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NON-LINEAR ANALYSIS OF LAMINATED PLATES

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ABSTRACT. The governing equations of laminated plates taking into account the transverse shear deformation effects for large deformation are given. The formulation of Ritz method and finite element method for non-linear analysis of this problem is presented.

1. Introduction

Static and dynamic analysis of laminated plates for small deformation has been studied by many researchers [1, 4, 5,...]. Besides of classical theory of laminated plates one considers the problems in higher-order displacement model. Some authors in our group of project investigate the bending problem for thick layered composite plate based on a full third - order plate bending theory [7].

At present the analysis of laminated plates for large deformation attracts our attention. When the loading is large, the geometric non-linearity of the laminated plate will considerably affect the deformation. For multi-layered plates and shells with periodic structure, applying the homogenization method, we have considered linear and non-linear bending and buckling problems [2].

The purpose of present paper is to investigate the non-linear behavior of laminated plates, structure of which is not periodic.

2. Governing equations of laminated plates

Taking into account the shear deformation in the plate, the stress - strain relation for the k-th layer can be expressed as follows

$$\boldsymbol{\sigma} = \mathbf{Q}\bar{\boldsymbol{\varepsilon}}, \qquad (2.1)$$

where $\boldsymbol{\sigma}$ - a stress tensor,

$\bar{\boldsymbol{\varepsilon}}$ - a strain tensor,

$\mathbf{Q}$ - a tensor of the transformed elastic coefficients.

For convenience later the relation (2.1) can be rewritten in matrix form

$$\{\boldsymbol{\sigma}\} = [\mathbf{Q}]\{\bar{\boldsymbol{\varepsilon}}\}, \qquad (2.2)$$

where

$$\{\sigma\} = \left(\sigma_{xx}, \sigma_{yy}, \tau_{xy}, \tau_{yz}, \tau_{xz}\right)^T,$$
$$\{\bar{\varepsilon}\} = \left(\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\right)^T,$$
$$[\mathbf{Q}] = \begin{pmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{pmatrix},$$

$\{.\}$ denotes a column vector and $[\ldots]$ denotes a matrix.

The strain-displacement relations in non-linear theory are of the form

$$\begin{aligned} \varepsilon_{xx} &= \varepsilon^0_{xx} + z\chi_x, \\ \varepsilon_{yy} &= \varepsilon^0_{yy} + z\chi_y, \\ \gamma_{xy} &= \varepsilon^0_{xy} + z\chi_{xy}, \\ \gamma_{yz} &= \gamma^0_{yz}, \\ \gamma_{xz} &= \gamma^0_{xz}. \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \varepsilon^0_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ \varepsilon^0_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \gamma^0_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \\ \gamma^0_{yz} &= \psi_y + \frac{\partial w}{\partial y}, \\ \gamma^0_{xz} &= \psi_x + \frac{\partial w}{\partial x}, \\ \chi_x &= \frac{\partial \psi_x}{\partial x}, \\ \chi_y &= \frac{\partial \psi_y}{\partial y}, \\ \chi_{xy} &= \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}, \end{aligned} \tag{2.4}$$

u, v and w are the midplane displacements along the x, y and z axes respectively and ψ_x, ψ_y are the normal rotations. All these quantities depend on variables x and y.

We determine membrane forces N_x, N_y, N_{xy}, transversal shear forces Q_x, Q_y and flexion moments M_x, M_y, M_{xy} of the plate by integrating over the thickness

$$\begin{aligned}(N_x, N_y, N_{xy}) &= \int\limits_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) dz, \\ (Q_x, Q_y) &= \int\limits_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) dz, \\ (M_x, M_y, M_{xy}) &= \int\limits_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) z dz.\end{aligned} \tag{2.5}$$

Substituting (2.2) into (2.5) after some operations we obtain the constitutive equations of the laminated plate

$$\mathbf{\Sigma} = \mathbf{D}\boldsymbol{\varepsilon},$$

or in the matrix form

$$\{\mathbf{\Sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\} \tag{2.6}$$

where

$$\begin{aligned}\{\mathbf{\Sigma}\} &= (N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_y, Q_x)^T, \\ \{\boldsymbol{\varepsilon}\} &= (\varepsilon^0_{xx}, \varepsilon^0_{yy}, \varepsilon^0_{xy}, \chi_x, \chi_y, \chi_{xy}, \gamma^0_{yz}, \gamma^0_{xz})^T, \\ [\mathbf{D}] &= \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix}\end{aligned}$$

Elements of the matrix $[\mathbf{D}]$ are determined by

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int\limits_{h_{k-1}}^{h_k} \left(Q_{ij}\right)_k (1, z, z^2) dz \quad (i, j = 1, 2, 6),$$

$$(A_{44}, A_{45}, A_{55}) = \sum_{k=1}^{N} \int\limits_{h_{k-1}}^{h_k} \left(Q_{44}, Q_{45}, Q_{55}\right)_k dz,$$

where h_k is the thickness and $\left(Q_{ij}\right)_k$ - the material coefficients of the k-th layer of the plate, N is the number of layers.

For the non-linear static analysis of laminated plates we can apply Lagrange's variation principle

$$\delta U = \delta A \tag{2.7}$$

and for the non-linear dynamic analysis we can use Hamilton's principle

$$\int_{t_0}^{t_1} (\delta T - \delta U + \delta A) dt = 0, \tag{2.8}$$

where δU is a variation of the strain energy, δT is a variation of the kinetic energy of the plate and δA is a variation of the work done by external forces acting on the plate, δ denotes the first variation , t_0 and t_1 are two arbitrary time variables.

By definition we can express the variation δU, δA and δT in the formulae

$$\begin{aligned} \delta U &= \iint_S \mathbf{\Sigma} \cdot \delta \boldsymbol{\varepsilon} dxdy = \iint_S \{\delta\boldsymbol{\varepsilon}\}^T [\mathbf{D}] \{\boldsymbol{\varepsilon}\} dxdy; \\ \delta A &= \iint_S \mathbf{F} \cdot \delta \mathbf{u} dxdy = \iint_S \{\delta \mathbf{u}\}^T \{\mathbf{F}\} dxdy; \\ \delta T &= \iint_S \delta \dot{\mathbf{u}} \cdot \mathbf{M} \dot{\mathbf{u}} dxdy = \iint_S \{\delta \dot{\mathbf{u}}\}^T [\mathbf{M}] \{\dot{\mathbf{u}}\} dxdy, \end{aligned} \tag{2.9}$$

where the integration is taken over all the plate middle surface S,

$\{\mathbf{F}\}$ - the external generalized forces acting on the plate,

$\{\mathbf{u}\} = (u, v, w, \psi_x, \psi_y)^T$ - the displacement components of a point of the middle surface,

$\{\dot{\mathbf{u}}\} = (\dot{u}, \dot{v}, \dot{w}, \dot{\psi}_x, \dot{\psi}_y)^T$ - the velocity components, the dot denotes a derivative with respect to the variable time t;

$[\mathbf{M}]$ - the mass matrix, which is defined as follows [see 6]

$$[\mathbf{M}] = \begin{bmatrix} \rho_s & 0 & 0 & R & 0 \\ 0 & \rho_s & 0 & 0 & R \\ 0 & 0 & \rho_s & 0 & 0 \\ R & 0 & 0 & I_{xy} & 0 \\ 0 & R & 0 & 0 & I_{xy} \end{bmatrix},$$

$$(\rho_s, R, I_{xy}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \rho^{(k)} (1, z, z^2) dz,$$

$\rho^{(k)}$ is the mass density of the k - th layer.

The boundary conditions depend on the supports of plate edges. We consider a rectangular plate of edges $x = 0$, $x = a$ and $y = 0$, $y = b$. The boundary conditions may be

a) Simply - supported edges

$$\begin{aligned} &u = w = 0, \quad \psi_y = 0 \quad \text{at } x = 0,\ x = a, \\ &v = w = 0, \quad \psi_x = 0, \quad \text{at } y = 0,\ y = b, \end{aligned} \tag{2.10}$$

b) Clamped edges

$$\begin{aligned} &u = v = w = 0, \quad \psi_x = \psi_y = 0, \\ &\text{at } x = 0, \quad x = a \text{ and } y = 0,\ y = b. \end{aligned} \tag{2.11}$$

c) Mixed conditions, for example clamped - supported edges

$$\begin{aligned} &u = w = 0, \quad \psi_y = 0 \text{ at } x = 0,\ x = a, \\ &u = v = w = \psi_x = \psi_y = 0 \text{ at } y = 0,\ y = b. \end{aligned} \tag{2.12}$$

Depending on the method to be used we can approach to the solution of the problems (2.7) - (2.9).

From the equations (2.8) - (2.9) we can get the motion differential equations and static boundary conditions, while from (2.7) - (2.9) we can get the equilibrium differential equations and static boundary conditions.

They are the governing equations of laminated plates.

Here we consider the direct approximation methods for solving the static problems (2.7) - (2.9).

3. Ritz's method

We use the Ritz method to solve the static problems (2.7) - (2.9)

$$\delta U - \delta A = 0 \quad \text{or} \quad \delta(U - A) = 0,$$

which reduces to minimizing functional $J = U - A$

$$J = \frac{1}{2}\iint\limits_S \{\varepsilon\}^T[\mathbf{D}]\{\varepsilon\}dxdy - \iint\limits_S \{\mathbf{u}\}^T\{\mathbf{F}\}dxdy. \tag{3.1}$$

For simplifying application we denote the components of displacement vector $\mathbf{u}$ as follows

$$(u, v, w, \psi_x, \psi_y)^T = (u_1, u_2, u_3, u_4, u_5)^T,$$

which will be approximated in the form

$$u_i = \sum_{\alpha=1}^{n} a_{i\alpha}\varphi_{i\alpha}(x,y), \tag{3.2}$$

$$\text{or} \quad \{\mathbf{u}\}_{5\times 1} = [\boldsymbol{\phi}]_{5\times 5n}\,\{\mathbf{a}\}_{5n\times 1},$$

where $\{\mathbf{a}\}$ is a matrix $(5n \times 1)$ with elements $a_{i\alpha}$ $(i = 1 \div 5, \alpha = 1 \div n)$, $\varphi_{i\alpha}(x,y)$ are the set of linearly independent orthogonal functions, which must be chosen such that the kinematic boundary conditions (one of (2.10) - (2.12)) are satisfied, $a_{i\alpha}$ are arbitrary undetermined constants. Now calculate the components of the strain tensor $\{\boldsymbol{\varepsilon}\}$

$$\begin{aligned}
\varepsilon^0_{xx} &= \sum_{\alpha=1}^{n} a_{1\alpha}\frac{\partial\varphi_{1\alpha}}{\partial x} + \frac{1}{2}\Big(\sum_{\alpha=1}^{n} a_{3\alpha}\frac{\partial\varphi_{3\alpha}}{\partial x}\Big)^2 \\
&= \sum_{\alpha=1}^{n} a_{1\alpha}\frac{\partial\varphi_{1\alpha}}{\partial x} + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{3\alpha}a_{3\beta}\frac{\partial\varphi_{3\alpha}}{\partial x}\frac{\partial\varphi_{3\beta}}{\partial x}, \\
\varepsilon^0_{yy} &= \sum_{\alpha=1}^{n} a_{2\alpha}\frac{\partial\varphi_{2\alpha}}{\partial y} + \frac{1}{2}\Big(\sum_{\alpha=1}^{n} a_{3\alpha}\frac{\partial\varphi_{3\alpha}}{\partial y}\Big)^2 \\
&= \sum_{\alpha=1}^{n} a_{2\alpha}\frac{\partial\varphi_{2\alpha}}{\partial y} + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{3\alpha}a_{3\beta}\frac{\partial\varphi_{3\alpha}}{\partial y}\frac{\varphi_{3\beta}}{\partial y}, \\
\gamma^0_{xy} &= \sum_{\alpha=1}^{n}\Big(a_{1\alpha}\frac{\partial\varphi_{1\alpha}}{\partial y} + a_{2\alpha}\frac{\partial\varphi_{2\alpha}}{\partial x}\Big) + \Big(\sum_{\alpha=1}^{n} a_{3\alpha}\frac{\partial\varphi_{3\alpha}}{\partial x}\Big)\Big(\sum_{\alpha=1}^{n} a_{3\alpha}\frac{\partial\varphi_{3\alpha}}{\partial y}\Big) \\
&= \sum_{\alpha=1}^{n} a_{1\alpha}\frac{\partial\varphi_{1\alpha}}{\partial y} + a_{2\alpha}\frac{\partial\varphi_{2\alpha}}{\partial x} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{3\alpha}a_{3\beta}\frac{\partial\varphi_{3\alpha}}{\partial x}\frac{\partial\varphi_{3\beta}}{\partial y}, \\
\gamma^0_{yz} &= \sum_{\alpha=1}^{n}\Big(a_{5\alpha}\varphi_{5\alpha} + a_{3\alpha}\frac{\partial\varphi_{3\alpha}}{\partial y}\Big), \qquad (3.3) \\
\gamma^0_{xz} &= \sum_{\alpha=1}^{n}\Big(a_{4\alpha}\varphi_{4\alpha} + a_{3\alpha}\frac{\partial\varphi_{3\alpha}}{\partial x}\Big), \\
\chi_x &= \sum_{\alpha=1}^{n} a_{4\alpha}\frac{\partial\varphi_{4\alpha}}{\partial x}, \\
\chi_y &= \sum_{\alpha=1}^{n} a_{5\alpha}\frac{\partial\varphi_{5\alpha}}{\partial y}, \\
\chi_{xy} &= \sum_{\alpha=1}^{n}\Big(a_{4\alpha}\frac{\partial\varphi_{4\alpha}}{\partial y} + a_{5\alpha}\frac{\partial\varphi_{5\alpha}}{\partial x}\Big).
\end{aligned}$$

Note that they depend non-linearly on $a_{i\alpha}$, we can express the strain tensor in the form

$$\{\varepsilon\}_{8\times1} = \big[\mathbf{B}(\mathbf{a})(x,y)\big]_{8\times5n}\,\{\mathbf{a}\}_{5n\times1}. \tag{3.4}$$

where $\big[\mathbf{B}(\mathbf{a})(x,y)\big]$ depend on $\{\mathbf{a}\}$ of first degree.

Substituting (3.4) into the functional (3.1) we obtain

$$\begin{aligned} J = \frac{1}{2}\iint\limits_S \{\mathbf{a}\}^T[\mathbf{B}(\mathbf{a})(x,y)]^T[\mathbf{D}][\mathbf{B}(\mathbf{a})(x,y)]\{\mathbf{a}\}dxdy \\ - \iint\limits_S \{\mathbf{F}\}^T[\boldsymbol{\phi}]\{\mathbf{a}\}dxdu. \end{aligned} \tag{3.5}$$

Denote that

$$\iint\limits_S \big[\mathbf{B}(\mathbf{a})(x,y)\big]^T[\mathbf{D}]\big[\mathbf{B}(\mathbf{a})(x,y)\big]dxdy = \big[\boldsymbol{\mathcal{B}}(\mathbf{a})\big]_{5n\times5n}$$

and

$$\iint\limits_S \{\mathbf{F}\}^T[\boldsymbol{\phi}]dxdy = \{\boldsymbol{\mathcal{F}}\}^T_{1\times5n},$$

where $\big[\boldsymbol{\mathcal{B}}(\mathbf{a})\big]$ depends on $\{\mathbf{a}\}$ of second degree, J becomes a function of multi-variables $a_{i\alpha}$

$$J = \frac{1}{2}\{\mathrm{a}\}^T\big[\boldsymbol{\mathcal{B}}(\mathrm{a})\big]\{\mathrm{a}\} - \{\boldsymbol{\mathcal{F}}\}^T\{\mathrm{a}\},$$

or

$$J = \frac{1}{2}\mathcal{B}_{pq}a_p a_q - \mathcal{F}_p a_p \quad (p,q = 1 \div 5n),$$

where

$$\begin{aligned} \{\mathbf{a}\}^T &= (a_{11},\ldots,a_{1n},a_{21},\ldots,a_{2n},\ldots,a_{51},\ldots,a_{5n})^T \\ &\equiv (a_1,a_2,\ldots,a_{5n})^T. \end{aligned}$$

Minimization of J

$$\frac{\partial J}{\partial a_r} = 0, \quad (r = 1,2,\ldots,5n)$$

leads to a system of $5n$ algebraic equations for finding a_i

$$\frac{1}{2}\Big(\frac{\partial \mathcal{B}_{pq}}{\partial a_r}a_p + \mathcal{B}_{rq} + \mathcal{B}_{qr}\Big)a_q - \mathcal{F}_r = 0,$$

or

$$K_{rq}(\mathbf{a})a_q - \mathcal{F}_r = 0,$$

where

$$K_{rq}(\mathbf{a}) = \frac{1}{2}\Big(\frac{\partial \mathcal{B}_{pq}}{\partial a_r}a_p + \mathcal{B}_{rq} + \mathcal{B}_{qr}\Big),$$

This system of equations is rewritten in matrix form

$$\big[\mathbf{K}(\mathbf{a})\big]\big\{\mathbf{a}\big\} = \big\{\boldsymbol{\mathcal{F}}\big\}. \tag{3.6}$$

Because $[\mathbf{B}(\mathbf{a})]$ depends on $\{\mathbf{a}\}$, so the obtained system (3.6) is a nonlinear system of third degree, which usually can be solved by an iterative method:

$$\big[\mathbf{K}(\mathbf{a}^{(k-1)})\big]\big\{\mathbf{a}^{(k)}\big\} = \big\{\boldsymbol{\mathcal{F}}\big\}$$

where k - the number of iteration.

For illustration we restrict one term in series (3.2)

$$\begin{Bmatrix} u \\ v \\ w \\ \psi_x \\ \psi_y \end{Bmatrix} = \begin{bmatrix} \varphi_1 & 0 & 0 & 0 & 0 \\ 0 & \varphi_2 & 0 & 0 & 0 \\ 0 & 0 & \varphi_3 & 0 & 0 \\ 0 & 0 & 0 & \varphi_4 & 0 \\ 0 & 0 & 0 & 0 & \varphi_5 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix},$$

the strain tensor now has the form

$$\{\varepsilon\} = [\mathbf{B}(\mathbf{a})(x,y)]\{\mathbf{a}\},$$

where

$$[\mathbf{B}(\mathbf{a})(x,y)] = \begin{bmatrix} \frac{\partial \varphi_1}{\partial x} & 0 & \frac{1}{2}a_3\Big(\frac{\partial \varphi_3}{\partial x}\Big)^2 & 0 & 0 \\ 0 & \frac{\partial \varphi_2}{\partial y} & \frac{1}{2}a_3\Big(\frac{\partial \varphi_3}{\partial y}\Big)^2 & 0 & 0 \\ \frac{\partial \varphi_1}{\partial y} & \frac{\partial \varphi_2}{\partial x} & a_3\frac{\partial \varphi_3}{\partial x}\frac{\partial \varphi_3}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial \varphi_3}{\partial y} & 0 & \varphi_5 \\ 0 & 0 & \frac{\partial \varphi_3}{\partial x} & \varphi_4 & 0 \\ 0 & 0 & 0 & \frac{\partial \varphi_4}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \varphi_5}{\partial y} \\ 0 & 0 & 0 & \frac{\partial \varphi_4}{\partial y} & \frac{\partial \varphi_5}{\partial x} \end{bmatrix}$$

Certainly $[\mathbf{B}]$ depends on $\{\mathbf{a}\}$ of first degree.

4. Finite element formulation

In [4, 5] the finite element formulation was used in analysis of laminated plates for small deformations. One can use a three-noded isoparametric element with five degrees of freedom or more exactly a nine - noded isoparametric element with five degrees of freedom to discretize the plate for large deformations, such one was used in [3], but there are some incorrect expressions of stiffness matrices. Here we introduce a more precise calculation.

A laminated plate is discretized into L_e elements, for an element "e" a nine - noded two - dimensional shape functions $N_i^{(e)}(x,y)$ are adopted for interpolating the generalized displacements of the plate

$$\{\mathbf{u}^e\} = \sum_{i=1}^{9} [\mathbf{N}_i^e(x,y)]\{\mathbf{u}_i^e\}, \tag{4.1}$$

where we can take N_i^e [1] for quadrilateral element in (ξ,η) space as following

$$N_1^e = -\frac{1}{4}(1-\xi)(1-\eta)\xi\eta, \qquad N_2^e = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta,$$
$$N_3^e = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta, \qquad N_4^e = \frac{1}{2}(1+\xi)(1-\eta)\xi,$$
$$N_5^e = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta, \qquad N_6^e = \frac{1}{2}(1-\xi^2)(1+\eta)\eta,$$
$$N_7^e = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta, \qquad N_8^e = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi,$$
$$N_9^e = (1-\xi^2)(1-\eta^2),$$

$$[\mathbf{N}_i^e] = N_i^e \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$\{\mathbf{u}_i^e\} = (u_i, v_i, w_i, \varphi_{xi}, \varphi_{yi})^T$ - are the nodal values.
Similarly (3.2), the equation (4.1) can also be written as

$$\{\mathbf{u}^e\} = [\mathbf{N}^e]\{\mathbf{d}^e\}, \tag{4.2}$$

where

$$[\mathbf{N}^e] = \big[\, [\mathbf{N}_1^e]\ [\mathbf{N}_2^e]\ \ldots\ [\mathbf{N}_9^e]\, \big]_{5\times 45},$$

$$\{\mathbf{d}^e\} = \big(\{\mathbf{u}_1^e\}, \{\mathbf{u}_2^e\}, \dots, \{\mathbf{u}_9^e\}\big)^T = \begin{Bmatrix} \{\mathbf{u}_1^e\} \\ \{\mathbf{u}_2^e\} \\ \vdots \\ \{\mathbf{u}_9^e\} \end{Bmatrix}_{45\times 1}$$

Now calculate the components of the strain tensor $\{\boldsymbol{\varepsilon}\}$ by formulae (2.4) we get similarly (3.4)

$$\{\boldsymbol{\varepsilon}^e\}_{8\times1} = \big[\mathbf{B}^e(\mathbf{d})(x,y)\big]_{8\times45}\{\mathbf{d}^e\}_{45\times1}. \tag{4.3}$$

where $\big[\mathbf{B}(\mathbf{d})\big]$ depends on $\mathbf{d}$ of first degree.

The variation of the strain tensor can be written

$$\{\delta\boldsymbol{\varepsilon}^e\} = \big[\mathbf{B}^e\big]\{\delta\mathbf{d}^e\} + \big[\delta\mathbf{B}^e\big]\{\mathbf{d}^e\}.$$

It is not difficult to show $\big[\delta\mathbf{B}^e\big]\{\mathbf{d}^e\} = \big[\overline{\mathbf{B}}^e\big]\{\delta\mathbf{d}^e\}$, where $\big[\overline{\mathbf{B}}^e\big]$ has the form such as $\big[\delta\mathbf{B}^e\big]$, but instead of δd_i we put d_i, thus we have

$$\{\delta\boldsymbol{\varepsilon}^e\} = \big[\mathbf{B}^e + \overline{\mathbf{B}}^e\big]\{\delta\mathbf{d}^e\} \equiv \big[\mathbf{B}_*^e\big]\{\delta\mathbf{d}^e\} \tag{4.4}$$

In general, the matrix $\big[\mathbf{B}_*\big]$ is different from the matrix $\big[\mathbf{B}\big]$. Because in the matrix $\big[\mathbf{B}\big]$ there are elements containing d_i and elements not containing d_i, after variation in the matrix $\big[\delta\mathbf{B}\big]$ the respective elements not containing d_i become null, but the others are different from zero, i.e. $\big[\delta\mathbf{B}\big] \neq 0$, consequently $\big[\mathbf{B}\big] \neq \big[\mathbf{B}_*\big]$. For a linear problem the matrix $\big[\delta\mathbf{B}\big] = 0$ and $\big[\mathbf{B}_*\big] = \big[\mathbf{B}\big]$. It is precisely this fact we have corrected in calculating the variation of strain tensor for a nonlinear problem.

Substituting (4.2), (4.3), (4.4) into (2.7), (2.9) and assembling the element equations we obtain

$$\{\delta\mathbf{d}\}^T[\mathbf{K}(\mathbf{d})]\{\mathbf{d}\} - \{\delta\mathbf{d}\}^T\{\mathbf{P}\} = 0, \tag{4.5}$$

where the stiffened plate system $[\mathbf{K}]$ must be calculated as follows

$$[\mathbf{K}] = \sum_{e=1}^{L_e}\iint_{S_e}[\mathbf{B}_*^e]^T[\mathbf{D}][\mathbf{B}^e]dxdy = \sum_{e=1}^{L_e}\iint_{S_e}\big[\mathbf{B}^e + \overline{\mathbf{B}}^e\big]^T\big[\mathbf{D}\big]\big[\mathbf{B}^e\big]dxdy. \tag{4.6}$$

The global acting force vector $\{\mathbf{P}\}$ is given by

$$\{\mathbf{P}\} = \sum_{e=1}^{L_e}\iint_{S_e}[\mathbf{N}^e]^T\{\mathbf{F}\}dxdy. \tag{4.7}$$

Since $\{\delta\mathbf{d}\}$ is an arbitrary value, from (4.4) we obtain

$$[\mathbf{K}(\mathbf{d})]\{\mathbf{d}\} = \{\mathbf{P}\}. \tag{4.8}$$

The system of equations (4.8) is a system of nonlinear algebraic equations of third degree, because of the dependence of [**K**] on {**d**}. By using an iterative method we can solve this system of equations:

$$[\mathbf{K}(\mathbf{d}^{(k-1)})]\{\mathbf{d}^{(k)}\} = \{\mathbf{P}\}.$$

Since the components of a strain tensor consists of a linear part and a non-linear part, thus the relation (4.3) can be rewritten as following

$$\{\varepsilon^e\} = [\mathbf{B}^e(\mathbf{d})]\{\mathbf{d}^e\} = [\mathbf{B}^e_L]\{\mathbf{d}^e\} + [\mathbf{B}^e_{NL}(\mathbf{d})]\{\mathbf{d}^e\}$$

where $[\mathbf{B}^e_L]$ - the representative matrix of a linear problem,

$[\mathbf{B}^e_{NL}]$ - the supplementary matrix appearing only in a nonlinear problem. It depends homogenously on $\{\mathbf{d}\}$ of first degree.

Then

$$\{\delta\varepsilon^e\} = [\mathbf{B}^e_L]\{\delta\mathbf{d}^e\} + [B^e_{NL}(\mathbf{d})]\{\delta\mathbf{d}^e\} + [\delta\mathbf{B}^e_{NL}]\{\mathbf{d}^e\},$$

one can prove that

$$[\delta\mathbf{B}^e_{NL}]\{\mathbf{d}^e\} = [\mathbf{B}^e_{NL}(\mathbf{d})]\{\delta\mathbf{d}^e\},$$

and

$$\{\delta\varepsilon^e\} = [\mathbf{B}^e_L]\{\delta\mathbf{d}^e\} + 2[\mathbf{B}^e_{NL}(\mathbf{d})]\{\delta\mathbf{d}^e\}.$$

The stiffness matrix of system is of the form

$$[\mathbf{K}(\mathbf{d})] = [\mathbf{K}_L] + [\mathbf{K}_{NL}(\mathbf{d})],$$

where

$$[\mathbf{K}_L] = \sum_{e=1}^{L_e} \iint_{S_e} [\mathbf{B}^e_L]^T [\mathbf{D}] [\mathbf{B}^e_L] dxdy,$$

$$[\mathbf{K}_{NL}(\mathbf{d})] = 2\sum_{e=1}^{L_e} \iint_{S_e} [\mathbf{B}^e_{NL}]^T [\mathbf{D}] [\mathbf{B}^e_{NL}] dxdy.$$

The procedure of an iterative method for solving the system of equations (4.8) may be performed as follows

$$[\mathbf{K}_L]\{\mathbf{d}^{(k)}\} = \{\mathbf{P}\} - [\mathbf{K}_{NL}(\mathbf{d}^{(k-1)})]\{\mathbf{d}^{(k-1)}\},$$

here $[\mathbf{K}_L]$ is the stiffness matrix of a linear problem.

5. Conclusion

In this pager we restrict ourselves in the formulation of the governing equations of laminated plates for large deformation, taking into account the shear deformation, introduce formulation of two methods - Ritz method and finite element method for solving this problem. In continuing we will consider concrete static and dynamic problems of laminated plates.

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TÍNH TOÁN PHI TUYẾN BẢN COMPOSITE LỚP

Để tính toán tĩnh và động của bản composite nhiều lớp khi có biến dạng lớn có tính đến biến dạng trượt ngang bài báo đề cập đến việc xây dựng các hệ thức xác định của vật liệu, thiết lập các phương trình cơ sở của bài toán dựa trên nguyên lý Lagrange và nguyên lý Hamilton. Đồng thời đưa ra thuật toán giải bài toán phi tuyến theo phương pháp Ritz và phương pháp phần tử hữu hạn.