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ON A BENDING PROBLEM OF THICK LAMINATED COMPOSITE PLATES

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ABSTRACT. A high-order displacement field in quadrilateral element with nine nodes and twelve-degrees-of-freedom per node is developed for bending analysis of thick arbitrary layered composite plates under transverse loads. Results for plate deformations, internal stress-resultants and stresses for selected examples are shown to compare well with the closed-form and other finite element solutions.

1. Introduction

The layered composite plate has been popular in many engineering applications since it has some beneficical properties such as large strength-to-weight ratios and desired directional strengths. Thus, the analysis of layered composite plates is under intensive research. Some studies [1, 2] have shown that the transverse shear effect was quite significant in the layered composite plates due to the high ratio of inplane modulus to transverse shear modulus. Consequently the classical plate theory is not suitable for layered composite plates of moderate thickness. Some researchers have used the Reissner-Mindlin plate bending theory [1, 2], which includes transverse shear deformations. In this theory, the transverse shear strains are constant through the thickness of plate. Thus, a transverse shear correction factor is introduced to the theory. The Reissner-Mindlin plate theory results in more accurate solutions than the classical plate theory when compared with the three-dimensional elasticity solutions. However, the Reissner-Mindlin solutions become quite unsatisfactory as the plate thickness-side length ratio increases. Thus, more refined high-order plate bending theories have been proposed [3-6]. For example, Reddy [4] presented a simple high-order theory, in which in-plane displacement components are expanded as cubic functions of the thickness coordinate and transverse displacement is constant through the plate thickness to obtain the closed form solution for the composite plate bending problem. Basing on the Reddy type of theory and by finite element method, Pandya and Kant [5] analyzed the unsymmetric laminated composite plates under transverse loads. Kwon and Akin [6] used linear displacements in x, y-directions, and quadratic transverse displacement in z-direction for the analysis of layered composite plates.

In this paper, a full third-order plate bending theory is used, in which both in-plane and out-of-plane displacement components are assumed to have cubic variations through the thickness of the plate. In the following sections, the formulation of a high-order plate bending equation and its quadrilateral finite elements with 108 degrees of freedom per element are given. Sections 4 and 5 present some finite element numerical results and conclusions.

2. A full third-order plate bending theory

2.1. Displacement and strain fields

The present analysis is based upon a displacement field in which the displacement components u, v, w are all of third-order in the thickness coordinate z [9], [10]:

$$\begin{aligned} u &= u_0(x,y) + z\Psi_x(x,y) + z^2\xi_x(x,y) + z^3\phi_x(x,y) \\ v &= v_0(x,y) + z\Psi_y(x,y) + z^2\xi_y(x,y) + z^3\phi_y(x,y) \\ w &= w_0(x,y) + z\Psi_z(x,y) + z^2\xi_z(x,y) + z^3\phi_z(x,y) \end{aligned} \tag{2.1}$$

Geometrically, u_0, v_0 and w_0 are the translations along the x, y and z axes respectively, and Ψ_x, Ψ_y denote rotations about the y and x axes, respectively. The rest of the unknown coefficients are the higher-order deformation terms in the Taylor series expansion and are also defined at the mid-plane.

The strain-displacement relations are obtained as folows:

$$(\varepsilon_x, \varepsilon_y, \varepsilon_z) = (\varepsilon_{0x}, \varepsilon_{0y}, \varepsilon_{0z}) + z(k_x, k_y, k_z) + z^2(\hat{\varepsilon}_{0x}, \hat{\varepsilon}_{0y}, \hat{\varepsilon}_{0z}) + z^3(\hat{k}_x, \hat{k}_y, \hat{k}_z). \tag{2.2a}$$

$$\begin{aligned} (\gamma_{yz}, \gamma_{xz}, \gamma_{xy}) = (\gamma_{0yz}, \gamma_{0xz}, \gamma_{0xy}) + z(k_{yz}, k_{xz}, k_{xy}) + z^2(\hat{\gamma}_{0yz}, \hat{\gamma}_{0xz}, \hat{\gamma}_{0xy}) \\ + z^3(\hat{k}_{yz}, \hat{k}_{xz}, \hat{k}_{xy}), \end{aligned} \tag{2.2b}$$

in which

$$\begin{aligned} &\varepsilon_{0x} = u_{0'x};\ \varepsilon_{0y} = v_{0'y};\ \varepsilon_{0z} = \Psi_z;\ \gamma_{0xy} = u_{0'y} + v_{0'x}; \\ &\gamma_{0xz} = w_{0'x} + \Psi_x;\ \gamma_{0yz} = w_{0'y} + \Psi_y \\ &k_x = \xi_{x'x};\ k_y = \xi_{y'y};\ k_z = 2\xi_z;\ k_{xy} = \Psi_{x'y} + \Psi_{y'x}; \\ &k_{xz} = \Psi_{x'z} + 2\xi_x;\ k_{yz} = \psi_{y'z} + 2\xi_y; \\ &\hat{\varepsilon}_{0x} = \xi_{x'x};\ \hat{\varepsilon}_{0y} = \xi_{y'y};\ \hat{\varepsilon}_{0z} = 3\phi_z; \\ &\hat{\gamma}_{0xy} = \xi_{x'y} + \xi_{y'x};\ \hat{\gamma}_{0xz} = \xi_{x'z} + 3\phi_x;\ \hat{\gamma}_{0yz} = \xi_{y'z} + 3\phi_y; \\ &\hat{k}_x = \phi_{x'x};\ \hat{k}_y = \phi_{y'y};\ \hat{k}_z = 0;\ \hat{k}_{xy} = \phi_{x'y} + \phi_{y'x};\ \hat{k}_{xz} = \phi_{x'z};\ \hat{k}_{yz} = \phi_{y'z}. \end{aligned} \tag{2.3}$$

2.2. Constitutive relations of thick laminated composite

Refering to the principal material coordinate axes $(1, 2, 3)$ and reference axes (x, y, z) of k^{th} orthotropic layer, the stress-strain relations can be written as:

$$\{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}_k^t = [C_{ij}]_k \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}, \gamma_{13}, \gamma_{12}\}_k^t \tag{2.4}$$

$$\{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}\}_k^t = [C'_{ij}]_k \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}_k^t \tag{2.5}$$

where: C_{ij} and C'_{ij} are the stiffness coefficients of k-material layer in $(1,2,3)$ and (x,y,z) axes, respectively. The relation between C_{ij} and C'_{ij} is expressed by:

$$[C'] = [T_\sigma^{-1}]_k [C]_k [T_\varepsilon]_k \tag{2.6}$$

in which, T_σ^{-1} is an inverse matrix of the stress transformation matrix T_σ; T_ε^{-1} is an inverse matrix of the strain transformation matrix, T_ε, and clearly the C_{ij} can be written in terms of the engineering constants.

Substituting equations (2.2a, 2.2b) into the relation (2.5) and performing intergrations with respect to the various powers $(0,1,2,3)$ of the thickness coordinate z, leads to the expressions [10]:

$$\begin{Bmatrix} N_x & \hat{N}_x & M_x & \hat{M}_x \\ N_y & \hat{N}_y & M_y & \hat{M}_y \\ N_z & \hat{N}_z & M_z & \hat{M}_z \\ N_{xy} & \hat{N}_{xy} & M_{xy} & \hat{M}_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \end{Bmatrix}_k (1, z^2, z, z^3) dz$$

$$\begin{Bmatrix} Q_x & \hat{Q}_x & S_x & \hat{S}_x \\ Q_y & \hat{Q}_y & S_y & \hat{S}_y \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}_k (1, z^2, z, z^3) dz. \tag{2.7}$$

We obtained the constitutive equations as following:

$$\begin{Bmatrix} \mathbf{N} \\ \hat{\mathbf{N}} \\ \cdots \\ \mathbf{M} \\ \hat{\mathbf{M}} \\ \cdots \\ \mathbf{Q} \\ \hat{\mathbf{Q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \vdots & \mathbf{B} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{B}^t & \vdots & \mathbf{D}_b & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \mathbf{D}_s \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \hat{\boldsymbol{\varepsilon}}_0 \\ \cdots \\ \mathbf{k} \\ \hat{\mathbf{k}} \\ \cdots \\ \boldsymbol{\gamma} \\ \hat{\boldsymbol{\gamma}} \end{Bmatrix} \quad \text{or} \quad \overline{\boldsymbol{\sigma}} = \mathbf{D}\overline{\boldsymbol{\varepsilon}} \tag{2.8}$$

where:

$$\begin{aligned} \mathbf{N} &= (N_x, N_y, N_z, N_{xy})^t; & \hat{\mathbf{N}} &= (\hat{N}_x, \hat{N}_y, \hat{N}_z, \hat{N}_{xy})^t \\ \mathbf{M} &= (M_x, M_y, M_z, M_{xy})^t; & \hat{\mathbf{M}} &= (\hat{M}_x, \hat{M}_y, \hat{M}_z, \hat{M}_{xy})^t \\ \mathbf{Q} &= (Q_x, Q_y)^t; & \hat{\mathbf{Q}} &= (S_x, S_y, \hat{Q}_x, \hat{Q}_y, \hat{S}_x, \hat{S}_y)^t \\ \boldsymbol{\varepsilon}_0 &= (\varepsilon_{0x}, \varepsilon_{0y}, \varepsilon_{0z}, \gamma_{0xy})^t; & \hat{\boldsymbol{\varepsilon}}_0 &= (\hat{\varepsilon}_{0x}, \hat{\varepsilon}_{0y}, \hat{\varepsilon}_{0z}, \hat{\gamma}_{0xy})^t \\ \mathbf{k} &= (k_x, k_y, k_z, k_{xy})^t; & \hat{\mathbf{k}} &= (\hat{k}_x, \hat{k}_y, \hat{k}_z, \hat{k}_{xy})^t \\ \boldsymbol{\gamma} &= (\gamma_{oxz}, \gamma_{oyz})^t; & \hat{\boldsymbol{\gamma}} &= (k_{xz}, k_{yz}, \hat{\gamma}_{oxz}, \hat{\gamma}_{oyz}, \hat{k}_{xz}, \hat{k}_{yz})^t \end{aligned} \tag{2.9}$$

and

$$
\mathbf{A}=\sum_{k=1}^{n}\begin{bmatrix}
C'_{11}H_1 & C'_{12}H_1 & C'_{13}H_1 & C'_{16}H_1 & C'_{11}H_3 & C'_{12}H_3 & C'_{13}H_3 & C'_{16}H_3\\
 & C'_{22}K_1 & C'_{23}H_1 & C'_{26}H_1 & C'_{12}H_3 & C'_{22}H_3 & C'_{23}H_3 & C'_{26}H_3\\
 & & C'_{33}H_1 & C'_{36}H_1 & C'_{13}H_3 & C'_{23}H_3 & C'_{33}H_3 & C'_{36}H_3\\
 & & & C'_{66}H_1 & C'_{16}H_3 & C'_{26}H_3 & C'_{36}H_3 & C'_{66}H_3\\
 & & & & C'_{11}H_5 & C'_{12}H_5 & C'_{13}H_5 & C'_{16}H_5\\
 & & & & & C'_{22}H_5 & C'_{23}H_5 & C'_{26}H_5\\
 & & & & & & C'_{33}H_5 & C'_{36}H_5\\
\text{Symmetric} & & & & & & & C'_{66}H_5
\end{bmatrix}_k \tag{2.10}
$$

$$
\mathbf{D}_s=\sum_{k=1}^{n}\begin{bmatrix}
C'_{55}H_1 & C'_{45}H_1 & C'_{55}H_2 & C'_{45}H_2 & C'_{55}H_3 & C'_{45}H_3 & C'_{55}H_4 & C'_{45}H_4\\
 & C'_{44}H_1 & C'_{45}H_2 & C'_{44}H_2 & C'_{45}H_3 & C'_{44}H_3 & C'_{45}H_4 & C'_{44}H_4\\
 & & C'_{55}H_3 & C'_{45}H_3 & C'_{55}H_4 & C'_{45}H_4 & C'_{55}H_5 & C'_{45}H_5\\
 & & & C'_{44}H_3 & C'_{45}H_4 & C'_{44}H_4 & C'_{45}H_5 & C'_{44}H_5\\
 & & & & C'_{55}H_5 & C'_{45}H_5 & C'_{55}H_6 & C'_{45}H_6\\
 & & & & & C'_{44}H_5 & C'_{45}H_6 & C'_{44}H_6\\
 & & & & & & C'_{55}H_7 & C'_{45}H_7\\
\text{Symmetric} & & & & & & & C'_{44}H_7
\end{bmatrix}_k \tag{2.11}
$$

The elements of the **B** matrix can be obtained by replacing H_1 by H_2, H_3 by H_4 and H_5 by H_6 in the **A** matrix. Similarly, the elements of the $\mathbf{D}_b$ matrix can be obtained by replacing H_1 by H_3, H_3 by H_5 and H_5 by H_7 in the **A** matrix, where

$$
H_i=(h_k^i-h_{k-1}^i)/i; \quad i=1,2,3,\ldots,7. \tag{2.12}
$$

3. Finite element formulation

In the well-established finite element method, the total studied domain is discretized into "K" elements such that

$$
\pi(\mathbf{d})=\sum_{e=1}^{K}\pi^e(\mathbf{d}) \tag{3.1}
$$

where π and π^e are the potential energies of the total domain and the element respectively; **d** is the displacement vector inside the element and is defined by:

$$
\mathbf{d}=(u_0,v_0,w_0,\Psi_x,\Psi_y,\Psi_z,\xi_x,\xi_y,\xi_z,\phi_x,\phi_y,\phi_z)^t \tag{3.2}
$$

The potential energy for an element "e" can be expressed in terms of the internal strain energy: U^e, and the external work done: W^e, such that

$$\pi^e(\mathbf{d}) = U^e - W^e. \tag{3.3}$$

In this study, nine-node C^0 two-dimensional shape functions N_i $(i = 1, 2, \ldots, 9)$ are adopted for interpolating both the generalized displacements and geometry such that:

$$\begin{aligned} \mathbf{d} &= \sum_{i=1}^{9} N_i d_i \\ (x, y) &= \sum_{i=1}^{9} N_i (x_i, y_i). \end{aligned} \tag{3.4}$$

The explicit expressions of N_i are shown in reference [10].

Now, refering to the expressions in equation (2.3), the extensional strains $(\boldsymbol{\varepsilon}_0, \hat{\boldsymbol{\varepsilon}}_0)$, the bending curvatures $(\mathbf{k}, \hat{\mathbf{k}})$ and the transverse shear strains $(\boldsymbol{\gamma}, \hat{\boldsymbol{\gamma}})$ can be written in terms of the displacements $\mathbf{d}$ using the matrix notations as follows:

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \hat{\boldsymbol{\varepsilon}}_0 \end{Bmatrix} = L_E \mathbf{d}; \quad \begin{Bmatrix} \mathbf{k} \\ \hat{\mathbf{k}} \end{Bmatrix} = L_B \mathbf{d}; \quad \begin{Bmatrix} \gamma \\ \hat{\gamma} \end{Bmatrix} = L_s \mathbf{d} \tag{3.5}$$

in which the subscripts E, B and S refer to extension, bending and shear respectively and the matrices L_E, L_B and L_S contain the shape functions and their derivatives.

Knowing the generalized displacement vector, $\mathbf{d}$, at all points within the element, the generalized strain vectors at any point are determined as follows:

$$\begin{aligned} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \hat{\boldsymbol{\varepsilon}}_0 \end{Bmatrix} &= L_E \mathbf{d} = L_E \sum_{i=1}^{9} \mathbf{N}_i \mathbf{d}_i = \sum_{i=1}^{9} B_{iE} \mathbf{d}_i = \mathbf{B}_E \mathbf{q} \\ \begin{Bmatrix} \mathbf{k} \\ \hat{\mathbf{k}} \end{Bmatrix} &= L_B \mathbf{d} = L_B \sum_{i=1}^{9} \mathbf{N}_i \mathbf{d}_i = \sum_{i=1}^{9} B_{id} \mathbf{d}_i = \mathbf{B}_B \mathbf{q} \\ \begin{Bmatrix} \boldsymbol{\gamma} \\ \hat{\boldsymbol{\gamma}} \end{Bmatrix} &= L_s \mathbf{d} = L_s \sum_{i=1}^{9} \mathbf{N}_i d_i = \sum_{i=1}^{9} B_{iS} d_i = \mathbf{B}_s \mathbf{q} \end{aligned} \tag{3.6}$$

in which

$$\begin{aligned} B_{iE} &= L_E \mathbf{N}_i; \quad \mathbf{B}_E = \big[[B_{1E}], [B_{2E}], \ldots, [B_{9E}]\big] \\ B_{iB} &= L_B \mathbf{N}_i; \quad \mathbf{B}_B = \big[[B_{1B}], [B_{2B}], \ldots, [B_{9E}]\big] \\ B_{iS} &= L_S \mathbf{N}_i; \quad \mathbf{B}_S = \big[[B_{1S}], [B_{2S}], \ldots, [B_{9S}]\big] \end{aligned} \tag{3.7}$$

and

$$\mathbf{q} = \left(\mathbf{d}_1^t, \mathbf{d}_2^t, \ldots, \mathbf{d}_9^t\right)^t \tag{3.8}$$

Combining the expressions in equation (3.7), the $\mathbf{B}$ matrix for the i^{th} node can be written as

$$B_i = \begin{bmatrix} B_{iE} \\ B_{iB} \\ B_{iS} \end{bmatrix} \tag{3.9}$$

The internal energy of an element is determined by integrating the products of in-plane, moment and shear stress resultants with the extensional, bending and shear strains, respectively, over the area of an element. This is expressed as

$$\mathbf{U}^e = \frac{1}{2}\int_A \left[\left(\boldsymbol{\varepsilon}_0^t, \hat{\boldsymbol{\varepsilon}}_0^t\right)\begin{Bmatrix}\mathbf{N}\\ \hat{\mathbf{N}}\end{Bmatrix} + \left(\mathbf{k}^t, \hat{\mathbf{k}}^t\right)\begin{Bmatrix}\mathbf{M}\\ \hat{\mathbf{M}}\end{Bmatrix} + \left(\boldsymbol{\gamma}^t, \hat{\boldsymbol{\gamma}}^t\right)\begin{Bmatrix}\mathbf{Q}\\ \hat{\mathbf{Q}}\end{Bmatrix}\right] dA. \tag{3.10}$$

Replacing stress-resultants by the product of rigidity matrix and strain in the strain energy expression in equation (3.10), we get

$$\begin{aligned} U^e = \frac{1}{2}\int_A & \left[(\boldsymbol{\varepsilon}_0^t, \hat{\boldsymbol{\varepsilon}}_0^t)\mathbf{A}\begin{Bmatrix}\boldsymbol{\varepsilon}_0\\ \hat{\boldsymbol{\varepsilon}}_0\end{Bmatrix} + (\boldsymbol{\varepsilon}_0^t, \hat{\boldsymbol{\varepsilon}}_0^t)\mathbf{B}\begin{Bmatrix}\mathbf{k}\\ \hat{\mathbf{k}}\end{Bmatrix} + (\mathbf{k}^t, \hat{\mathbf{k}}^t)\mathbf{B}^t\begin{Bmatrix}\boldsymbol{\varepsilon}_0\\ \hat{\boldsymbol{\varepsilon}}_0\end{Bmatrix}\right. \\ & \left. + (\mathbf{k}^t, \hat{\mathbf{k}}^t)\mathbf{D}_b\begin{Bmatrix}\mathbf{k}\\ \hat{\mathbf{k}}\end{Bmatrix} + (\boldsymbol{\gamma}^t, \hat{\boldsymbol{\gamma}}^t)\mathbf{D}_s\begin{Bmatrix}\boldsymbol{\gamma}\\ \hat{\boldsymbol{\gamma}}\end{Bmatrix}\right] dA \end{aligned} \tag{3.11}$$

The internal strain energy expression in terms of the nodal displacements is derived by substituting relations in equation (3.6) into equation (3.11). The result is

$$\begin{aligned} U^e = \frac{1}{2}\int_A & \left[\mathbf{q}^t\mathbf{B}_E^t\mathbf{A}\mathbf{B}_E\mathbf{q} + \mathbf{q}^t\mathbf{B}_B^t\mathbf{B}\mathbf{B}_E\mathbf{q} + \mathbf{q}^t\mathbf{B}_E^t\mathbf{B}\mathbf{B}_B\mathbf{q} + \mathbf{q}^t\mathbf{B}_B^t\mathbf{D}_B\mathbf{B}_B\mathbf{q}\right. \\ & \left. + \mathbf{q}^t\mathbf{B}_S^t\mathbf{D}_S\mathbf{B}_S\mathbf{q}\right] dA \end{aligned} \tag{3.12}$$

$$\text{or} \quad \mathbf{U}^e = \frac{1}{2}\mathbf{q}^t\mathbf{K}^e\mathbf{q} \tag{3.13}$$

in which $\mathbf{K}^e$ is the *elememt stiffness matrix* and is expressed as

$$\mathbf{K}^e = \int_A (\mathbf{B}_E^t\mathbf{A}\mathbf{B}_E + \mathbf{B}_B^t\mathbf{B}\mathbf{B}_E + \mathbf{B}_E^t\mathbf{B}\mathbf{B}_B + \mathbf{B}_B^t\mathbf{D}_B\mathbf{B}_B + \mathbf{B}_S^t\mathbf{D}_S\mathbf{B}_S) dA \tag{3.14}$$

The integral is evaluated numerically using the 3×3 Gauss quadrature rule [11]:

$$K_{ij}^e = \int_{-1}^{1}\int_{-1}^{1} B_i^t D B_j |J| d\xi d\eta = \sum_{k=1}^{3}\sum_{\ell=1}^{3} W_k W_\ell |J| B_i^t D B_j \tag{3.15}$$

in which W_k and W_ℓ are weighting coefficients and $|J|$ is the determinant of the jacobian matrix. The subscrips i and j vary from 1 to 9. The B_i and D are given by eqns (3.9) and (2.8) respectively and B_j is obtained by replacing i by j.

For the flexual analysis, the total external work done by the transverse load is given by:

$$\mathbf{W}^e = \mathbf{q}^t \int_A N_i^t p dA \tag{3.16}$$

in which, i varies from 1 to 9; p: uniform distributed load intensity acting over an element e in the z direction.

The integral of eqn (3.16) is also evaluated numerically using the Gauss quadrature rule as follows

$$\mathbf{F}_i = \sum_{k=1}^{3}\sum_{\ell=1}^{3} W_k W_\ell |J| N_i^t p\{0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\}^t. \tag{3.17}$$

Assembling all elements yields the equilibrium equations system as follows:

$$\mathbf{K}\,\mathbf{Q} = \mathbf{F} \tag{3.18}$$

in which $\mathbf{K}$ is the global stiffness matrix without displacement constraints, $\mathbf{Q}$ and $\mathbf{F}$ are the assembled nodal displacement vector and the assembled nodal force vector, respectively.

After solving the linear algebraic equations (3.18), the generalized strains and the generalized resultant vectors can be obtained by (3.6) and (2.8). Once the generalized strains are obtained, six strain components and six stress components are evaluated from eqn (2.2a)-(2.2b) and (2.4)-(2.5).

4. Finite element results

Plates with several layers are solved by the present method and their solutions are compared with the other finite element solutions as well as the closed-form solutions. The material properties used for each lamina in this study are presented in Table 1.

Each plate is discretized with four nine-noded quadrilateral elements; twelve-degrees-of-freedom per node.

Example 1. A simply-supported unsymmetric angle-ply (15°/-15°) square plate under uniform transverse load is considered for comparisons of maximum deflection and stress-resultants with those obtained by Pandya & Kant [5] and Turvey [7] in Table 2 for various ratios of a/h = 5, 10 and 40. Table 2 shows that the present method yields exellent results.

Example 2. A simply-supported symmetric cross-ply $(90°/0°/90°/0°)_s$ square plate under uniform transverse load is considered. The distribution of the stresses at midplate is presented in Figure 1. With the present displacement model, it is not possible to satisfy the zero transverse shear stress conditions on the bounding plane of the plate. There are also the discontinuities on the interfaces for the interlaminar stresses and in-plane lamina stresses.

Table 1. Material properties for each layer of laminated composite plates

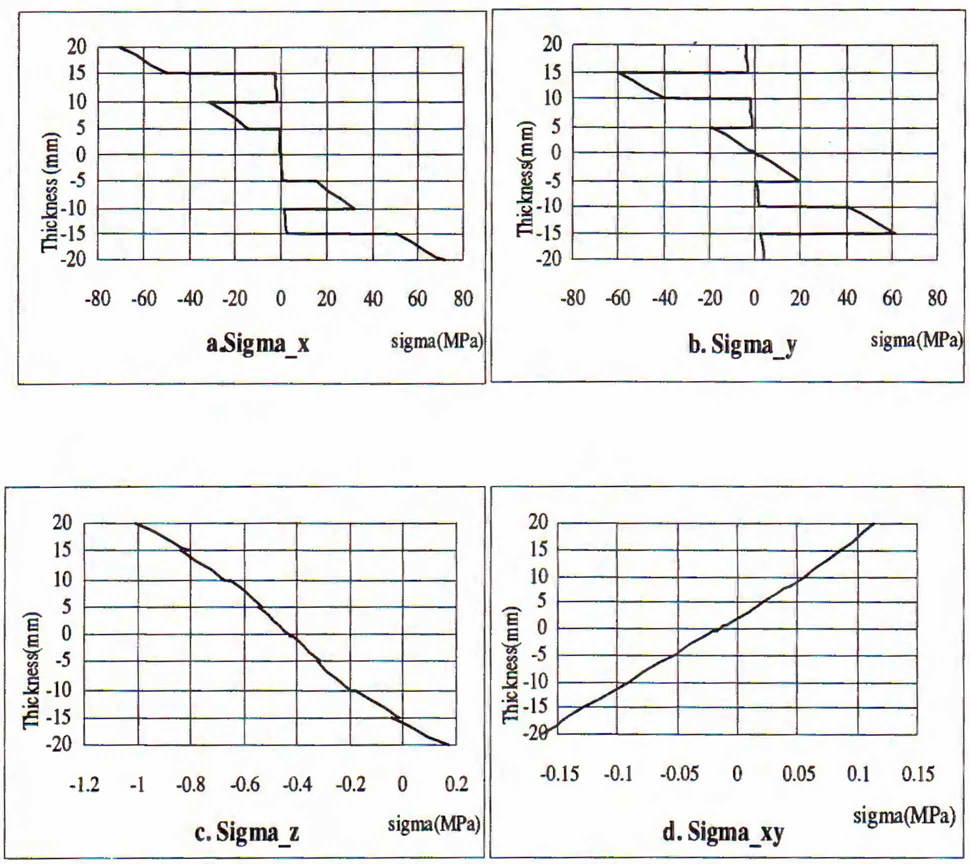
	Conf.	Thick. δ(mm)	Load q(MPa)	E_1 (GPa)	E_2 (GPa)	E_3 (GPa)	G_1 (GPa)	G_2 (GPa)	G_3 (GPa)	v_{12}	v_{13}	v_{23}
Ex.1	(1)	40;20;5	0.0001	280	7	7	4.2	4.2	3.5	0.25	0.25	0.25
Ex.2	(2)	5	.1	181	10.3	10.3	7.17	7.17	2.87	0.28	0.28	0.33

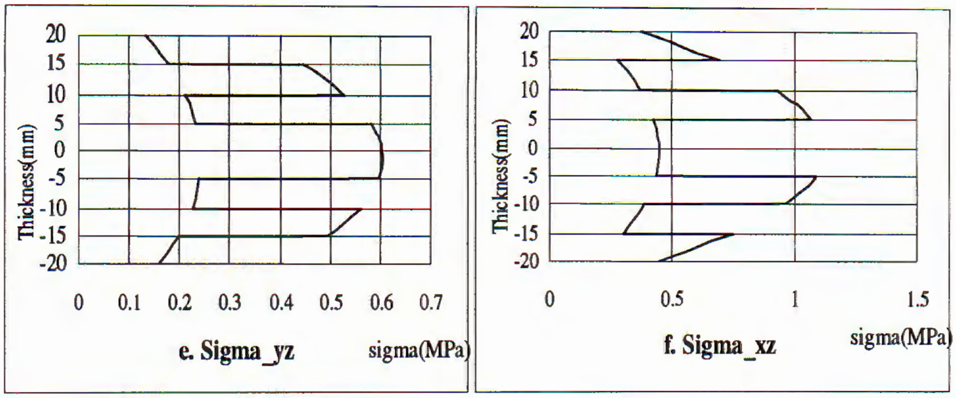
Conf. : Configuration
Thick. : Thickness
(1) : $[15°/-15°]$
(2) : $[90°/0°/90°/0°]_s$

Table 2. Maximum Deflection and Stress Resultants for a Simply-Supported Unsymmetric Angle-ply (15°/ -15°) Square Plate under Uniform Transverse Load $q(x,y) = 0.0001\,\text{MPa}$.

Source	a/h	w_0(m) $(\frac{a}{2},\frac{a}{2})$	M_x(KN) $(\frac{a}{2},\frac{a}{2})$	M_y(KN) $(\frac{a}{2},\frac{a}{2})$	$-M_{xy}$(KN) (0.0)
Present		1.61753E-8	0.0017952	0.0002949	0.0002179
Pandya & Kant [5]	5	1.10021E-8	0.0017328	0.0002504	0.0002843
Turvey [7]		1.00614E-8	0.0016693	0.0002339	–
Present		5.75572E-8	0.0017609	0.0002120	0.0002114
Pandya & Kant [5]	10	5.24971E-8	0.0018241	0.0002049	0.0002541
Turvey [7]		5.09414E-8	0.0017842	0.0002004	–
Present		2.66322E-6	0.0017827	0.0001781	0.0002744
Pandya & Kant [5]	40	2.61632E-6	0.0018512	0.0001896	0.0002441
Turvey [7]		2.60644E-6	0.0018218	0.0001886	–

Source	a/h	N_x(KN/m) $(0,0)$	N_{xy} $(\frac{a}{2},\frac{a}{2})$	Q_y $(0,\frac{a}{2})$	Q_y $(\frac{a}{2},0)$
Present		0.010144	0.00763018	0.0177515	0.0057565
Pandya & Kant [5]	5	0.006904	0.006712	0.016836	0.00576
Turvey [7]		–	–	–	–
Present		0.027529	0.0169409	0.0189025	0.00561243
Pandya & Kant [5]	10	0.016056	0.01536	0.017208	0.005124
Turvey [7]		–	–	–	–
Present		0.200569	0.0660086	0.0258864	0.00497648
Pandya & Kant [5]	40	0.067161	0.06408	0.017308	0.004864
Turvey [7]		–	–	–	–





Figures 1. Distribution of stresses at midplate for a Simply-Supported symmetric $[90°/0°/90°/0°]_s$ Square Plate under Uniform Transverse Load $q = 0.1$ MPa.

5. Conclusions

A simple $C°$ isoparametric formulation of a full third-order displacement model was developed for bending analysis of thick composite plates. The present shear deformable theory does not require the usual shear correction factors generally associated with the Mindlin-Reissner type of theory. In general, the agreement of the present finite element solution is exellent for bending problem of the thin layered composite plates when compared with other finite element and the closed-form solution. For the thick ($a/h = 5$ and 10) simply-supported unsymmetric angle-ply $(15°/-15°)$ square plate, the present finite element solution overpredicts central deflections.

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VỀ BÀI TOÁN UỐN TẤM COMPOSITE LỚP DÀY

Trường chuyển vị bậc cao trong phần tử tứ giác 9 nút, 12 bậc tự do ở mỗi nút được phát triển cho bài toán uốn tấm omposite lớp dày bất kỳ, chịu tải trọng uốn ngang. Biến dạng, nội lực và ứng suất thu được trong các ví dụ được so sánh với lời giải giải tích và một số kết quả PTHH khác.