Vietnam Journal of Mechanics, NCST of Vietnam Vol. **25**, 2003, No 3 (129 – 136)

CONNECTED OSCILLATIONS IN A NONLINEARLY COUPLED SYSTEM

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A single-mass plane system subjected to symmetric restoring forces can strongly oscillate in the direction which is free from the external excitation. The raising oscillation is called the connected one. This phenomenon exists only in nonlinear systems under certain resonance conditions and was first investigated by Kononenko V. O. [1].

In this paper the conditions, under which the connected oscillations occur, will be considered by means of the method of small parameter. The amplitude of the connected oscillation will be determined by the asymptotic method of nonlinear dynamics.

1. Statement of problem. Equations of motion

Let us consider a single-mass plane system shown in Fig. 1, where the springs are located symmetrically and the external force $f\sin(\omega t + \alpha)$ is directed along the y-axis. Due to the symmetry of the springs, in the Taylor's expansion of the potential energy U of these springs the odd terms relative to the coordinates x and y are absent:

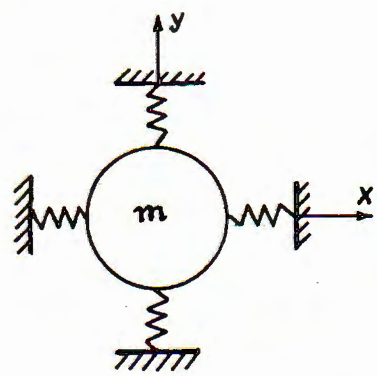

Fig. 1. An oscillating system

$$U = \frac{1}{2}u_{20}x^2 + \frac{1}{2}u_{02}y^2 + \frac{1}{4}u_{40}x^4 + \frac{1}{4}u_{04}y^4 + \frac{1}{2}u_{22}x^2y^2 + \dots,$$

where u_{ij} are constants, u_{20} and u_{02} are linear spring coefficients, x and y are displacements from the equilibrium position.

Using the Lagrange's equations and taking into consideration the friction forces and external excitation one can write the equations of motion of the mass m as [2]:

$$\begin{aligned} &m\ddot{x} + u_{20}x + H_1\dot{x} + u_{40}x^3 + u_{22}xy^2 = 0, \\ &m\ddot{y} + u_{02}y + H_2\dot{y} + u_{04}y^3 + u_{22}yx^2 = f\sin(\omega t + \alpha), \end{aligned} \tag{1.1}$$

where we limit ourselves by nonlinear terms lower than four relative to x and y; i.e. we consider only small oscillations of the mass m, and H_i are positive viscous damping coefficients, f and ω are the intensity and the frequency of the external force, α is its initial phase.

Assuming that the nonlinear terms and the friction force $H_1\dot{x}$ in (1.1) are small and introducing a small positive parameter ε, which will be set to unity after calculations, we can write equations (1.1) in the form

$$\begin{aligned} &\ddot{x} + \lambda_1^2 x + \varepsilon(h_1\dot{x} + \gamma_1 x^3 + \alpha x y^2) = 0, \\ &\ddot{y} + \lambda_2^2 y + h_2\dot{y} + \varepsilon(\gamma_2 y^3 + \alpha y x^2) = f_0 \sin(\omega t + \alpha), \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} &\lambda_1^2 = \frac{u_{20}}{m}, \quad \lambda_1^2 = \frac{u_{02}}{m}, \quad \varepsilon h_1 = \frac{H_1}{m}, \quad h_2 = \frac{H_2}{m}, \\ &\varepsilon\gamma_1 = \frac{u_{40}}{m}, \quad \varepsilon\gamma_2 = \frac{u_{04}}{m}, \quad \varepsilon\alpha = \frac{u_{22}}{m}, \quad f_0 = \frac{f}{m}. \end{aligned} \tag{1.3}$$

Here γ_i and α can be positive or negative.

Obviously, equation (1.2) have partial solution

$$x_* = 0, \quad y_* \neq 0, \tag{1.4}$$

which corresponds to the motion of the mass m only along the y-direction. This is natural for linear system, because then two equations of (1.2) are separate

$$\begin{aligned} &\ddot{x} + \lambda_1^2 x + \varepsilon h_1 \dot{x} = 0, \\ &\ddot{y} + \lambda_2^2 y + h_2\dot{y} = f_0 \sin(\omega t + \alpha). \end{aligned}$$

The oscillation of the coordinate y does not influence on the coordinate x and this coordinate will be at rest. The situation is changed for the nonlinear system described by equation (1.2). Under certain resonance conditions the oscillation of x-coordinate arises and sometimes becomes more strong than that of y-coordinate. In the next paragraphs we will clear up these conditions.

It is noted that in the case $x = 0$, the y-coordinate satisfies the Duffing's equation

$$\ddot{y} + \lambda_2^2 y + h_2\dot{y} + \varepsilon\gamma_2 y^3 = f_0 \sin(\omega t + \alpha). \tag{1.5}$$

The initial phase α can be chosen so that the solution of (1.5) has a simple form

$$y = R \sin \omega t, \tag{1.6}$$

where R and α are determined by equations

$$R = \frac{f_0}{h_2^2\omega^2 + \left(\lambda_2^2 - \omega^2 + \frac{3}{4}\varepsilon\gamma_2 R^2\right)^2}, \quad \alpha = \operatorname{arctg} \frac{h_2\omega}{\lambda_2^2 - \omega^2 + \frac{3}{4}\varepsilon\gamma_2 R^2}. \tag{1.7}$$

2. Conditions for arising the oscillation of the x-coordinate. Parametric resonance

In order to study the stability of the solution (1.4):

$$x_* = 0, \quad y_* = R\sin\omega t \tag{2.1}$$

of equations (1.2), we consider their arbitrary solutions x, y which are deviated from (2.1) and are presented in the form:

$$x = u, \quad y = R\sin\omega t + v, \tag{2.2}$$

where u and v are new and small variables. If all u and v tend to zero when $t \to \infty$, then the solution (2.1) of equations (1.2) is asymptotically stable. If at least one of u and v grows infinitely when $t \to \infty$, then the solution (2.1) is unstable. So, the problem on the stability of the solution (2.1) leads to the problem on the stability of the zero solution $u = v = 0$ of the corresponding equations for u and v. These equations can be obtained by substituting (2.2) into (1.2). We have

$$\begin{aligned} &\ddot{u} + \lambda_1^2 u + \varepsilon[h_1\dot{u} + (\alpha R^2\sin^2\omega t)u] + \varepsilon^2\cdots = 0, \\ &\ddot{v} + \lambda_2^2 v + h_2\dot{v} + (3\varepsilon\gamma_2 R^2\sin^2\omega t)v + \varepsilon^2\cdots = 0, \end{aligned} \tag{2.3}$$

where the non-written terms are at least second degree in (u, v), so that all linear terms are explicitly shown.

Assuming that there is a resonance relationship between λ_1 and ω:

$$\lambda_1^2 = \omega^2 + \varepsilon\Delta, \tag{2.4}$$

where Δ is a detuning parameter and there is no relation of type $n_1\lambda_1 + n_2\lambda_2 = 0$ between λ_1 and λ_2, where n_1 and n_2 are integers. When $\varepsilon = 0$, and for small $h_2 : h_2 < 2\lambda_2$, equations (2.3) have characteristic roots $\pm i\omega$, $-\frac{1}{2}h_2 \pm \frac{i}{2}\sqrt{4\lambda_2^2 - h_2^2}$, the first two of which $\pm i\omega$ are critical and other two are non-critical. The problem on the stability of the solution $u = v = 0$ of the equations (2.3) with periodic coefficients will be solved by considering the real parts of the characteristic indexes [3]. It is easy to prove that these indexes for non-critical characteristic roots have negative real parts. Hence, the stability of zero solution of (2.3) depends only on the sign of the real parts of characteristic exponents of corresponding critical roots $\pm i\omega$.

We use the following change of variables

$$u = Ue^{\sigma t}, \quad v = Ve^{\sigma t}, \tag{2.5}$$

where U and V are some periodic functions of t with period $\dfrac{2\pi}{\omega}$ and σ is a characteristic exponent

$$\sigma = \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots \tag{2.6}$$

The unknown quantities U, V, σ_1 and σ_2 will be determined in the further calculations. By substituting expressions (2.5) and (2.6) into (2.3) and dividing to $e^{\sigma t}$ we get.

$$\ddot{U} + \omega^2 U + \varepsilon\{\Delta U + 2\sigma_1 \dot{U} + h_1 \dot{U} + (\alpha R^2 \sin^2 \omega t)U\} + \varepsilon^2 \cdots = 0,$$
$$\ddot{V} + \lambda_2^2 V + h_2 \dot{V} + \varepsilon\{2\sigma_1 \dot{V} + (3\gamma_2 R^2 \sin^2 \omega t)V\} + \varepsilon^2 \cdots = 0.$$

The periodic solution with the period $\dfrac{2\pi}{\omega}$ of the last equations will be found in the series

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots,$$
$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \dots.$$

By comparing coefficients of equal powers of ε, it is easy to obtain the following equations for U_0 and U_1:

$$\begin{cases} \ddot{U}_0 + \omega^2 U_0 = 0, \\ \ddot{U}_1 + \omega^2 U_1 = -\{\Delta U_0 + (h_1 + 2\sigma_1)\dot{U}_0 + (\alpha R^2 \sin^2 \omega t)U_0\}, \\ V_0 = 0, \dots \end{cases}$$

The periodic solution with period $\dfrac{2\pi}{\omega}$ for U_0 is

$$U_0 = N \cos \omega t + M \sin \omega t,$$

where N and M are constants, which will be determined from the conditions of periodicity of the function U_1. We have

$$\ddot{U}_1 + \omega^2 U_1 = -\Big\{(\Delta + \frac{\alpha}{4}R^2)N + (h_1 + 2\sigma_1)\omega M\Big\} \cos \omega t$$
$$+ \Big\{(h_1 + 2\sigma_1)\omega N - (\Delta + \frac{3}{4}\alpha R^2)M\Big\} \sin \omega t + \dots$$

So, the conditions of periodicity of U_1 are

$$(\Delta + \frac{\alpha}{4}R^2)N + (h_1 + 2\sigma_1)\omega M = 0,$$
$$(h_1 + 2\sigma_1)\omega N - (\Delta + \frac{3}{4}\alpha R^2)M = 0.$$

From here it follows the equation for σ_1:

$$4\omega^2 \sigma_1^2 + 4\omega^2 h_1 \sigma_1 + S = 0,$$
$$S = h_1^2 \omega^2 + (\Delta + \frac{\alpha}{4}R^2)(\Delta + \frac{3}{4}\alpha R^2). \tag{2.7}$$

The stability condition for the solution $u = v = 0$ is $\mathrm{Re}\sigma_1 < 0$, or the same

$$S > 0. \tag{2.8}$$

If this condition is satisfied, the zero solution of equations (2.3) and therefore the solution (2.1) of equations (1.2) will be asymptotically stable and there will be no oscillation of mass m in x-direction. We will consider the case, when the condition (2.8) is not satisfied. The oscillation of the mass m in x-direction may be occurred. The determination of the amplitude of this oscillation and the study of its stability will be given in the next paragraph.

3. Oscillations of the mass m in x-direction

Coming back to the equations (1.2) with resonance condition (2.4) and using the transformation of variables

$$\begin{cases} x = a\cos(\omega t + \varphi), \\ \dot{x} = -a\omega\sin(\omega t + \varphi), \\ y = R\sin\omega t, \end{cases} \tag{3.1}$$

where a and φ are new variables which are substituted for x and $\dot{x}$. By substituting (3.1) in (1.2) and solving for $\dot{a}$ and $\dot{\varphi}$ we get

$$\begin{aligned} \omega\dot{a} &= \varepsilon F\sin\psi, \\ \omega a\dot{\varphi} &= \varepsilon F\cos\psi, \end{aligned} \tag{3.2}$$

where $\psi = \omega t + \varphi$,

$$F = \Big(\Delta + \frac{\alpha R^2}{2}\Big)x + h_1\dot{x} + \gamma_1 x^3 - \frac{\alpha}{2}R^2(\cos 2\omega t)x. \tag{3.3}$$

Equations (3.2) belong to the standard form, for which the averaging principle is applied [4]. The averaged equations for (3.2) are

$$\begin{aligned} \omega\dot{a} &= -\frac{\varepsilon}{8}a(4\omega h_1 + \alpha R^2\sin 2\varphi), \\ \omega a\dot{\varphi} &= \frac{\varepsilon}{8}a\Big[4(\Delta + \frac{\alpha}{2}R^2) + 3\gamma_1 a^2 - \alpha R^2\cos 2\varphi\Big]. \end{aligned} \tag{3.4}$$

There exist two stationary solutions $a = a_0 = \mathrm{const}$, $\varphi = \varphi_0 = \mathrm{const}$ of these equations:

1) $a_0 = 0$, φ_0 is arbitrary, which corresponds to the state at rest of the x-coordinate

2) $a = a_0 \neq 0$, $\varphi = \varphi_0$ which corresponds to the oscillation of the x-coordinate.

The non-trivial solutions $a = a_0 \neq 0$, $\varphi = \varphi_0$ of (3.4) satisfy the equations

$$4\omega h_1 + \alpha R^2 \sin 2\varphi_0 = 0,$$
$$4\big(\Delta + \frac{\alpha}{2}R^2\big) + 3\gamma_1 a_0^2 - \alpha R^2 \cos 2\varphi_0 = 0.$$

By eliminating the phase φ_0 we obtain a relationship for the amplitude a_0:

$$W(a_0^2, \omega^2) = 0, \tag{3.5}$$

where

$$W(a_0^2, \omega^2) = \Big[4\big(\Delta + \frac{\alpha}{2}R^2\big) + 3\gamma_1 a_0^2\Big]^2 + 16h_1^2\omega^2 - \alpha^2 R^4. \tag{3.6}$$

The phase φ_0 is determined by the formula

$$\varphi_0 = \operatorname{arctg}\Big\{\frac{-4h_1\omega}{4(\Delta + \frac{\alpha}{2}R^2) + 3\gamma_1 a_0^2}\Big\}. \tag{3.7}$$

To study the stability of the solution a_0, φ_0 of (3.4) we let in which

$$a = a_0 + \xi, \quad \varphi = \varphi_0 + \eta.$$

The variations ξ and η satisfy the following equations of the first approximation

$$\omega\frac{d\xi}{dt} = -\frac{\varepsilon}{4}\alpha a_0 R^2 \cos 2\varphi_0 \cdot \eta,$$
$$\omega a_0 \frac{d\eta}{dt} = \frac{3}{4}\varepsilon\gamma_1 a_0^2 \xi + \frac{\varepsilon}{4}\alpha a_0 R^2 \sin 2\varphi_0 \cdot \eta.$$

The characteristic equation for this system can be presented in the form

$$\omega^2\rho^2 + \varepsilon\omega^2 h_1 \rho + \frac{\varepsilon^2}{32}a_0^2 \frac{\partial W}{\partial a_0^2} = 0. \tag{3.8}$$

Hence, the stability condition for stationary solution a_0, φ_0 is

$$\frac{\partial W}{\partial a_0^2} > 0. \tag{3.9}$$

By the rule stated in [3], it is easy to identify the stable branches of the resonance curves, basing on the expressions (3.6) and (3.9).

It is noted that, the expression (3.6) can be written in the form

$$W(a_0^2, \omega^2) = 9\gamma_1^2 a_0^4 + 24\gamma_1\big(\Delta + \frac{\alpha}{2}R^2\big)a_0^2 + 16S, \tag{3.10}$$

here S is of the form (2.7). On the segment I of ω^2-axis ($a_0 = 0$) of the plane (a_0^2, ω^2) the function $W(0, \omega^2)$ is negative, the expression S is negative too. Remember that as it is shown at the end of the paragraph §2, this means that the segment I corresponds to the instability of the state at rest of the x-coordinate. Hence, where this state is unstable, the parametric resonance of the x-coordinate occurs.

Introducing the notations

$$\Omega = \frac{\omega}{\lambda_1}, \quad \gamma = \frac{\gamma_1}{\lambda_1^2}, \quad \beta = \frac{\varepsilon\alpha}{2\lambda_1^2}, \quad h = \frac{\varepsilon h_1}{\lambda_1}, \tag{3.11}$$

we can solve equation (3.5) relative to a_0^2:

$$a_0^2 = \frac{4}{3\varepsilon\gamma}\left[\Omega^2 - 1 - \beta R^2 \pm \sqrt{\frac{\beta^2}{4}R^4 - h^2\Omega^2}\right]. \tag{3.12}$$

For $\gamma > 0$, we set

$$\frac{3}{4}\varepsilon\gamma a_0^2 = A_0^2. \tag{3.13}$$

For $\gamma < 0$, we set

$$-\frac{3}{4}\varepsilon\gamma a_0^2 = \overline{A}_0^2. \tag{3.14}$$

We have

$$A_0^2 = \Omega^2 - 1 - \beta R^2 \pm \sqrt{\frac{1}{4}\beta^2 R^4 - h^2\Omega^2} \tag{3.15}$$

$$\overline{A}_0^2 = 1 - \Omega^2 + \beta R^2 \pm \sqrt{\frac{1}{4}\beta^2 R^4 - h^2\Omega^2}, \tag{3.16}$$

and for the phase φ_0:

$$\varphi_0 = -\mathrm{arctg}\frac{h\Omega}{1 - \Omega^2 + \beta^2 R^2 + A_0^2}, \tag{3.17}$$

$$\varphi_0 = -\mathrm{arctg}\frac{h\omega}{1 - \Omega^2 + \beta^2 R^2 - \overline{A}_0^2}. \tag{3.18}$$

The dependence on Ω of the amplitude a_0 of oscillation of the x-coordinate is given in Fig. 2 for $\beta = -0.1$, $R = 1.5$ and for various values of parameters h: $h = 0$, $h = 0.100$ and $h = 0.105$.

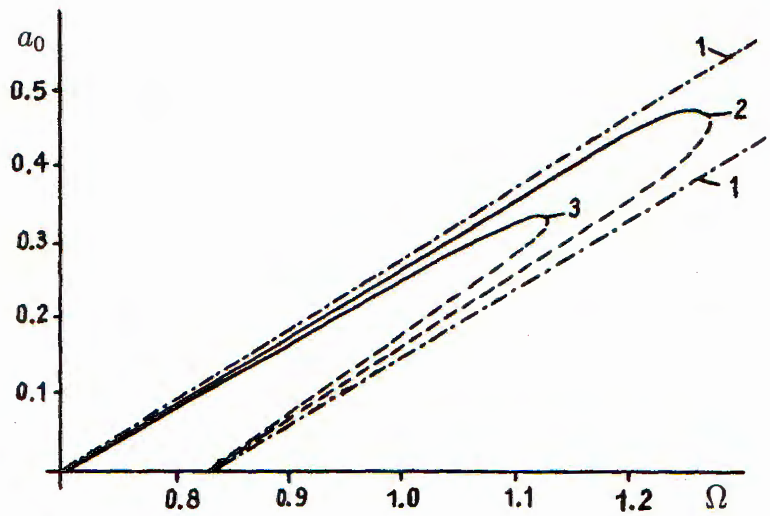

Fig. 2. Resonance curves: 1. $h = 0$, 2. $h = 0.100$, 3. $h = 0.105$

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Received May 14, 2003

DAO ĐỘNG QUAN LIÊN
TRONG MỘT HỆ PHI TUYẾN NỐI KẾT ĐỐI XỨNG

Bài báo đề cập đến một vấn đề cổ điển của dao động quan liên trong mô hình dao động hoàn toàn đối xứng nhằm làm sáng tỏ những điều kiện, theo đó dao động của vật thể sẽ xảy ra theo phương không có ngoại lực tác động trực tiếp (xem điều kiện $S < 0$) và xác định biên độ dao động của vật thể theo phương này (Biểu thức (3.15), (3.16)). Phương pháp tham số bé và phương pháp tiệm cận của lý thuyết dao động phi tuyến đã được sử dụng để giải quyết bài toán đặt ra.