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SYNCHRONIZATION IN THE SECOND APPROXIMATION

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It is well-known the phenomenon of synchronization (frequency entrainment) in self-excited systems subjected to external or parametric excitations; A lot of monographs [2, 3, 4] has analyzed several important systems for which the synchronization effect occurs and can be determined in the first approximation. However, there exist also certain systems possessing a stable-excited oscillation obtained in the first approximation, which may be synchronized only in the second approximation. This article deals with some systems of the mentioned type; the asymptotic method [1] is applied for this purpose.

1. System under consideration and the solution in the second approximation

Let us consider the system governed by the differential equation of the form:

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, n\omega t) = \varepsilon\{h(1 - x^2)\dot{x} + g(x, n\omega t)\}, \tag{1.1}$$

where x is an oscillatory variable; overdots denote differentiation with respect to time t; 1 is the own frequency; $\varepsilon > 0$ is a small formal parameter; $h > 0$ is the intensity of the self-excitation $h(1-x^2)\dot{x}$; $g(x, n\omega t)$ represents external or parametric excitations of frequency $n\omega$ (g will be given below for each case examined).

Assuming that ω is very close to the own frequency (pratically, the system is in exact resonance) we rewrite (1.1) as:

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, n\omega t) + \varepsilon^2 \Delta x, \tag{1.2}$$

where $\varepsilon^2\Delta = \omega^2 - 1$ is the detuning parameter of order ε^2.

According to the asymptotic method, following expansions are used:

$$x = a\cos\psi + \varepsilon u_1(a, \theta, \psi) + \varepsilon^2 u_2(a, \theta, \psi) + \dots, \quad \psi = \omega t + \theta, \tag{1.3}$$

$$\dot{a} = \varepsilon A_1(a, \theta) + \varepsilon^2 A_2(a, \theta) + \dots, \tag{1.4}$$

$$\dot{\theta} = \varepsilon B_1(a, \theta) + \varepsilon^2 B_2(a, \theta) + \dots, \tag{1.5}$$

where a, θ are full amplitude and dephase of the first harmonic; A_i, B_i ($i = 1, 2, \dots$) are functions of a, θ; and u_i ($i = 1, 2, \dots$) are functions of a, θ, ψ, 2π-periodic with respect to ψ and do not contain first harmonics $\cos\psi$, $\sin\psi$. With regard to (1.4), (1.5), substituting (1.3) into (1.2), expanding $f(x,\dot{x},n\omega t)$ in Taylor serie of ε, equating the terms of like powers of ε yield.

In the first approximation:

$$-2\omega A_1\sin\psi - 2\omega a B_1\cos\psi + \omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = f(a\cos\psi_1 - \omega a\sin\psi_1 n\psi - n\theta). \tag{1.6}$$

In the second approximation

$$\begin{aligned} &- 2\omega A_2\sin\psi - 2\omega a B_2\cos\psi_1 + \omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) = \\ &= -A - 1\frac{\partial A_1}{\partial a}\cos\psi - 2\omega A_1\frac{\partial^2 u_1}{\partial\psi\partial a} + u_i f_x(a\cos\psi_1 - \omega a\sin\psi_1 n\psi - n\theta) \\ &\quad + \Big(A_1\cos\psi + \omega\frac{\partial u_1}{\partial\psi}\Big) f_{\dot{x}}(a\cos\psi_1 - \omega a\sin\psi_1 n\psi - n\theta) + \delta a\cos\psi + \ldots, \end{aligned} \tag{1.7}$$

where f_x, $f_{\dot{x}}$ are partial derivatives with respect to x, $\dot{x}$ and non-written terms contain B_1 as a factor. A_1, B_1, u_1 and A_2, B_2, u_2 are obtained by equating the terms of like harmonics in both sides of (1.6), (1.7). If A_1 does not contain θ and $B_1 \equiv 0$, the amplitude and the dephase of synchronized oscillations (if it exists) are determined by the equations:

$$\varepsilon A_1(a) + \varepsilon^2 A_2(a,\theta) = 0, \tag{1.8}$$

$$\varepsilon^2 B_2(a,\theta) = 0. \tag{1.9}$$

2. Synchronization under external excitation in subharmonic resonance of order 1/3

First, consider the case of an external excitation of intensity $e > 0$ and frequency 3ω, that is

$$g(x, n\omega t) = e\cos 3\omega t. \tag{2.1}$$

In the first approximation we have

$$\begin{aligned} &- 2\omega A_1\sin\psi - 2\omega a B_1\cos\psi + \omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = \\ &= -h\omega a\Big(1 - \frac{a^2}{4}\Big)\sin\psi + \frac{h\omega a^3}{4}\sin 3\psi + e\cos(3\psi - 3\theta), \end{aligned} \tag{2.2}$$

$$A_1 = \frac{ha}{2}\Big(1 - \frac{a^2}{4}\Big), \quad B_1 = 0, \tag{2.3}$$

$$u_1 = -\frac{h\omega a^2}{32\omega^2}\sin 3\psi - \frac{e}{8\omega^2}\cos(3\psi - 3\theta). \tag{2.4}$$

Hence, the expansions (1.4), (1.5) are:

$$\dot{a} = \frac{\varepsilon h a}{2}\Big(1 - \frac{a^2}{4}\Big), \quad \dot{\theta} = 0. \tag{2.5}$$

Evidently, in the first approximation, there exists only a stationary self-excited oscillation with amplitude $a_* = 2$ and arbitrary (indetermined) dephase θ; this oscillation is stable (in amplitude) since $\dfrac{\partial A_1(a_*)}{\partial a} = -h < 0$, it is perturbed by small oscillations (2.5); the latter are due to the'external excitation and also to the non-linear character of the self-excitation. The synchronization cannot be revealed by the calculation in the first approximation.

In the second approximation (with regard that $B_1 = 0$)

$$- 2\omega A_2 \sin\psi - 2\omega a B_2 \cos\psi + \omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2 = -A_1\frac{\partial A_1}{\partial a}\cos\psi - 2\omega A_1\frac{\partial^2 u_1}{\partial\psi\partial a}$$
$$+ u_1 \cdot h\omega a^2 \sin 2\psi + \Big(A_1\cos\psi + \omega\frac{\partial u_1}{\partial\psi}\Big)h(1 - a^2\cos^2\psi) + \Delta a\cos\psi, \tag{2.6}$$

$$A_2 = \frac{hea^2}{64\omega^2}\cos 3\theta, \tag{2.7}$$

$$B_2 = \frac{-1}{2\omega a}\Big\{\frac{h\omega e a^2}{32\omega^2}\sin 3\theta + \frac{h^2\omega^2 a^5}{128\omega^2} + (\omega^2 - 1)a - A_1\frac{\partial A_1}{\partial a} + hA_1\Big(1 - \frac{3a^2}{4}\Big)\Big\}. \tag{2.8}$$

By a_s, θ_s we denote the amplitude and dephase of synchronized oscillations; they are determined (as it has been noted above) by the equations (1.8), (1.9).

By letting

$$a_s = a_* + \varepsilon a_1 = 2 + \varepsilon a_1, \tag{2.9}$$

and expanding $A_1(a_* + \varepsilon a_1)$ in Taylor serie of ε, the equation (1.8) becomes

$$\varepsilon A_1(a_*) + \varepsilon^2 a_1\frac{\partial A_1(a_\alpha)}{\partial a} + \varepsilon^2 A_2(a_*, \theta_s) = 0. \tag{2.10}$$

With regard that $A_1(a_*) = A_1(2) = 0$, from (2.10), one obtains:

$$a_1 = -A_2(a_*, \theta_s)\Big/\frac{\partial A_1(a_*)}{\partial a} = \frac{e\cos 3\theta_s}{16\omega^2}. \tag{2.11}$$

The dephase θ_s is given by the equation (1.9), that is:

$$\frac{h\omega e a_*^2}{32\omega^2}\sin 3\theta_s + \frac{h^2\omega^2 a_*^5}{128\omega^2} + (\omega^2 - 1)a_* = 0$$

or

$$\sin 3\theta_s = \frac{-2\omega}{he}[h^2 + 8(\omega^2 - 1)] \tag{2.12}$$

with the condition that

$$-he \leq 2\omega[h^2 + 8(\omega^2 - 1)] \leq he.$$

If $\omega^2 = 1$ (exact resonance), $\sin 3\theta_s = -\dfrac{2h}{e}$ with the condition that $e \geq 2h$, $a_1 = \pm\dfrac{e}{16}\sqrt{1 - \dfrac{4h^2}{e^2}}$, $a_s = 2 \pm \dfrac{e}{16}\sqrt{1 - \dfrac{4h^2}{e^2}}$. Note that there exist two amplitudes: the larger corresponds to $\cos 3\theta_s > 0$ i.e. $a_1 > 0$, the smaller corresponds to $\cos 3\theta_s < 0$ i.e. $a_1 < 0$.

3. Synchronization under linear parametric excitation in fundamental resonance

The second case to be examined is that of a linear parametric excitation of intensity $2p > 0$ and frequency ω i.e.

$$g(x, n\omega t) = 2px\cos\omega t. \tag{3.1}$$

In the first approximation:

$$\begin{aligned} &-2\omega A_1 \sin\psi - 2\omega a B_1 \cos\psi + \omega^2\Big(\frac{\partial^2 u_1}{\partial \psi_2} + u_1\Big) = -h\omega a\Big(1 - \frac{a^2}{4}\Big)\sin\psi \\ &\qquad + \frac{h\omega a^3}{4}\sin 3\psi + pa\cos\theta + pa\cos(2\psi - \theta), \end{aligned} \tag{3.2}$$

$$A_1 = \frac{ha}{2}\Big(1 - \frac{a^2}{4}\Big), \quad B_1 = 0, \tag{3.3}$$

$$u_1 = -\frac{h\omega a^3}{32\omega^2}\sin 3\psi + \frac{1}{\omega^2}pa\cos\theta - \frac{1}{3\omega^2}pa\cos(2\psi - \theta). \tag{3.4}$$

In the second approximation

$$\begin{aligned} &-2\omega A_2 \sin\psi - 2\omega a B_2 \cos\psi + \omega^2\Big(\frac{\partial^2 u_2}{\partial \psi^2} + u_2\Big) \\ &= -A_1\frac{\partial A_1}{\partial a}\cos\psi - 2\omega A_1\frac{\partial^2 u_1}{\partial\psi\partial a} + u_1\big\{h\omega a^2\sin 2\psi + 2p\cos(\psi - \theta)\big\} \\ &\quad + \Big(A_1\cos\psi + \omega\frac{\partial u_1}{\partial\psi}\Big)h(1 - a^2\cos^2\psi) + \delta a\cos\psi, \end{aligned} \tag{3.5}$$

$$A_2 = -\frac{p^2 a\sin 2\theta}{4\omega^3}, \tag{3.6}$$

$$B_2 = \frac{-1}{2\omega a}\Big\{\frac{p^2 a}{\omega^2}\cos^2\theta - \frac{p^2 a}{3\omega^2} + \frac{h^2\omega^2 a^5}{128\omega^2} + (\omega^2 - 1)a - A_1\frac{\partial A_1}{\partial a} + hA_1\Big(1 - \frac{3a^2}{4}\Big\}, \tag{3.7}$$

$$\cos^2\theta_s = \frac{1}{3} - \frac{h^2\omega^2}{8p^2} - \frac{\omega^2(\omega^2 - 1)}{p^2}, \quad a_1 = -\frac{p^2\sin 2\theta_s}{4h\omega^3}, \quad a_s = 2 - \frac{p^2\sin 2\theta_s}{4h\omega^3}. \tag{3.8}$$

If $\omega^2 = 1$, $a_s = 2 - \dfrac{p^2 \sin 2\theta_s}{4h}$, $\cos^2\theta_s = \dfrac{1}{3} - \dfrac{h^2}{8p^2}$ on condition that $p^2 \geq \dfrac{3}{8}h^2$.

4. Synchronization under quadratic parametric excitation in subharmonic resonance of order $\frac{1}{2}$

For the third example, consider the case

$$g(x, n\omega t) = 2px^2 \cos 2\omega t, \tag{4.1}$$

where $2p > 0$ and 2ω are intensity and frequency of a quadratic parametric excitation in subharmonic resonance of order $\frac{1}{2}$.

In the first approximation

$$\begin{aligned} &-2\omega A_1 \sin\psi - 2\omega a B_1 \cos\psi + \omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = -h\omega a\Big(1 - \frac{a^2}{4}\Big)\sin\psi \\ &\quad + \frac{h\omega a^3}{4}\sin 3\psi + \frac{1}{2}pa^2\cos 2\theta + pa^2\cos(2\psi - 2\theta) + \frac{1}{2}pa^2\cos(4\psi - 2\theta), \end{aligned} \tag{4.2}$$

$$A_1 = \frac{ha}{2}\Big(1 - \frac{a^2}{4}\Big), \quad B_1 = 0, \tag{4.3}$$

$$u_1 = -\frac{h\omega a^3}{32\omega^2}\sin 3\psi + \frac{1}{2\omega^2}pa^2\cos 2\theta - \frac{pa^2}{3\omega^2}\cos(2\psi - 2\theta) - \frac{1}{30\omega^2}(2a^2\cos(4\psi - 2\theta). \tag{4.4}$$

In the second approximation

$$\begin{aligned} &-2\omega A_2 \sin\psi - 2\omega a B_2 \cos\psi + \omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) = -A_1\frac{\partial A_1}{\partial a}\cos\psi - 2\omega A_1\frac{\partial^2 u_1}{\partial\psi\partial a} \\ &\quad + u_2\{h\omega a^2\sin 2\psi + 4pa\cos\psi\cos(2\psi - 2\theta)\} \\ &\quad + \Big(A_1\cos\psi + \omega\frac{\partial u_1}{\partial\psi}\Big)h(1 - A^2\cos^2\psi) + \Delta a\cos\psi, \end{aligned} \tag{4.5}$$

$$A_2 = \frac{-1}{8\omega^3}p^2a^3\sin^4\theta, \tag{4.6}$$

$$\begin{aligned} B_2 = \frac{-1}{2\omega a}\Big\{&\frac{p^2a^3}{2\omega^2}\cos^2 2\theta - \frac{21p^2a^3}{30\omega^2} + \frac{h^2\omega^2a^5}{128\omega^2} \\ &+ (\omega^2 - 1)a - A_1\frac{\partial A_1}{\partial a} + hA_1\Big(1 - \frac{3a^2}{4}\Big)\Big\}, \end{aligned} \tag{4.7}$$

$$a_1 = -\frac{p^2\sin 4\theta_s}{h\omega^3}, \quad a_s = 2 - \frac{p^2\sin 4\theta_s}{h\omega^3}, \quad \cos^2 2\theta_s = \frac{7}{5} - \frac{h^2\omega^2}{16p^2} - \frac{\omega^2(\omega^2 - 1)}{2p^2}.$$

If $\omega^2 = 1$, $a_s = 2 - \dfrac{p^2\sin 4\theta_s}{h}$, $\cos^2 2\theta_s = \dfrac{7}{5} - \dfrac{h^2}{16p^2}$ with the condition that

$$\frac{5h^2}{112} \leq p^2 \leq \frac{5h^2}{32}.$$

5. Synchronization under the interaction between quadratic nonlinearity and excitation in subharmonic resonance of order $\frac{1}{2}$

The last example is devoted to the case in which $g(x, n\omega t)$ consists of the quadratic nonlinearity $-\beta x^2$ ($\beta > 0$ and the external excitation $e\cos 2\omega t$

$$g(x, n\omega t) = -\beta x^2 + e\cos 2\omega t. \tag{5.1}$$

If $\beta = 0$, the self-excited oscillation cannot be synchronized; if $\beta > 0$, under certain condition, the synchronization may occur in the second approximation.

In the first approximation

$$-2\omega A_1 \sin\psi - 2\omega a B_1 \cos\psi + \omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = -h\omega a\Big(1 - \frac{a^2}{4}\sin\psi\Big)$$
$$+ \frac{h\omega a^3}{4}\sin 3\psi - \beta a^2\cos^2\psi + e\cos(2\psi - 2\theta), \tag{5.2}$$

$$A_1 = \frac{ha}{2}\Big(1 - \frac{a^2}{4}\Big), \quad B_1 = 0, \tag{5.3}$$

$$u_1 = -\frac{h\omega a^3}{32\omega^2}\sin 3\psi - \frac{\beta a^2}{2\omega^2} + \frac{\beta a^2}{6\omega^2}\cos 2\psi - \frac{e}{3\omega^2}\cos(2\psi - 2\theta). \tag{5.4}$$

In the second approximation:

$$-2\omega A_2\sin\psi - 2\omega a B_2\cos\psi + \omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) = -A_1\frac{\partial A_1}{\partial a}\cos\psi - 2\omega A_1\frac{\partial^2 u_1}{\partial\psi\partial a}$$
$$+ u_1\{h\omega a^2\sin 2\psi - 2\beta a\cos\psi\} + \Big(A_1\cos\psi + \omega\frac{\partial u_1}{\partial\psi}\Big)h(1 - a^2\cos^2\psi)$$
$$+ \Delta a\cos\psi, \tag{5.5}$$

$$A_2 = -\frac{\beta a e}{6\omega^3}\sin 2\theta, \tag{5.6}$$

$$B_2 = -\frac{1}{2\omega a}\Big\{\frac{\beta a e}{2\omega^2}\cos 2\theta + \frac{\beta^2 a^3}{3\omega^2} + \frac{h^2\omega^2 a^5}{128\omega^2} + (\omega^2 - 1)a - A_1\frac{\partial A_1}{\partial a} + hA_1\Big(1 - \frac{3a^2}{4}\Big)\Big\}, \tag{5.7}$$

$$a_1 = -\frac{\beta e}{3h\omega^3}\sin 2\theta_s,\ a_s = 2 - \frac{\beta e}{3h\omega^3}\sin 2\theta_s,\ \cos 2\theta_s = -\frac{4\beta}{e} - \frac{3h^2\omega^2}{8\beta e} - \frac{3\omega^2(\omega^2 - 1)}{\beta e}.$$

If $\omega^2 = 1$, $a_s = 2 - \dfrac{\beta e}{3h}\sin 2\theta_s$, $\cos 2\theta_s = \dfrac{-4\beta}{e} - \dfrac{3h^2}{8\beta e}$ on condition that $8\beta e \geq 32\beta^2 + 3h^2$.

6. Stability conditions

The stability study is based on the equations of variation

$$
\begin{aligned}
(\delta a)^{\bullet} &= \Big(\varepsilon\frac{\partial A_1}{\partial a} + \varepsilon^2\frac{\partial A_2}{\partial a}\Big)\delta a + \varepsilon^2\frac{\partial A_2}{\partial \theta}\cdot\delta\theta,\\
(\delta\theta)^{\bullet} &= \varepsilon^2\frac{\partial B_2}{\partial a}\delta a + \varepsilon^2\frac{\partial B_2}{\partial\theta}\delta\theta,
\end{aligned}
\tag{6.1}
$$

from which the characteristic equation can be established

$$
\lambda^2 - \Big\{\varepsilon\frac{\partial A_1}{\partial a} + \varepsilon^2\Big(\frac{\partial A_2}{\partial a} + \frac{\partial B_2}{\partial\theta}\Big)\Big\}\lambda + \Big\{\varepsilon^3\frac{\partial A_1}{\partial a}\frac{\partial B_2}{\partial\theta} + \varepsilon^4\dots\Big\} = 0. \tag{6.2}
$$

The two sufficient stability conditions are:

$$
\varepsilon\frac{\partial A_1}{\partial a} + \varepsilon^2\Big(\frac{\partial A_2}{\partial a} + \frac{\partial B_2}{\partial\theta}\Big) < 0, \tag{6.3}
$$

$$
\varepsilon^3\frac{\partial A_1}{\partial a}\cdot\frac{\partial B_2}{\partial\theta} + \varepsilon^4\dots\dots > 0. \tag{6.4}
$$

With regard that $a_s = a_* + \varepsilon a_1$ the condition (6.2) can be written as

$$
\varepsilon\frac{\partial A_1(a_*)}{\partial a} + \varepsilon^2\Big\{a_1\frac{\partial^2 A_1(a_*)}{\partial a^2} + \frac{\partial A_2(a_*)}{\partial a} + \frac{\partial B_2(a_*,\theta_s)}{\partial\theta}\Big\} < 0, \tag{6.5}
$$

or, in pratice (by neglecting the terms of order ε^2)

$$
\frac{\partial A_1(a_*)}{\partial a} < 0. \tag{6.6}
$$

Since $A_1 = ha\Big(1 - \dfrac{a^2}{4}\Big)$ and $a_* = 2$ we have $\dfrac{\partial A_1(a_*)}{\partial a} = -h < 0$ so that the first condition of stability is always satisfied.

By neglecting the terms of order ε^4, the second stability condition can be simplified as

$$
\begin{aligned}
&\frac{\partial A_1(a_*)}{\partial a}\cdot\frac{\partial B_2(a_*,\theta_s)}{\partial\theta} = -h\frac{\partial B_2(a_*,\theta_s)}{\partial\theta} > 0\\
&\text{i.e.}\qquad \frac{\partial B_2(a_*,\theta_s)}{\partial\theta} < 0.
\end{aligned}
\tag{6.7}
$$

As an illustration let us form the stability conditions of the system examined in §2 (the case of external excitation in subharmonic resonance of order $\dfrac{1}{3}$). We have

$$
\frac{\partial B_2}{\partial\theta} = \frac{-3hea\cos 3\theta}{64\omega^2},\qquad \frac{\partial B_2(a_*,\theta_s)}{\partial\theta} = \frac{-3he\cos 3\theta_s}{32\omega^2} = \frac{-3ha_1}{2\omega^2}.
$$

The stability condition (6.7) is satisfied with $a_1 > 0$ and does not satisfied with $a_1 < 0$. This means that synchronized oscillations corresponding to a larger amplitude ($a_s = 2 + \varepsilon a_1 > 2$) are stable; those corresponding to a smaller amplitude ($a_s = 2 - \varepsilon a_1 < 2$) are unstable.

Conclusion

The examples examined above show that there exist self-excited systems for which the synchronization occurs only in the second approximation. The asymptotic method can successfully be used to study these systems; the stability conditions can easily be established.

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HIỆN TƯỢNG ĐỒNG BỘ Ở XẤP XỈ THỨ HAI

Xét hiện tượng đồng bộ ở một số hệ tự chấn chịu kích động cưỡng bức hoặc thông số. Đặc điểm các hệ này là ở xấp xỉ thứ nhất có chế độ tự chấn dừng ổn định và chế độ này được đồng bộ hóa ở xấp xỉ thứ hai.