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FREQUENCY OF A VANDERPOL OSCILLATOR WITH LARGE CUBIC RESTORING NONLINEARITY

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In [2], a modified Poincaré method has been presented and applied to evaluate the period (frequency) of free oscillation in an undamped Duffing oscillator with large cubic restoring nonlinearity. There, the system examined is supposed to be near certain linear one with unknown (to be evaluated) frequency.

In this article, the same modified method is used to study the case of a Vanderpol oscillator: the self-excitation is assumed to be weak while the cubic restoring nonlinearity is assumed to be strong enough. It is shown that stationary self-excited oscillation nearly retains its amplitude while its frequency strongly depends on the cubic nonlinearity. The frequency obtained from the standard Poincaré method can only be used for small cubic nonlinearity; that obtained from the modified method is practically acceptable even for large cubic nonlinearity.

1. Systems under consideration. Original amplitude and frequency

Consider a Vanderpol oscillator described by the differential equation

$$\ddot{x} + x = -\gamma x^3 + h(1 - x^2)\dot{x}, \tag{1.1}$$

where x is oscillatory variable; overdots denote differentiation with respect to time t; $\gamma > 0$ is coefficient of the cubic restoring nonlinearity which may not be small; $h > 0$ is a small intensity of the self-excitation $h(1 - x^2)\dot{x}$.

For $\gamma = 0$ (without cubic nonlinearity), in the first approximation, stationary self-excited oscillation is a stable harmonic one

$$x = a\cos\psi \tag{1.2}$$

with amplitude $a = 2$ and with frequency $\omega = \dot{\psi} = 1$. For small γ, the amplitude and the frequency are lightly modified. The question to be answered is how the amplitude and the frequency depend on the cubic nonlinearity if the later is strong enough.

2. Frequency from the standard Poincaré method

For the sake of comparison, the standard Poincaré method [1] is used first to evaluate the frequency in the case of small γ.

Introducing the new dimensionless time $\tau = \omega t$, we rewrite the differential equation (1.1) in the form

$$\omega^2 x'' + x = \varepsilon\{-\gamma x^3 + h(1 - x^2)\omega x'\}, \tag{2.1}$$

where primes denote differentiation with respect to τ; ω is the unknown frequency; ε is a formal parameter indicating the smallness of the right hand side.

The problem of interest is to determine the frequency of stationary self-excited oscillation satisfying initial condition

$$x'(\tau)\big|_{\tau=0} = x'(0) = 0. \tag{2.2}$$

According to the standard Poincaré method, both unknowns x and ω are expanded in powers of ε, that is

$$x = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots, \tag{2.3}$$
$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots. \tag{2.4}$$

Corresponding to the initial condition (2.2), those of x_0, x_i $(i = 1, 2, \dots)$ are

$$x_0'(0) = 0, \quad x_i'(0) = 0 \quad (i = 1, 2, \dots). \tag{2.5}$$

Substituting (2.3), (2.4) into (2.1), equating the terms of like powers of ε yield

$$x_0'' + x_0 = 0, \tag{2.6}$$
$$x_1'' + x_1 = -2\omega_1 x_0'' - \gamma x_0^3 + h(1 - x_0^2)x_0', \tag{2.7}$$
$$\begin{aligned} x_2'' + x_2 = {} & -2\omega_1 x_1'' - (\omega_1^2 + 2\omega_2)x_0'' - 3\gamma x_0^2 x_1 \\ & + h(1 - x_0^2)(x_1' + \omega_1 x_0') - 2h x_0 x_1 x_0', \end{aligned} \tag{2.8}$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

With regard to the initial condition $x_0'(0) = 0$, the general solution of the differential equation (2.6) is

$$x_0 = a_0 \cos\tau, \quad a_0 = \text{const}. \tag{2.9}$$

Using (2.9), the differential equation (2.7) becomes

$$\begin{aligned} x_1'' + x_1 = {} & 2\omega_1 a_0 \cos\tau - \gamma a_0^3\Big(\frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau\Big) \\ & - h a_0\Big(1 - \frac{a_0^2}{4}\Big)\sin\tau + \frac{1}{4}h a_0^3 \sin 3\tau. \end{aligned} \tag{2.10}$$

To eliminate secular terms in (2.10), the unknowns a_0 and ω_1 should be taken as:

$$a_0 = 2, \quad \omega_1 = \frac{3}{8}\gamma a_0^2 = \frac{3}{2}\gamma. \tag{2.11}$$

Then

$$x_1'' + x_1 = -\frac{\gamma a_0^3}{4}\cos 3\tau + \frac{h a_0^3}{4}\sin 3\tau = -2\gamma\cos 3\tau + 2h\sin 3\tau, \tag{2.12}$$

$$x_1 = \frac{\gamma}{4}\cos 3\tau - \frac{h}{4}\sin 3\tau + a_1\cos\tau + b_1\sin\tau, \tag{2.13}$$

where a_1, b_1 are constants to be determined.

To satisfy the condition (2.5) for $i = 1$, b_1 should be taken

$$b_1 = \frac{3h}{4}\cdot \tag{2.14}$$

With regard to (2.9), (2.11), (2.13), (2.14), the differential equation (2.8) can be simplified as:

$$x_2'' + x_2 = \left(4\omega_2 - 6\gamma a_1 + \frac{15\gamma^2 + h^2}{4}\right)\cos\tau + h(2a_1 + \gamma)\sin\tau + \dots, \tag{2.15}$$

where non-written terms do not contain the first harmonics.

Eliminating secular terms in (2.15) gives

$$a_1 = -\frac{\gamma}{2}, \quad \omega_2 = \frac{-1}{16}(2\gamma^2 + h^2). \tag{2.16}$$

Thus, the first two formulas for frequency are:

$$\omega_I = 1 + \frac{3\gamma}{2}, \tag{2.17}$$

$$\omega_{II} = 1 + \frac{3\gamma}{2} - \frac{27\gamma^2 + h^2}{16}\cdot \tag{2.18}$$

Preliminary remark: for large γ, the frequency ω_{II} becomes negative ($\omega_{II} = 0$ if $\gamma \approx 1.3333...$); this announces that the frequency obtained from the standard Poincaré method is not reliable for large γ.

3. Frequency from a modified Poincaré method

In this section, the problem of interest is treated by the modified Poincaré method as presented in [2].

Assuming that the strongly nonlinear Vanderpol oscillator considered is near certain linear one of unknown frequency ω, we rewrite the differential equation (1.1) as:

$$\ddot{x} + \omega^2 x = \mu\{(\omega^2 - 1)x - \gamma x^3 + h(1 - x^2)\dot{x}\}, \tag{3.1}$$

or, with the new time $\tau = \omega t$:

$$\omega^2(x'' + x) = \mu\{(\omega^2 - 1)x - \gamma x^3 + h(1 - x^2)\omega x'\}, \tag{3.2}$$

where μ is a small formal parameter introduced to indicate the smallness of the right hand side.

Both unknowns x and ω are now expanded in powers of μ, that is:

$$x = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \dots, \tag{3.3}$$

$$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \dots, \tag{3.4}$$

where ω_0 is the approximate frequency of order μ^0 (not 1 as in the standard Poincaré method)

Substituting (3.3), (3.4) into (3.2), equating the terms of like powers of μ yield

$$\omega_0^2(x_0'' + x_0) = 0, \tag{3.5}$$

$$\omega_0^2(x_1'' + x_1) = (\omega_0^2 - 1)x_0 - \gamma x_0^3 + h(1 - x_0^2)\omega_0 x_0', \tag{3.6}$$

$$\begin{aligned}\omega_0^2(x_2'' + x_2) = &-2\omega_0\omega_1(x_1'' + x_1) + (\omega_0^2 - 1)x_1 + 2\omega_0\omega_1 x_0 - 3\gamma x_0^2 x_1 \\ &+ h(1 - x_0^2)(\omega_0 x_1' + \omega_1 x_0') - 2hx_0x_1\omega_0 x_0',\end{aligned} \tag{3.7}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

The general solution x_0 satisfying the initial conditions $x_0'(0) = 0$ is

$$x_0 = a_0 \cos\tau, \quad a_0 = \text{const}. \tag{3.8}$$

Substituting (3.8) into (3.6) gives:

$$\begin{aligned}\omega_0^2(x_1'' + x_1) = (\omega_0^2 - 1)a_0\cos\tau - \gamma a_0^3\Big(\frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau\Big) \\ - h\omega_0 a_0\Big(1 - \frac{a_0^2}{4}\Big)\sin\tau + \frac{h\omega_0 a_0^3}{4}\sin 3\tau.\end{aligned} \tag{3.9}$$

To eliminate secular terms in (3.9), a_0 and ω_0 should be taken as

$$a_0 = 2, \quad \omega_0^2 = 1 + \frac{3\gamma}{4}a_0^2 = 1 + 3\gamma. \tag{3.10}$$

Using (3.8), (3.10), the differential equation governing x_1 becomes

$$\omega_0^2(x_1'' + x_1) = -\frac{\gamma a_0^3}{4}\cos 3\tau + \frac{h\omega_0 a_0^3}{4}\sin 3\tau = -2\gamma\cos 3\tau + 2h\omega_0\sin 3\tau \tag{3.11}$$

and the general solution x_1 is:

$$x_1 = \frac{\gamma}{4\omega_0^2}\cos 3\tau - \frac{h\omega_0}{4\omega_0^2}\sin 3\tau + a_1\cos\tau + b_1\sin\tau, \tag{3.12}$$

where a_1, b_1 are two constants to be determined.

To satisfy the initial condition (2.5) for $i = 1$, b_1 should be taken as

$$b_1 = \frac{3h_0\omega_0}{4\omega_0^2}. \tag{3.13}$$

The differential equation (3.7) can be simplified as:

$$\omega_0^2(x_2'' + x_2) = h\omega_0\Big(2a_1 + \frac{\gamma}{\omega_0^2}\Big)\sin\tau + \Big(4\omega_0\omega_1 - 6\gamma a_1 - \frac{3\gamma^2 + 6h^2\omega_0^2}{4\omega_0^2}\Big)\cos\tau$$
$$+ \text{(higher harmonic terms)}. \tag{3.14}$$

Eliminating secular terms in (3.14) gives:

$$a_1 = -\frac{\gamma}{2\omega_0^2}, \quad \omega_1 = -\frac{9\gamma^2 + h^2\omega_0^2}{16\omega_0^3}. \tag{3.15}$$

Thus, the two formulas for frequency are

$$\omega_{I*} = \omega_0 = \sqrt{1 + 3\gamma}, \tag{3.16}$$

$$\omega_{II*} = \omega_0 + \omega_1 = \sqrt{1+3\gamma}\Big\{1 - \frac{9\gamma^2 + h^2(1+3\gamma)}{(1+3\gamma)^2}\Big\}, \tag{3.17}$$

or, by neglecting h^2 (if γ is large enough) we have

$$\omega_{II*} = \sqrt{1+3\gamma}\Big\{1 - \frac{9\gamma^2}{(1+3\gamma)^2}\Big\}. \tag{3.18}$$

4. Comparison and conclusion

For $h = 0.05$ and for different values γ, using (2.17) and (3.16), (3.17), approximate frequencies ω_I, ω_{II} (from the standard Poincaré method) and ω_{I*}, ω_{II*} (from the modified Poincaré method) are shown in the third, fourth and fifth, sixth columns, of the table standing at the end of this article, respectively. The frequencies ω_c in the second column are obtained by analyzing the amplitude spectrum [3] of stationary self-excited oscillations $x(t)$ ($t \in [0, 12\pi]$, number of times $N = 2048$). It can be seen that, the first approximate frequencie ω_I obtained from the standard Poincaré method can be used only for $\gamma \ll 1$ (for $\gamma = 1$, the relative error is 20%), the second approximation ω_{II} must be rejected. On the contrary, the frequencies ω_{I*}, ω_{II*} obtained from the modified Poincaré method are practically acceptable and ω_{II*} is better than ω_{I*} (for $\gamma = 20$, the relative error of ω_{II*} is of order 2.2%).

Table 1

γ	ω_c	ω_I	ω_{II}	ω_{I*}	ω_{II*}
0	1.0000	1.0000	0.9999	1.0000	0.9999
1	2.0000	2.5000	0.1324	2.0000	1.9296
2	2.5000	4.0000	–	2.6457	2.5342
3	3.0000	5.5000	–	3.1623	3.0021
4	3.5000	7.0000	–	3.6055	3.4450
5	3.8333	8.5000	–	4.0000	3.7802
6	4.1666	10.0000	–	4.3589	4.1144
7	4.5000	11.5000	–	4.6904	4.4233
8	4.8333	13.0000	–	5.0000	4.7120
9	5.1666	14.5000	–	5.2915	4.9840
10	5.3333	15.0000	–	5.5678	5.2418
12	5.8333	19.0000	–	6.0828	5.7228
14	6.3333	22.0000	–	6.5574	6.1664
16	6.6666	25.0000	–	7.0000	6.5802
18	7.1666	28.0000	–	7.4162	6.9694
20	7.5000	31.0000	–	7.8102	7.3380

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TẦN SỐ CỦA CHẤN TỬ VAN-DER-POL CÓ PHI TUYẾN HỒI PHỤC MẠNH BẬC BA

Khảo sát chấn tử Vanderpol có phi tuyến hồi phục mạnh bậc ba và kích động tự chấn yếu. Xét quan hệ giữa tần số của chế độ tự chấn dừng với hệ số phi tuyến hồi phục. Phương pháp chuẩn Poincaré không áp dụng được nên một biến thể của nó trình bày ở [2] được sử dụng. Kết quả ở xấp xỉ thứ hai thực tế chấp nhận được.