to that partial derivatives of the expression e_1 with respect to variables $k_{eq,m}$ and λ_m are equal to zero

$$\frac{\partial e_1}{\partial k_{eq,m}} = 0, \quad \frac{\partial e_1}{\partial \lambda_m} = 0, (m = 1, 2, \ldots, M). \tag{18}$$

The system (18) yields a set of algebraic equations of variables $k_{eq,m}$ and λ_m, $(m = 1, 2, \ldots, M)$ as follows

$$\begin{aligned} &\left((1+\rho)\,\omega_m^2 E\left[w_m^2\right]\right) k_{eq,m} = \left(E\left[w_m G_m\right]\right)\left(1 + \rho\lambda_m\right), \\ &\lambda_m = \left(\omega_m^2 \frac{E\left[w_m G_m\right]}{E\left[G_m^2\right]}\right) k_{eq,m}. \end{aligned} \tag{19}$$

Solving the system (19) for unknowns $k_{eq,m}$ and λ_m, we arrive at

$$k_{eq,m} = \frac{1}{\omega_m^2}\frac{1}{1+\rho-\rho\eta_m}\frac{E\left[w_m G_m\right]}{E\left[w_m^2\right]}, \tag{20}$$

$$\lambda_m = \frac{\eta_m}{1+\rho-\rho\eta_m}, \tag{21}$$

where

$$\eta_m = \frac{\left(E\left[w_m G_m\right]\right)^2}{E\left[w_m^2\right] E\left[G_m^2\right]}. \tag{22}$$

It is observed that, using the dual criterion (17), the original nonlinear Eq. (14) of modes of beam vibration is replaced by its linearization version (16), in which linearization coefficients $k_{eq,m}$ $(m = 1, 2, \ldots, M)$ are determined from expressions (20)-(22). In the framework of this article, the dual criterion (17) is elucidated for random vibrations of a simply supported beam under a space-wise and time-wise white noise loading.

4. RESPONSES OF A SIMPLY SUPPORTED BEAM AT BOTH ENDS

4.1. Equivalent linearization coefficients

For a simply supported beam, one has the following expression for the modal shape ϕ_m [25]

$$\phi_m(\xi) = \sqrt{2}\sin(m\pi\xi). \tag{23}$$

Using the expressions (10) and (23), one obtain

$$K_{nm} = K_{mn} = \int_0^1 \frac{d\phi_n}{d\xi}\frac{d\phi_m}{d\xi}d\xi = \begin{cases} \pi^2 m^2 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{24}$$

Eq. (14) becomes

$$\ddot{w}_m + \frac{\beta}{\mu A}\dot{w}_m + \omega_m^2\left(1 + \frac{\alpha}{m^4}\right)w_m + \frac{\omega_m^2}{2R^2m^2}\sum_{n=1}^{M} n^2 w_n^2 w_m = b_m(t), \tag{25}$$

where

$$\omega_m^2 = \omega_0^2 m^4, \quad \omega_0^2 = \frac{EI\pi^4}{\mu AL^4}, \ \alpha = \frac{K_f}{\mu A\omega_0^2}, \quad R = \sqrt{\frac{I}{A}}. \tag{26}$$

The nonlinear functions G_m ($m = 1, 2, \ldots, M$) in Eq. (25) take the form

$$G_m = \frac{\omega_m^2}{2R^2m^2}\sum_{n=1}^{M} n^2 w_n^2 w_m. \tag{27}$$

Substituting expressions G_m ($m = 1, 2, \ldots, M$) from Eq. (27) into Eq. (20) yields

$$k_{eq,m} = \frac{1}{2R^2m^2}\frac{1}{1 + \rho - \rho\eta_m}\frac{\sum\limits_{n=1}^{M} n^2 E\left[w_n^2 w_m^2\right]}{E\left[w_m^2\right]}, \tag{28}$$

where

$$\eta_m = \frac{\sum\limits_{i=1}^{M}\sum\limits_{j=1}^{M} i^2 j^2 E\left[w_i^2 w_m^2\right] E\left[w_j^2 w_m^2\right]}{E\left[w_m^2\right]\sum\limits_{i=1}^{M}\sum\limits_{j=1}^{M} i^2 j^2 E\left[w_i^2 w_j^2 w_m^2\right]}. \tag{29}$$

It is noted that, to calculate higher-order moments in Eqs. (28) and (29), we employ the following generalized formula expressed in terms of second-order moments of Gaussian random processes with zero-mean [30]

$$E\left[z_1 z_2 \ldots z_{2m}\right] = \sum_{\text{all independent pairs}}\left(\prod_{j\neq k} E\left[z_j z_k\right]\right), \tag{30}$$

where the number of independent pairs is equal to $(2m)!/(2^m m!)$. Particularly, in view of the present dual method for beam vibration, the following higher-order moment terms will appear

$$\begin{aligned} E\left[w_m^2 w_n^2\right] &= E\left[w_m^2\right] E\left[w_n^2\right] + 2\left(E\left[w_m w_n\right]\right)^2 = y_{mm} y_{nn} + 2y_{mn}^2, \\ E\left[w_j^2 w_j^2 w_m^2\right] &= E\left[w_i^2\right] E\left[w_j^2\right] E\left[w_m^2\right] + 2\left(E\left[w_i w_j\right]\right)^2 E\left[w_m^2\right] \\ &\quad + 2\left(E\left[w_j w_m\right]\right)^2 E\left[w_i^2\right] + 2\left(E\left[w_i w_m\right]\right)^2 E\left[w_j^2\right] \\ &\quad + 8E\left[w_i w_j\right] E\left[w_j w_m\right] E\left[w_i w_m\right] \\ &= y_{ii} y_{jj} y_{mm} + 2y_{ij}^2 y_{mm} + 2y_{jm}^2 y_{ii} + 2y_{im}^2 y_{jj} + 8y_{ij} y_{jm} y_{im}, \end{aligned} \tag{31}$$

where the notation y_{mn} for the second moment of w_m is introduced

$$y_{mn} = y_{nm} = E\left[w_m w_n\right]. \tag{32}$$

Substituting expressions (31) in Eq. (28), we arrive at

$$k_{eq,m} = \frac{1}{2R^2 m^2} \frac{1}{1 + \rho - \rho\eta_m} \sum_{n=1}^{M} n^2 \frac{y_{nn} y_{mm} + 2y_{mn}^2}{y_{mm}}, \tag{33}$$

where

$$\eta_m = \frac{\sum\limits_{i=1}^{M} \sum\limits_{j=1}^{M} i^2 j^2 \left(y_{ii} y_{mm} + 2y_{im}^2\right)\left(y_{jj} y_{mm} + 2y_{jm}^2\right)}{y_{mm} \sum\limits_{i=1}^{M} \sum\limits_{j=1}^{M} i^2 j^2 \left(y_{ii} y_{jj} y_{mm} + 2y_{ij}^2 y_{mm} + 2y_{jm}^2 y_{ii} + 2y_{im}^2 y_{jj} + 8y_{ij} y_{jm} y_{im}\right)}. \tag{34}$$

For purpose of comparing the present dual criterion with other methods, in this paper, we also present two known results of the conventional linearization [21, 22], and energy method [24, 25]. From the linearized system (16), the following expressions of the equivalent linearization coefficients $k_{eq,m}$ $(m = 1, 2, \ldots, M)$ are obtained using the conventional linearization method

$$k_{eq,m\_}\text{conventional} = \frac{1}{2R^2 m^2} \sum_{n=1}^{M} n^2 \frac{y_{nn} y_{mm} + 2y_{nm}^2}{y_{mm}}. \tag{35}$$

It is observed that the expression (35) is also obtained from (33) by taking the detuning parameter ρ be zero. The linearization method based on energy criterion gives the equivalent coefficients $k_{eq,m}$ $(m = 1, 2, \ldots, M)$ obtained from the following system [24,25]

$$\begin{aligned} &\left\{1 + \alpha + k_{1,eq},\ 2^4\left(1 + \frac{\alpha}{2^4} + k_{2,eq}\right), \ldots, M^4\left(1 + \frac{\alpha}{M^4} + k_{M,eq}\right)\right\}^T \\ &\qquad = \frac{2}{\omega_0^2} \mathbf{A}^{-1} \left\{E\left[w_1^2 U\right] \quad E\left[w_2^2 U\right] \ \ldots\ E\left[w_M^2 U\right]\right\}^T, \end{aligned} \tag{36}$$

where the matrix **A** and potential energy U of the system are determined as follows

$$\mathbf{A} = \begin{bmatrix} E\left[w_1^2 w_1^2\right] & E\left[w_1^2 w_2^2\right] & \dots & E\left[w_1^2 w_M^2\right] \\ E\left[w_2^2 w_1^2\right] & E\left[w_2^2 w_2^2\right] & \dots & E\left[w_2^2 w_M^2\right] \\ \dots & \dots & \dots & \dots \\ E\left[w_M^2 w_1^2\right] & E\left[w_M^2 w_2^2\right] & \dots & E\left[w_M^2 w_M^2\right] \end{bmatrix}, \tag{37}$$

$$U = \frac{\omega_0^2}{2}\left[\sum_{m=1}^{M}\left(\alpha + m^4\right) w_m^2 + \frac{1}{4R^2}\left(\sum_{m=1}^{M} m^2 w_m^2\right)^2\right]. \tag{38}$$

To get a closed form of the equivalent linearization coefficients $k_{eq,m}$ $(m = 1, 2, \dots, M)$ in Eq. (33), we utilize responses of the linearized (16) via spectral density of the external excitations.

4.2. Responses of the linearized system

For the linearized system (16) under the random excitation b, one can obtain second-order moments $E\left[w_m w_n\right]$ of the responses w_m as follows (see [19] for details)

$$E\left[w_m w_n\right] = \int_{-\infty}^{\infty} H_m\left(-\omega\right) B_{mn}\left(\omega\right) H_n\left(\omega\right) d\omega, \tag{39}$$

where

$$B_{mn}\left(\omega\right) = \frac{S_m \delta_{mn}}{\left(\mu A\right)^2}, \tag{40}$$

and the frequency-response function $H_m\left(\omega\right)$ is given by

$$H_m\left(\omega\right) = \frac{1}{\left(1 + \frac{\alpha}{m^4} + k_{eq,m}\right)\omega_m^2 - \omega^2 + i\frac{\beta}{\mu A}\omega}. \tag{41}$$

Because $B_{mn}\left(\omega\right)$ given by (40) are constants, moments $E\left[w_m w_n\right]$ can be rewritten as

$$E\left[w_m w_n\right] = B_{mn}\int_{-\infty}^{\infty} H_m\left(-\omega\right) H_n\left(\omega\right) d\omega. \tag{42}$$

Introducing (41) into the right-hand side of the expression (42) and employing residual theorem in theory of complex variable functions, we get

$$\begin{aligned} y_{mn} = E\left[w_m w_n\right] = \frac{4\pi\beta B_{mn}}{\mu A}&\left\{\left[\left(1 + \frac{\alpha}{m^4} + k_{eq,m}\right)\omega_m^2 - \left(1 + \frac{\alpha}{n^4} + k_{eq,n}\right)\omega_n^2\right]^2\right. \\ &\left.+2\left(\frac{\beta}{\mu A}\right)^2\left[\left(1 + \frac{\alpha}{m^4} + k_{eq,m}\right)\omega_m^2 + \left(1 + \frac{\alpha}{n^4} + k_{eq,n}\right)\omega_n^2\right]\right\}^{-1}. \end{aligned} \tag{43}$$

It is seen that a closed system of nonlinear algebraic equations for unknowns $k_{eq,m}$ $(m = 1, 2, \dots, M)$ is obtained by substituting (43) in to the right hand side of Eq. (33). As noted before, because the contribution of the first modes of the system is significant, we here restrict our calculations for modal responses of the beam in two cases: single-, and two-mode using three approaches: the conventional linearization, energy method, and present dual criterion.

5. NUMERICAL RESULTS AND DISCUSSIONS

It is seen that, in Eq. (25) for vibrational modes, as R tends to infinity, the effect of nonlinear terms $G_m (m = 1, 2, \ldots, M)$ disappear. Therefore, one can view the magnitude of the quantity $1/R$ as the parameter related to the magnitude of nonlinearity of the original nonlinear system (25). Assume that spectral densities $(m = 1, 2, \ldots, M)$ of the stochastic processes $q_m(t)$ have the same value, i.e. $S_1 = S_2 = \ldots = S_M = S_0$. For this assumption, from the Fokker-Planck equation corresponding to the system (25), the exact expression of the probability density function can be obtained [24]

$$\begin{aligned} P(w_1, w_2, \ldots, w_M) &= \frac{1}{C} \exp\left\{ -\frac{\beta\mu A\omega_0^2}{2\pi S_0} \left[\sum_{m=1}^{M} \left(\alpha + m^4\right) w_m^2 + \frac{1}{4R^2} \left(\sum_{m=1}^{M} m^2 w_m^2 \right)^2 \right] \right\} \\ &= \frac{1}{C} \exp\left\{ -\frac{\beta\mu A}{\pi S_0} U(w_1, w_2, \ldots, w_M) \right\}, \end{aligned} \tag{44}$$

where $U = U(w_1, w_2, \ldots, w_M)$ is the potential energy of the system (25) given by (38), and C is the normalization constant

$$C = \underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{M-fold} \exp\left\{ -\frac{\beta\mu A\omega_0^2}{2\pi S_0} \left[\sum_{m=1}^{M} \left(\alpha + m^4\right) w_m^2 + \frac{1}{4R^2} \left(\sum_{m=1}^{M} m^2 w_m^2 \right)^2 \right] \right\} dw_1 \ldots dw_M. \tag{45}$$

The exact modal mean-square response of Eq. (25) is evaluated by the following multiple integral with M-fold

$$E\left[w_m^2\right]_{\text{exact}} = \frac{1}{C} \underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{M-\text{fold}} w_m^2 \exp\left\{ -\frac{\beta\mu A\omega_0^2}{2\pi S_0} \left[\sum_{m=1}^{M} \left(\alpha + m^4\right) w_m^2 + \frac{1}{4R^2} \left(\sum_{m=1}^{M} m^2 w_m^2 \right)^2 \right] \right\} dw_1 \ldots dw_M. \tag{46}$$

In general, the multiple integral (46) must be calculated using a numerical method. In the following computation, we use the exact solution (46) in the case of single-mode $(M = 1)$ and of two-mode $(M = 2)$ to elucidate the accuracy of the proposed dual criterion method (33), and other methods for comparison purpose.

5.1. The case of single-mode

For the single-mode, $M = 1$, the governing equation of the simply supported beam (25) takes the form

$$\ddot{w}_1 + \frac{\beta}{\mu A} \dot{w}_1 + \omega_0^2 (1 + \alpha) w_1 + \frac{\omega_0^2}{2R^2} w_1^3 = \frac{1}{\mu A} q_1(t). \tag{47}$$

This is the well-known Duffing oscillator subjected to random excitation [4, 5]. From Eq. (46), one can get an exact solution of mean-square response of w_1 in Eq. (47) as follows (see also [22, 26])

$$E\left[w_1^2\right]_{\text{exact},1} = \frac{\int\limits_{-\infty}^{\infty} w_1^2 \exp\left\{-\frac{\beta\mu A}{\pi S_0}\left(\frac{1}{2}\omega_0^2\left(1+\alpha\right)w_1^2 + \frac{\omega_0^2}{8R^2}w_1^4\right)\right\}dw_1}{\int\limits_{-\infty}^{\infty} \exp\left\{-\frac{\beta\mu A}{\pi S_0}\left(\frac{1}{2}\omega_0^2\left(1+\alpha\right)w_1^2 + \frac{\omega_0^2}{8R^2}w_1^4\right)\right\}dw_1}$$
$$= (1+\alpha)R^2\left[K_{3/4}\left(\frac{(1+\alpha)^2}{4}\frac{R^2}{R_{01}^2}\right) - K_{1/4}\left(\frac{(1+\alpha)^2}{4}\frac{R^2}{R_{01}^2}\right)\right] \Big/ K_{1/4}\left(\frac{(1+\alpha)^2}{4}\frac{R^2}{R_{01}^2}\right), \tag{48}$$

where $K_\nu\left(y\right)$ is the modified Bessel function of the second kind of order ν of the variable y, and the quantity R_{01} is given by

$$R_{01} = \sqrt{\frac{\pi S_0}{\beta\left(\mu A\right)\omega_0^2}}. \tag{49}$$

Using Eqs. (43) and (28) for $M = 1$, we get the following single-mode approximate mean-square response $E\left[w_1^2\right]$ depending upon the nonlinear parameter $1/R$ in the following form

$$E\left[w_1^2\right]_{\text{dual},1} = \frac{2R_{01}^2}{1+\alpha+\sqrt{\left(1+\alpha\right)^2 + \frac{30}{7}\frac{R_{01}^2}{R^2}}}. \tag{50}$$

Similarly, approximate mean-square responses of w_1 corresponding to the conventional linearization (35) and energy method (36) in the case of single-mode are obtained, respectively,

$$E\left[w_1^2\right]_{\text{conventional},1} = \frac{2R_{01}^2}{1+\alpha+\sqrt{\left(1+\alpha\right)^2 + \frac{6R_{01}^2}{R^2}}}, \tag{51}$$

$$E\left[w_1^2\right]_{\text{energy},1} = \frac{2R_{01}^2}{1+\alpha+\sqrt{\left(1+\alpha\right)^2 + \frac{5R_{01}^2}{R^2}}}. \tag{52}$$

Numerical results for the first mode of beam vibration in the case of single-mode using four methods, including the exact solution (48), conventional linearization (51), energy method (52), and dual criterion method (50) are illustrated in Tabs. 1 and 2. The system parameters are $\omega_0 = 1$, $\beta = 0.1$, $\mu A = 1$. Tab. 1 shows a comparison of relative errors between results obtained from approximate and exact solution methods. The stiffness parameter α of the system is fixed at 1, whereas the nonlinearity parameter $1/R$ varies from 0.01 to 10.0. It is seen that, for small values of $1/R$, for example $1/R = 0.01,\ 0.02,\ 0.05$, the conventional linearization yields quite small errors, about 0.05%, whereas errors of the energy and dual criterion methods are larger. In the range $[1, 10]$ of $1/R$, the error of conventional linearization becomes larger 10% while that of the energy method and

Table 1. Mean-square response of w_1 of the simply supported beam in case of single-mode with $\omega_0 = 1$, $\alpha = 1$, $\beta = 0.1$, $\mu A = 1$, $S_0 = 1$ and various values of $1/R$ (CL: Conventional Linearization; EM: Energy Method; DM: Dual Criterion Method)

$1/R$	$E\left[w_1^2\right]_{\text{exact},1}$	$E\left[w_1^2\right]_{\text{CL},1}$	Error (%)	$E\left[w_1^2\right]_{\text{EM},1}$	Error (%)	$E\left[w_1^2\right]_{\text{DM},1}$	Error (%)
0.01	15.6895	15.6895	0.0001	15.6926	0.0195	15.6948	0.0335
0.02	15.6349	15.6346	0.0014	15.6468	0.0761	15.6554	0.1317
0.05	15.2778	15.2707	0.0466	15.3403	0.4086	15.3907	0.7388
0.10	14.2613	14.1964	0.4549	14.4101	1.0437	14.5706	2.1690
0.20	11.9200	11.6419	2.3324	12.0674	1.2370	12.4086	4.0991
0.50	7.3952	6.8668	7.1448	7.3248	0.9518	7.7220	4.4191
1.00	4.4131	3.9581	10.3102	4.2767	3.0909	4.5614	3.3618
2.00	2.4261	2.1276	12.3034	2.3146	4.5968	2.4842	2.3928
5.00	1.0294	0.8890	13.6400	0.9712	5.6599	1.0463	1.6388
10.00	0.5251	0.4510	14.1093	0.4934	6.0421	0.5322	1.3564

Table 2. Mean-square response of w_1 of the simply supported beam in case of single-mode with $\omega_0 = 1$, $\beta = 0.1$, $\mu A = 1$, $S_0 = 1$, $R = 1$ and various values of α

α	$E\left[w_1^2\right]_{\text{exact},1}$	$E\left[w_1^2\right]_{\text{CL},1}$	Error (%)	$E\left[w_1^2\right]_{\text{EM},1}$	Error (%)	$E\left[w_1^2\right]_{\text{DM},1}$	Error (%)
1.0	4.4131	3.9581	10.3102	4.2767	3.0909	4.5614	3.3618
2.0	4.0310	3.6844	8.5978	3.9549	1.8889	4.1930	4.0181
3.0	3.6976	3.4334	7.1448	3.6624	0.9518	3.8610	4.4191
4.0	3.4056	3.2038	5.9244	3.3975	0.2382	3.5629	4.6202
5.0	3.1489	2.9944	4.9075	3.1581	0.2921	3.2960	4.6714
6.0	2.9224	2.8036	4.0656	2.9422	0.6756	3.0573	4.6147
7.0	2.7218	2.6300	3.3715	2.7475	0.9442	2.8438	4.4840
8.0	2.5433	2.4721	2.8008	2.5719	1.1241	2.6528	4.3056
9.0	2.3840	2.3284	2.3324	2.4135	1.2370	2.4817	4.0991
10.0	2.2412	2.1975	1.9480	2.2703	1.3000	2.3281	3.8785

dual criterion are remaining about 6%. For large values of $1/R$, for instance $1/R = 5$, $1/R = 10$, however, the error of the dual criterion method is smallest (about 2%).

In Tab. 2, the parameter $1/R$ is taken to be 1, the stiffness parameter α varies from 1.0 to 10.0, other parameters have the same values as in Tab. 1. It is observed that, the dual criterion method gives a good prediction on response errors (about 5%) as α varies, and the error of energy method is smallest.

5.2. The case of two-mode

Numerical computations used the fixed-point iteration method [25, 26] to find approximate mean-square response of the first mode of beam vibration are carried out for the nonlinear algebraic system (33) and (43) with unknowns $k_{m,eq}$ in the case of two-mode. The exact solution is obtained from the multiple integral (46) with $M = 2$. The obtained numerical results are presented in Tabs. 3 and 4. Tab. 3 shows that, the error of dual criterion method is in good agreement with that of the energy method. In Tab. 4, in general, the dual criterion and energy method yield values that are close to the exact solutions with different value of the stiffness parameter α.

Table 3. Mean-square response of $w_1$of the simply supported beam in case of two-mode with $\omega_0 = 1$, $\alpha = 1$, $\beta = 0.1$, $\mu A = 1$, $S_0 = 1$ and various values of $1/R$

$1/R$	$E\left[w_1^2\right]_{\text{exact},1}$	$E\left[w_1^2\right]_{\text{CL},1}$	Error (%)	$E\left[w_1^2\right]_{\text{EM},1}$	Error (%)	$E\left[w_1^2\right]_{\text{DM},1}$	Error (%)
0.01	15.6865	15.6866	0.0007	15.6901	0.0232	15.6921	0.0358
0.02	15.6233	15.6232	0.0006	15.6372	0.0887	15.6449	0.1385
0.05	15.2125	15.2050	0.0494	15.2847	0.4744	15.3290	0.7658
0.10	14.0574	13.9915	0.4688	14.2343	1.2587	14.3672	2.2039
0.20	11.4744	11.2121	2.2857	11.6951	1.9235	11.9358	4.0207
0.50	6.7777	6.3387	6.4772	6.8873	1.6175	7.0652	4.2423
1.00	3.9033	3.5576	8.8571	3.9627	1.5219	4.0385	3.4636
2.00	2.0923	1.8799	10.1519	2.1284	1.7243	2.1529	2.8951
5.00	0.8714	0.7764	10.9030	0.8891	2.0276	0.8936	2.5492
10.00	0.4414	0.3923	11.1312	0.4510	2.1828	0.4522	2.4529

Table 4. Mean-square response of $w_1$of the simply supported beam in case of two-mode with $\omega_0 = 1$, $\beta = 0.1$, $\mu A = 1$, $S_0 = 1$, $R = 1$ and various values of α

α	$E\left[w_1^2\right]_{\text{exact},1}$	$E\left[w_1^2\right]_{\text{CL},1}$	Error (%)	$E\left[w_1^2\right]_{\text{EM},1}$	Error (%)	$E\left[w_1^2\right]_{\text{DM},1}$	Error (%)
1.0	3.9033	3.5576	8.8571	3.9627	1.5219	4.0385	3.4636
2.0	3.5878	3.3203	7.4557	3.6862	2.7440	3.7299	3.9603
3.0	3.3112	3.1035	6.2725	3.4338	3.7025	3.4526	4.2703
4.0	3.0677	2.9058	5.2784	3.2038	4.4375	3.2038	4.4375
5.0	2.8526	2.7256	4.4511	2.9947	4.9812	2.9808	4.4929
6.0	2.6616	2.5616	3.7582	2.8047	5.3755	2.7807	4.4741
7.0	2.4915	2.4121	3.1849	2.6321	5.6419	2.6010	4.3960
8.0	2.3393	2.2760	2.7071	2.4752	5.8105	2.4394	4.2811
9.0	2.2026	2.1517	2.3090	2.3326	5.9016	2.2938	4.1422
10.0	2.0793	2.0383	1.9738	2.2027	5.9355	2.1623	3.9922

6. CONCLUSIONS

The method of equivalent linearization is one of effective tools in solving random vibration problems of mechanical systems. In this study, the modal responses of a simply supported beam subjected to a space-wise and time-wise white noise loading are carried out by the dual criterion of equivalent linearization. Our calculations are restricted in two cases of single- and two-mode of beam vibrations. The exact solutions of the original modal equation system obtained by Fokker-Planck equation are available for both cases. A closed form of nonlinear algebraic system is obtained by the dual approach associated with the frequency-response function method for the linearized modal system. In the case of single-mode, the analytical solution of the first mode of the beam is easy to solve explicitly for four methods considered (the exact solution, energy method, conventional linearization and dual criterion method). Also, in the case of two-mode, the closed system is solved by the fixed-point iteration method. Numerical results show that the dual criterion gives a good prediction on the random responses of the beam, especially in the range of strong linearity of system parameters. Further investigations for random vibrations of other beam systems seem to be appropriate in order to verify the advantages of the dual criterion.

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