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ANALYSIS OF CRACKED PLATE MEMBRANE PROBLEM USING METIS FINITE ELEMENT MODEL

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ABSTRACT. A conformable and convergent finite element technique is presented for calculation of stress intensity factors for cracked plate membrane problem, which is based on the formulation of the hybrid displacement finite element method, named "Metis elements". In order to achieve a high convergence, this element is combined with an isoparametric element of Barsoum, in which, the mid-side nodes are moved to quarter-point position. Many examples are numerically tested for evaluation this model, show that the element HSM has a good performance for calculation of stress intensity factors.
Key words. Stress intensity factors, Hybrid metis Singular element of Membrane plate (HSM).

1. Introduction

Use of the stress intensity factor in examining cracked structural problem requires an accurate knowledge of the stress field in the vicinity of the crack tip for the structural geometry, loading and boundary condition. Unfortunately, analytical solutions only exist for selecting, relatively simple cases wherein: 1) the domain is considered to be infinite, 2) the material is homogeneous, in most cases, is isotropic, and 3) the boundary conditions are not complicated. To deal with practical problems of mechanics of fracture in flawed structures of finite size, arbitrary shape, complicated boundary condition, and arbitrary material properties, numerical techniques such as finite elements and boundary integral methods are mandatory.

This work deals with constructing a special singular finite element named HSM, which is based on combination of the Metis element proposed by Nguyen Dang Hung [1] and an isoparametric element of Barsoum [5], to analyze the cracked plate membrane problem.

2. Basic fracture mechanics

2.1. Definition

Fracture mechanics admits the existence of an initial fissure in the structure under consideration which defines a local and irreversible separation of a continuous milieu at two borders, calls crack surfaces. The external solicitation tends to extend the fissures to different modes. Irwin observed that there are three independent kinematic movements of the upper and lower crack surfaces with respect to each other and these are categorized as:

Opening mode (mode I): in which the two crack surfaces are pulled apart in their direction, but where the deformations are symmetric about the $x-z$ and $x-y$ planes.

Shearing mode (mode II): in which the two crack surfaces slide over each other in the x-direction, but where the deformations are symmetric about the $x-y$ plan and skew symmetric about $x-z$ plane.

Tearing mode (mode III): in which the two crack surfaces slide over each other in the z-direction, but where the deformations are skew symmetric about the $x-y$ and $x-z$ planes.

The appropriate superposition of these three modes allows to describe the field of displacement, strain and stress in the vicinity of crack-tip in general case. In two dimension problems, only mode I and mode II are existent.

2.2. Stress intensity factors for an isotropic material

For cracked plates subjected to in-plane loads, if the fissure is localized in polar coordinate (r,θ), the solutions of stress and displacement in the vicinity of crack-tip are always the form:

$$\sigma_{ij} = \frac{K_M}{\sqrt{2\pi r}} f_{ij}(\theta), \quad u_i = K_M\sqrt{\frac{2r}{\pi}}\, g_i(E,\nu,\theta) \tag{2.1}$$

where E and ν are Young's modulus and Poisson's coefficient respectively, K_M $(M = I, II)$ are the **stress intensity factors**. If their values are known, they permit to determine completely the stress and displacement field near the point of fissure.

3. Modelisation of isoparametric displacement metis element in fracture mechanics

3.1. Variational formulation of displacement metis elements

Consider the finite elements V_m ($1 \leq m \leq N$, N is number of element) of the structure, $B(\sigma_{ij})$ is the complementary energy density, T_j are the imposed tractions on the part S_{σ_m} of the element boundary and u_j are the imposed displacements on the part S_{um} of the element boundary. If volumetric force is absent, correspondent total potential energy is:

$$\Pi_{MD} = \sum_{m=1}^{N}\Big\{ \int_{V_m} \frac{1}{2}\big[B(\sigma_{ij}) - \sigma_{ij}(u_{i,j}+u_{j,i})\big]dV + \int_{S_{\sigma_m}} \overline{T}_j u_j dS\Big\}. \tag{3.1}$$

The hybrid element is based on the variation of (3.1) leads to a class named "**displacement metis elements**".

3.2. Discretisation formulation

We choose a discretized field σ in V_m contains a regular part and a singular part:

$$\sigma = \sigma^R + \sigma^S = S_1 b_1 + S_2 b_2 \tag{3.2}$$

where S_1 is the polynomial interpolation matrix of regular stresses of element; b_1 is the vector of unknown parameters whose number (ND) is associated to select degree of the regular discretisation. S_2 is the singular matrix presented by (2.1). It defines the singular field of stresses. b_2 is the vector of unknown parameters of singular part of discretisation stress field:

$$b_2^T = \langle K_I, K_{II} \rangle \tag{3.3}$$

For the discretisation of displacement field, we take:

$$\mathbf{u} = \mathbf{N}_s \mathbf{q} \tag{3.4}$$

where $\mathbf{N}_s$ is the matrix of interpolation functions and $\mathbf{q}$ is the vector of indetermined displacements at nodes of elements.

With (3.2), (3.4), the function (3.1) becomes:

$$\Pi_{MD} = \sum_{m=1}^{K} \Pi_{MD}^R + \sum_{m=K+1}^{N} \Pi_{MD}^S \tag{3.5}$$

where K is the number of regular elements of structure (the number of singular elements $= N - K$) and the symbol R and S signify respectively the regular and singular parts.

4. Method of calculation of stress intensity factors

The matrix of stiffness and vector of global solicitations $[\mathbf{K}]$ and $[\mathbf{F}]$ are constructed by assemble operation of all the structure:

$$\begin{bmatrix} K_1 & \dots & K_{12} \\ \vdots & \ddots & \vdots \\ K_{12}^T & \dots & K_2 \end{bmatrix} \begin{Bmatrix} q \\ b_2 \end{Bmatrix} = \begin{Bmatrix} F \\ \vdots \\ 0 \end{Bmatrix} \tag{4.1}$$

where K_1 is the element stiffness matrix of regular part; K_2 is the element stiffness matrix of singular part and K_{12} is the element stiffness matrix of regular-singular part.

The resolution of equation system (4.1) furnishes the unknown displacements q at nodes of elements and the parameters of singular part b_2. Then, the stress intensity factors can be obtained from b_2 by equation (3.3). This method is called the direct method (MCI).

5. Analysis of element HSM in cracked plate membrane problem

5.1. Description of element HSM

5.1.1. The interpolation function of stress field

5.1.1.1. Regular stress field S_1

S_1 is the discretisation matrix of the regular part of stresses σ^R of the equation (3.2). Using of a stress function of Airy [2], we have:

$$\underbrace{\begin{Bmatrix} \sigma_x^R \\ \sigma_y^R \\ \tau_{xy}^R \end{Bmatrix}}_{[3\times 1]} = \underbrace{\begin{bmatrix} \dots & n_i(n_i-1)x^{m_i-n_i}y^{n_i-2} & \dots \\ \dots & (m_i-n_i)(m_i-n_i-1)x^{m_i-n_i-2}y^{n_i} & \dots \\ \dots & n_i(n_i-m_i)x^{m_i-n_i-1}y^{n_i-1} & \dots \end{bmatrix}}_{[3\times ND]} \underbrace{\begin{Bmatrix} b_1 \\ \vdots \\ b_{ND} \end{Bmatrix}}_{[ND\times 1]} \tag{5.1}$$

where the index i varies from 1 to ND. We can find the relations between m and n in term of i:

$$m_i = \text{interger part of}\{1.99 + 0.5(\sqrt{25+8i} - 5)\}, \quad n_i = i + 2 - 0.5\,m(m+1). \tag{5.2}$$

Remark. Pian [3, 4] shown that the necessary condition so that the element stiffness matrix is regular is:

$$ND \geq N_q - N_r \tag{5.3}$$

where ND is the number of parameters b, N_q is the number of nodal displacements and N_r is the number of rigid body modes. For a quadrangle element of 8 nodes with 16 degree of freedom, the minimum value of ND is 13. Thus the complete polynomial base is minimum cubic in order to satisfy the invariant condition and the optimal number of parameters b is 18.

5.1.1.2. Singular field of generalized stress S_2

S_2 is the discretisation matrix of singular part of stress σ^S of the equation (2.1):

$$\underbrace{\begin{Bmatrix} \sigma_x^S \\ \sigma_y^S \\ \tau_{xy}^S \end{Bmatrix}}_{[3\times 1]} = [S_2] \begin{Bmatrix} b_2 \\ \vdots \\ b_{2NF} \end{Bmatrix} = \underbrace{\begin{bmatrix} S_x^I & S_x^{II} \\ S_y^I & S_y^{II} \\ S_{xy}^I & S_{xy}^{II} \end{bmatrix}}_{[3\times 2NF]} \underbrace{\begin{Bmatrix} \beta_I \\ \vdots \\ \beta_{II} \\ \vdots \end{Bmatrix}}_{[2NF\times 1]} \tag{5.4}$$

where NF is the number of unknown parameters which depends on degree of polynomial of discretisation S_2.

We have the analytic solutions of singular stress field for the plane problem of fissure in the case of isotropic material if $NF = 1$:

$$[S_2] = \frac{1}{4\sqrt{r}} \begin{bmatrix} 3\cos\frac{\theta}{2} + \cos\frac{5\theta}{2} & -7\sin\frac{\theta}{2} - \sin\frac{5\theta}{2} \\ 5\cos\frac{\theta}{2} - \cos\frac{5\theta}{2} & -\sin\frac{\theta}{2} + \sin\frac{5\theta}{2} \\ -\sin\frac{\theta}{2} + \sin\frac{5\theta}{2} & 3\cos\frac{\theta}{2} + \cos\frac{5\theta}{2} \end{bmatrix} \tag{5.5}$$

5.1.2. The interpolation function of displacement field

5.1.2.1. Regular displacement field

The interpolation functions of displacement field (3.4) for the regular element of HSM is interpolated by the follow form functions:

$$\tilde{u} = \sum_{i=1}^{8} N_i(\xi,\eta)u_i, \quad \tilde{v} = \sum_{i=1}^{8} N_i(\xi,\eta)v_i \tag{5.6}$$

where the form functions are:

$$\begin{aligned} N_i = {} & \frac{1}{4}\xi_i^2\eta_i^2\big[(1+\xi\xi_i)(1+\eta\eta_i) - (1-\xi^2)(1+\eta\eta_i) - (1-\eta^2)(1+\xi\xi_i)\big] \\ & + \frac{1}{2}\eta_i^2(1-\xi^2)(1+\eta\eta_i)(1-\xi_i^2) + \frac{1}{2}\xi_i^2(1-\eta^2)(1+\xi\xi_i)(1-\eta_i^2) \end{aligned} \tag{5.7}$$

where: ξ_i, $\eta_i = \pm 1$ at vertex nodes and $= 0$ at mid-side nodes.

5.1.2.2. Singular displacement field

As in the element of Barsoum [5], we obtain the singularity of strain of a triangular element by degrading an edge of the quadrilateral element and moving the mid-side nodes to the quarter point adjacent to the crack-tip (Fig. 1)

5.2. Numerical results and discussion

5.2.1. Central fissure square plate subjected to uniform tension

In the first example, we examine a square plate of horizontally central fissure subjected to an uniform traction. We choose the material properties as follows:

- The module of Young: $E = 5250$ (MN/m^2)
- The Poisson's coefficient: $\nu = 0.3$
- Demi-length of plate: $L = 3.6$ (m)
- Demi-length of fissure: $a = 1.8$ (m)
- Intensity of traction over unit length: $p = 1$ (MN/m)

The results by the direct method (MCI) are in the Table 1, compare with the exact solution K_I of Bowie and Neal [6] and the error rate for each case.

As the demonstration in the Table 1, the obtained results are well accordant with the exact value of K_I. The convergence is good despite of the mesh is quite coarse. The error is 0.82 % for a mesh of only 10 elements (74 degrees of freedom). Pin

Tong, Pian and Larsy [7] are obtained a value of K_I with error 5 % by utilizing 70 degrees of freedom and error 0.5% by utilizing 184 degrees of freedom. We remark that the MCI method is better than the indirect method (MCII) [8].

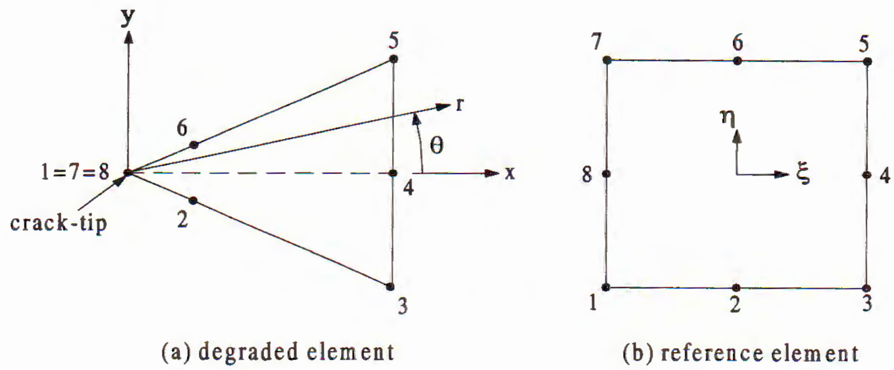

Fig. 1. Element of Barsoum

Table 1. Results of the central fissure square plate

Mesh	Number of element	Degree of freedom	K_I	K_I (exact)	Error (%)
A	10	74	3.197		0.82
B	18	134	3.183	3.171	0.38
C	38	270	3.178		0.22

5.2.2. Symmetric fissures square plate subjected to uniform tension

In the second example, we examine a square plate of horizontally symmetric fissures subjected to an uniform traction. We choose material properties same as the first example, except $\nu = 0.2$.

The results by MCI method are in the Table 2, compare with the exact solution K_I of Bowie and Neal [6] and the error rate for each case. As the demonstration in the Table 2, the obtained results are well accordant with the exact value of K_I. The error is 2.18 % for a mesh of only 10 elements (74 degrees of freedom).

Table 2. Results of the symmetric fissure square plate

Mesh	Number of element	Degree of freedom	K_I	K_I (exact)	Error (%)
A	10	74	3.138		2.18
B	18	134	3.137	3.071	2.15
C	38	270	3.101		0.98

5.2.3. Fissure rectangular plates subjected to uniform tension

In the third example, we examine two rectangular plates of horizontally central and symmetric fissures subjected to an uniform traction. We choose the material properties same as the second example, the length and width of plate are $12a \times 4a$.

The results by MCI method are in the Table 3 and 4, compare with the exact solution K_I of Bowie and Neal [6] and the error rate for each case.

Table 3. Results of the central fissure square plate

Mesh	Number of element	Degree of freedom	K_I	K_I (exact)	Error (%)
A	14	110	2.874		1.59
B	18	138	2.835	2.829	0.21
C	22	162	2.839		0.35

Table 4. Results of the symmetric fissure square plate

Mesh	Number of element	Degree of freedom	K_I	K_I (exact)	Error (%)
A	14	110	2.859		4.45
B	18	138	2.774	2.737	1.35
C	22	162	2.779		1.53

According to the demonstration in the Table 3 and 4, the obtained results are well accordant with the exact value of K_I. We can arrive to the follow conclusions: the size of the singular elements is very important for the calculation of stress intensity factors, but the size of the regular elements influence lightly to the convergence of K_I and the fan shape meshes are very effective.

6. Conclusions

It can be concluded that the metis finite element model presented in this paper is an efficient model for the analysis of cracked membrane plate problem. For further development, the present method can be used to calculate the stress intensity factors of cracked plate bending as well as shell problem.

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PHÂN TÍCH BÀI TOÁN TẤM MÀNG CÓ VẾT NỨT
SỬ DỤNG MÔ HÌNH PHẦN TỬ HỮU HẠN LAI METIS

Một kỹ thuật phần tử hữu hạn hội tụ và thích hợp, dựa trên dạng lai chuyển vị, được trình bày để tính toán các hệ số cường độ ứng suất của bài toán tấm màng có vết nứt, có tên là "Phần tử Metis". Để đạt được một sự hội tụ cao, phần tử này sẽ được kết hợp với phần tử đẳng tham số Barsoum, trong đó các nút ở giữa các cạnh được chuyển về vị trí một phần tư. Rất nhiều khảo sát bằng số đã được thực hiện đối với mô hình này cho thấy phần tử HSM có một phẩm chất rất tốt trong việc tính toán các hệ số cường độ ứng suất.

TÁC GIẢ GỬI BÀI ĐĂNG CHÚ Ý

Từ năm 2002 Tạp chí Cơ học (VIETNAM JOURNAL OF MECHANICS) được in với chất lương cao. Nhà in nhận in từ file.dvi được soạn thảo bằng Latex (trong đó có đủ hình vẽ ở dạng file.WMF hoặc file.BMP).

Vậy khi gửi bài các tác giả cần gửi cho tòa soạn các file hình dưới dạng file.WMF hoặc file.BMP.

Mỗi hình là một file riêng, kích thước trang nền để hình vừa bằng kích thước hình (để dễ ghép vào bài) với chiều ngang không quá 15cm, chiều đứng không quá 19cm. (xem tham khảo tạp chí năm 2002 các số 1-4)