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ON A TECHNIQUE FOR DERIVING EXPLICIT TRANSFER MATRICES OF ORTHOTROPIC LAYERS

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Abstract. This paper presents a technique by which the transfer matrix in explicit form of an orthotropic layer can be easily obtained. This transfer matrix is applicable for both the wave propagation problem and the reflection/transmission problem. The obtained transfer matrix is then employed to derive the explicit secular equation of Rayleigh waves propagating in an orthotropic half-space coated by an orthotropic layer of arbitrary thickness.

Keywords: Orthotropic layer, layer transfer matrix.

1. INTRODUCTION

Multilayered materials can be encountered in various branches of physics, studies of wave propagation in layered media therefore plays an important role in practical applications. Applications of these studies include such technologically important areas as earthquake prediction, underground fault mapping, oil and gas exploration, architectural noise reduction, and the design of ultrasonic transducer. To compute the displacement and stress field of waves propagating in the layered system consisting of an arbitrary number of different homogeneous layers are applied the transfer matrix method [1, 2], the stiffness matrix method [3], the impedance surface method [4], the R/T method [5], the global matrix method [6, 7], and among them the transfer matrix method is the simplest and was earliest proposed. For the transfer matrix method, the transfer matrix of the layered system (called ”global transfer matrix”) is obtained by simple multiplication of each layer transfer matrix. Therefore, it is needed to derive explicit expressions of elements of the layer transfer matrices. Thompson [1] derived explicit expressions of the transfer matrix elements for an isotropic layer. It is not easy to obtain explicit expressions of the transfer matrix elements for an anisotropic layer as mentioned in Crampin [8]. For an orthotropic layer, explicit expressions of the transfer matrix elements were reported in [9], but without the detail derivation. They were reported again in [10] in a more convenient form for calculations. According to the author of the paper [9], in order to get these explicit expressions, first, the solution in exponential form of an orthotropic layer was employed, then a four-order system of linear algebraic equations must be solved. Since the explicit solution expressions of that system are rather cumbersome, it is not easy to arrive at the explicit expressions of elements of the orthotropic-layer transfer matrix. It seems that approach is not feasible to get explicit transfer matrix for a monoclinic layer with the symmetry plane $x_1 = 0$ or $x_2 = 0$.

In this paper, we provide a technique by which explicit expressions of the transfer matrix elements for an orthotropic layer are easily derived. This technique is based on the layer solution expressed in terms of hyperbolic-sin and hyperbolic-cos functions, sinh(.) and cosh(.). In order to derive these explicit expressions we only have to solve two second-order systems of linear algebraic equations. The derivation is therefore simple. The obtained expressions look more compact in form than those reported in [9], and they can be conveniently used for both the problem of wave reflection/transmission and the one of wave propagation. With this technique we can derive explicit expressions of the transfer matrix elements for a monoclinic layer with the symmetry $x_1 = 0$ or $x_2 = 0$. These expressions will be reported elsewhere.

As an application of the obtained explicit expressions, they are employed along with the effective boundary condition method [11,12] to derive the explicit secular equation of Rayleigh waves in an orthotropic half-space coated an orthotropic layer with arbitrary thickness. This secular equation recovers the one derived by Ben-Menahem and Singh [13] for the isotropic case as a special case. The converting of the obtained secular equation to the secular equation (16) in Ref. [14] reveals some misprints of the latter.

2. DERIVATION OF EXPLICIT EXPRESSIONS OF ELEMENTS OF THE TRANSFER MATRIX FOR AN ORTHOTROPIC LAYER

Consider a compressible orthotropic elastic layer with uniform thickness h occupying the domain $a \leq x_2 \leq b$, $b - a = h$. We are interested in the plane strain such that

$$\bar{u}_i = \bar{u}_i(x_1, x_2, t), \; i = 1, 2, \; \bar{u}_3 \equiv 0, \tag{1}$$

where $\bar{u}_i$ are displacement components of the layer, t is the time. In the absence of body forces the equations of motion are

$$\bar{\sigma}_{11,1} + \bar{\sigma}_{12,2} = \bar{\rho}\ddot{\bar{u}}_1, \; \bar{\sigma}_{12,1} + \bar{\sigma}_{22,2} = \bar{\rho}\ddot{\bar{u}}_2, \tag{2}$$

where $\bar{\sigma}_{ij}$ are stress components of the layer, commas signify differentiation with respect to x_k, a dot indicates differentiation with respect to t. For an orthotropic material the strain-stress relation is of the form

$$\bar{\sigma}_{11} = \bar{c}_{11}\bar{u}_{1,1} + \bar{c}_{12}\bar{u}_{2,2}, \; \bar{\sigma}_{22} = \bar{c}_{12}\bar{u}_{1,1} + \bar{c}_{22}\bar{u}_{2,2}, \; \bar{\sigma}_{12} = \bar{c}_{66}(\bar{u}_{1,2} + \bar{u}_{2,1}), \tag{3}$$

where $\bar{c}_{ij}$ are material constants of the layer. Substituting (3) into (2) and taking into account (1) yield

$$\begin{aligned} &\bar{c}_{11}\bar{u}_{1,11} + \bar{c}_{66}\bar{u}_{1,22} + (\bar{c}_{12} + \bar{c}_{66})\bar{u}_{2,12} = \bar{\rho}\ddot{\bar{u}}_1 \;, \\ &(\bar{c}_{12} + \bar{c}_{66})\bar{u}_{1,12} + \bar{c}_{66}\bar{u}_{2,11} + \bar{c}_{22}\bar{u}_{2,22} = \bar{\rho}\ddot{\bar{u}}_2 \;, \end{aligned} \tag{4}$$

Now we consider the propagation of a plane wave traveling in the x_1-direction with velocity c and wave number k. It is not difficult to verify that the displacement components of the wave, that satisfy Eqs. (4), are given by

$$\bar{u}_1 = \bar{U}_1(x_2)\mathrm{e}^{ik(x_1-ct)},\ \bar{u}_2 = \bar{U}_2(x_2)\mathrm{e}^{ik(x_1-ct)}, \tag{5}$$

where

$$\begin{aligned} \bar{U}_1(x_2) &= A_1\mathrm{ch}\bar{b}_1y + A_2\mathrm{sh}\bar{b}_1y + A_3\mathrm{ch}\bar{b}_2y + A_4\mathrm{sh}\bar{b}_2y, \\ \bar{U}_2(x_2) &= i\big[\alpha_1\big(A_1\mathrm{sh}\bar{b}_1y + A_2\mathrm{ch}\bar{b}_1y\big) + \alpha_2\big(A_3\mathrm{sh}\bar{b}_2y + A_4\mathrm{ch}\bar{b}_2y\big)\big], \end{aligned} \tag{6}$$

$y = k(x_2 - b)$, A_1, A_2, A_3, A_4 are constants, $\bar{\alpha}_k$ and $\bar{b}_k$ are given by

$$\begin{aligned} \bar{\alpha}_k &= -\frac{(\bar{c}_{12}+\bar{c}_{66})\bar{b}_k}{\bar{c}_{22}\bar{b}_k^2 - \bar{c}_{66} + \bar{X}},\ k = 1,2,\ \bar{X} = \bar{\rho}c^2, \\ \bar{b}_1 &= \sqrt{\frac{\bar{S}+\sqrt{\bar{S}^2-4\bar{P}}}{2}},\ \bar{b}_2 = \sqrt{\frac{\bar{S}-\sqrt{\bar{S}^2-4\bar{P}}}{2}}, \\ \bar{S} &= \frac{\bar{c}_{22}(\bar{c}_{11}-\bar{X}) + \bar{c}_{66}(\bar{c}_{66}-\bar{X}) - (\bar{c}_{12}+\bar{c}_{66})^2}{\bar{c}_{22}\bar{c}_{66}}, \\ \bar{P} &= \frac{(\bar{c}_{11}-\bar{X})(\bar{c}_{66}-\bar{X})}{\bar{c}_{22}\bar{c}_{66}}. \end{aligned} \tag{7}$$

Note that $\bar{b}_1$ and $\bar{b}_2$ are complex in general and no requirements are imposed on their real and imaginary parts. On use of Eqs. (5)-(7) into (3) we have

$$\bar{\sigma}_{12} = k\bar{\Sigma}_1(x_2)\mathrm{e}^{ik(x_1-ct)},\ \bar{\sigma}_{22} = k\bar{\Sigma}_2(x_2)\mathrm{e}^{ik(x_1-ct)}, \tag{8}$$

where

$$\begin{aligned} \bar{\Sigma}_1(x_2) &= \bar{\beta}_1\big(A_1\mathrm{sh}\bar{b}_1y + A_2\mathrm{ch}\bar{b}_1y\big) + \bar{\beta}_2(A_3\mathrm{sh}\bar{b}_2y + A_4\mathrm{ch}\bar{b}_2y), \\ \bar{\Sigma}_2(x_2) &= i\big[\bar{\gamma}_1\big(A_1\mathrm{ch}\bar{b}_1y + A_2\mathrm{sh}\bar{b}_1y\big) + \bar{\gamma}_2\big(A_3\mathrm{ch}\bar{b}_2y + A_4\mathrm{sh}\bar{b}_2y\big)\big], \end{aligned} \tag{9}$$

and

$$\bar{\beta}_n = \bar{c}_{66}(\bar{b}_n - \bar{\alpha}_n),\ \bar{\gamma}_n = \bar{c}_{12} + \bar{c}_{22}\bar{b}_n\bar{\alpha}_n,\ n = 1,2. \tag{10}$$

Remark 1:

For the wave propagation problem c is the wave velocity (to be determined) of Rayleigh, Stoneley or Lamb wave and $k = \omega/c$ is the wave number (ω is the given wave circular frequency), while for the reflection and/or transmission problem $c = c_0/\sin\theta_0$ (is given) where c_0 is the velocity of incident wave, θ_0 $(0 < \theta_0 \leq \pi/2)$ is the incident angle and $k = k_0\sin\theta_0$, $k_0 = \omega/c_0$, ω is also given.

Putting $x_2 = b$ in Eqs. (6) and (9) leads to

$$\begin{aligned} &\bar{U}_1(b) = A_1 + A_3,\ \bar{U}_2(b) = i(\bar{\alpha}_1A_2 + \bar{\alpha}_2A_4), \\ &\bar{\Sigma}_1(b) = \bar{\beta}_1A_2 + \bar{\beta}_2A_4,\ \bar{\Sigma}_2(b) = i(\bar{\gamma}_1A_1 + \bar{\gamma}_2A_3). \end{aligned} \tag{11}$$

Solving the system (11) for A_1, A_2, A_3, A_4 we have

$$\begin{aligned} A_1 &= \frac{\bar{\gamma}_2}{[\bar{\gamma}]}\bar{U}_1(b) + \frac{i}{[\bar{\gamma}]}\bar{\Sigma}_2(b),\ A_2 = \frac{i\bar{\beta}_2}{[\bar{\alpha};\bar{\beta}]}\bar{U}_2(b) + \frac{\bar{\alpha}_2}{[\bar{\alpha};\bar{\beta}]}\bar{\Sigma}_1(b), \\ A_3 &= -\frac{\bar{\gamma}_1}{[\bar{\gamma}]}\bar{U}_1(b) - \frac{i}{[\bar{\gamma}]}\bar{\Sigma}_2(b),\ A_4 = -\frac{i\bar{\beta}_1}{[\bar{\alpha};\bar{\beta}]}\bar{U}_2(b) - \frac{\bar{\alpha}_1}{[\bar{\alpha};\bar{\beta}]}\bar{\Sigma}_1(b), \end{aligned} \tag{12}$$

here, for the seeking of simplicity, we use the notations

$$[f;g] := f_2g_1 - f_1g_2,\ [f;g]^{(+)} := f_2g_1 + f_1g_2,\ [f] := f_2 - f_1,\ [f]^{(+)} := f_2 + f_1. \tag{13}$$

Substitution of (12) into (6), (9) and taking $x_2 = a$ yields

$$\xi(a) = T\xi(b), \tag{14}$$

where $\xi(.) = [\bar{U}_1(.)\ \bar{U}_2(.)\ \bar{\Sigma}_1(.)\ \bar{\Sigma}_2(.)]^T$ and

$$T = \begin{bmatrix} \dfrac{[\bar{\gamma};\text{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{-i[\bar{\beta};\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-[\bar{\alpha};\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-i[\text{ch}\varepsilon]}{[\bar{\gamma}]} \\ \dfrac{-i[\bar{\gamma};\bar{\alpha}\text{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{[\bar{\alpha}\text{ch}\varepsilon;\bar{\beta}]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-i\bar{\alpha}_1\bar{\alpha}_2[\text{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-[\bar{\alpha}\text{sh}\varepsilon]}{[\bar{\gamma}]} \\ \dfrac{-[\bar{\gamma};\bar{\beta}\text{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{-i\bar{\beta}_1\bar{\beta}_2[\text{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\alpha};\bar{\beta}\text{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{i[\bar{\beta}\text{sh}\varepsilon]}{[\bar{\gamma}]} \\ \dfrac{-i\bar{\gamma}_1\bar{\gamma}_2[\text{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{[\bar{\beta};\bar{\gamma}\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-i[\bar{\alpha};\bar{\gamma}\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\gamma}\text{ch}\varepsilon]}{[\bar{\gamma}]} \end{bmatrix}, \tag{15}$$

here $\varepsilon_n = \varepsilon\bar{b}_n$, $n = 1,2$, $\varepsilon = kh$ and $[\text{ch}\varepsilon] = \text{ch}\varepsilon_2 - \text{ch}\varepsilon_1$, $[\bar{\alpha}\text{ch}\varepsilon] = \bar{\alpha}_2\text{ch}\varepsilon_2 - \bar{\alpha}_1\text{ch}\varepsilon_1$, $[\bar{\alpha};\bar{\beta}\text{sh}\varepsilon] = \bar{\alpha}_2\bar{\beta}_1\text{sh}\varepsilon_1 - \bar{\alpha}_2\bar{\beta}_1\text{sh}\varepsilon_1$, ... Matrix T given by (15) is the transfer matrix for a compressible orthotropic layer. It is not difficult to prove the equalities

$$t_{11} = t_{33},\ t_{12} = t_{43},\ t_{14} = t_{23},\ t_{21} = t_{34},\ t_{22} = t_{44},\ t_{32} = t_{41}\,, \tag{16}$$

where t_{ij} are components of the transfer matrix T. Analogously, using the solution (5), (6), (8), (9) with $y = k(x_2 - a)$ provides

$$\xi(b) = \hat{T}\xi(a), \tag{17}$$

where $\hat{T}$ is given by (15) in which shε is replaced by $-$shε. In particular, it is

$$\hat{T} = \begin{bmatrix} \dfrac{[\bar{\gamma};\text{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{i[\bar{\beta};\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\alpha};\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-i[\text{ch}\varepsilon]}{[\bar{\gamma}]} \\ \dfrac{i[\bar{\gamma};\bar{\alpha}\text{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{[\bar{\alpha}\text{ch}\varepsilon;\bar{\beta}]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-i\bar{\alpha}_1\bar{\alpha}_2[\text{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\alpha}\text{sh}\varepsilon]}{[\bar{\gamma}]} \\ \dfrac{[\bar{\gamma};\bar{\beta}\text{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{-i\bar{\beta}_1\bar{\beta}_2[\text{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\alpha};\bar{\beta}\text{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-i[\bar{\beta}\text{sh}\varepsilon]}{[\bar{\gamma}]} \\ \dfrac{-i\bar{\gamma}_1\bar{\gamma}_2[\text{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{-[\bar{\beta};\bar{\gamma}\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{i[\bar{\alpha};\bar{\gamma}\text{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\gamma}\text{ch}\varepsilon]}{[\bar{\gamma}]} \end{bmatrix}. \tag{18}$$

One can see that the following equalities are valid

$$\hat{t}_{11} = \hat{t}_{33},\ \hat{t}_{12} = \hat{t}_{43},\ \hat{t}_{14} = \hat{t}_{23},\ \hat{t}_{21} = \hat{t}_{34},\ \hat{t}_{22} = \hat{t}_{44},\ \hat{t}_{32} = \hat{t}_{41}, \tag{19}$$

where $\hat{t}_{ij}$ are components of the transfer matrix $\hat{T}$. From (14) and (17) it implies: $\hat{T} = T^{-1}$.

Remark 2:

(i) From (17) and (18) it follows

$$\eta(b) = A\eta(a), \tag{20}$$

where $\eta(.) = [\bar{v}_1(.)\ \bar{v}_2(.)\ \bar{\sigma}_{22}(.)\ \bar{\sigma}_{12}(.)]^T$ and

$$A = \begin{bmatrix} \dfrac{[\bar{\gamma};\mathrm{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{i[\bar{\beta};\mathrm{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-c[\mathrm{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{-ic[\bar{\alpha};\mathrm{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} \\ \dfrac{i[\bar{\gamma};\bar{\alpha}\mathrm{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{[\bar{\alpha}\mathrm{ch}\varepsilon;\bar{\beta}]}{[\bar{\alpha};\bar{\beta}]} & \dfrac{-ic[\bar{\alpha}\mathrm{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{-c\bar{\alpha}_1\bar{\alpha}_2[\mathrm{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} \\ \dfrac{\bar{\gamma}_1\bar{\gamma}_2[\mathrm{ch}\varepsilon]}{c[\bar{\gamma}]} & \dfrac{-i[\bar{\beta};\bar{\gamma}\mathrm{sh}\varepsilon]}{c[\bar{\alpha};\bar{\beta}]} & \dfrac{[\bar{\gamma}\mathrm{ch}\varepsilon]}{[\bar{\gamma}]} & \dfrac{i[\bar{\alpha};\gamma\mathrm{sh}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} \\ \dfrac{i[\bar{\gamma};\bar{\beta}\mathrm{sh}\varepsilon]}{c[\bar{\gamma}]} & \dfrac{\bar{\beta}_1\bar{\beta}_2[\mathrm{ch}\varepsilon]}{c[\bar{\alpha};\bar{\beta}]} & \dfrac{-i[\bar{\beta}\mathrm{sh}\varepsilon]}{[\bar{\gamma}]} & \dfrac{[\bar{\alpha};\bar{\beta}\mathrm{ch}\varepsilon]}{[\bar{\alpha};\bar{\beta}]} \end{bmatrix}, \tag{21}$$

$\bar{v}_1 = -i\omega\bar{u}_1, \bar{v}_2 = -i\omega\bar{u}_2$ are the components of the particle velocity.

From (19) it implies

$$A_{24} = A_{13},\ A_{33} = A_{22},\ A_{34} = A_{12},\ A_{42} = A_{31},\ A_{43} = A_{21},\ A_{44} = A_{11}\ , \tag{22}$$

where A_{ij} are components of the transfer matrix A. These relations were mentioned in Ref. [10].

Comparing the matrix A with the layer transfer matrix reported in [9] reveals that λ_{xzxz} in the expression for a_{11} in [9] must be replaced by λ_{xxzz}.

(ii) One can see that the expressions of elements of the transfer matrix A are simpler in form than the corresponding expressions obtained by Solyanik [9].

3. EXPLICIT SECULAR EQUATION OF RAYLEIGH WAVES IN AN ORTHOTROPIC HALF-SPACE COATED BY AN ORTHOTROPIC LAYER

Consider a compressible orthotropic elastic half-space $x_2 \geq 0$ overlaid by a compressible orthotropic elastic layer with arbitrary thickness h occupying the domain $-h \leq x_2 \leq 0$. It is assumed that the layer and the half-space are in welded contact with each other and the top surface of the layer $x_2 = -h$ is free from traction. Note that same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a bar if pertaining to the layer.

3.1. Effective boundary conditions

Consider the propagation of a Rayleigh wave traveling with velocity c and wave number k in the x_1-direction, decaying in the x_2-direction. From the traction-free condition: $\bar{\sigma}_{12} = \bar{\sigma}_{22} = 0$ at $x_2 = -h$, using (14), (15) with $a = -h$, $b = 0$ and taking into account the continuity of displacements and stresses through the interface $x_2 = 0$ we have

$$\begin{aligned} t_{31}U_1(0) + t_{32}U_2(0) + t_{33}\Sigma_1(0) + t_{34}\Sigma_2(0) = 0, \\ t_{41}U_1(0) + t_{42}U_2(0) + t_{43}\Sigma_1(0) + t_{44}\Sigma_2(0) = 0. \end{aligned} \tag{23}$$

The relations (23) is called the effective boundary conditions because the entire effect of the layer on the half-space is exactly replaced with these conditions.

3.2. Explicit secular equation

Now we can ignore the layer and consider the propagation of a Rayleigh wave traveling along surface $x_2 = 0$ of the half-space in the x_1-direction with velocity c, wave number k and decaying in the x_2-direction and satisfying the effective boundary conditions (23). According to Vinh & Ogden [15], the displacements of the Rayleigh wave in the half-space $x_2 > 0$ are given by

$$u_1 = U_1(y)\mathrm{e}^{ik(x_1-ct)},\ u_2 = U_2(y)\mathrm{e}^{ik(x_1-ct)},\ y = k, x_2 \tag{24}$$

where

$$U_1(y) = B_1e^{-b_1y} + B_2e^{-b_2y},\ U_2(y) = i(\alpha_1B_1e^{-b_1y} + \alpha_2B_2e^{-b_2y}), \tag{25}$$

B_1 and B_2 are constants to be determined, and

$$\alpha_k = \frac{(c_{12}+c_{66})b_k}{c_{22}b_k^2 - c_{66} + X},\ k = 1,2,\ X = \rho c^2, \tag{26}$$

b_1 and b_2 are two roots with positive real part of the following equation

$$b^4 - Sb^2 + P = 0, \tag{27}$$

S and P are calculated by (7) without the bar symbol. It follows from (27) that

$$b_1^2 + b_2^2 = 2S,\ b_1^2b_2^2 = P. \tag{28}$$

It is not difficult to show that if a Rayleigh wave exists (→ the real parts of b_1 and b_2 must be positive), then (see [15])

$$0 < X < \min\{c_{66}, c_{11}\}, \tag{29}$$

and (see [16])

$$P > 0,\ S + P > 0,\ b_1b_2 = \sqrt{P},\ b_1 + b_2 = \sqrt{S + 2\sqrt{P}}. \tag{30}$$

Using expressions (24) and (25) into the strain-stress relation (3) provides

$$\sigma_{12} = k\Sigma_1(y)\mathrm{e}^{ik(x_1-ct)},\ \sigma_{22} = k\Sigma_2(y)\mathrm{e}^{ik(x_1-ct)}, \tag{31}$$

where

$$\Sigma_1(y) = \beta_1B_1e^{-b_1y} + \beta_2B_2e^{-b_2y},\ \Sigma_2(y) = i(\gamma_1B_1e^{-b_1y} + \gamma_2B_2e^{-b_2y}), \tag{32}$$

where

$$\beta_k = -c_{66}(b_k + \alpha_k),\ \gamma_k = c_{12} - c_{22}b_k\alpha_k,\ k = 1,2. \tag{33}$$

Taking $x_2 = 0$ in (25) and (32) gives

$$\begin{aligned} U_1(0) &= B_1 + B_2,\ U_2(0) = i(\alpha_1B_1 + \alpha_2B_2), \\ \Sigma_1(0) &= \beta_1B_1 + \beta_2B_2,\ \Sigma_2(0) = i(\gamma_1B_1 + \gamma_2B_2). \end{aligned} \tag{34}$$

Substituting (34) into (23) leads to two linear equations for B_1 and B_2, namely

$$\begin{aligned} f(b_1)B_1 + f(b_2)B_2 &= 0, \\ F(b_1)B_1 + F(b_2)B_2 &= 0, \end{aligned} \tag{35}$$

where

$$\begin{aligned} f(b_k) &= t_{33}\beta_k + it_{34}\gamma_k + t_{31} + it_{32}\alpha_k \\ F(b_k) &= t_{43}\beta_k + it_{44}\gamma_k + t_{41} + it_{42}\alpha_k \end{aligned} \quad (k = 1,\, 2) \tag{36}$$

Due to $B_1^2 + B_2^2 \neq 0$, the determinant of coefficients of the homogeneous system (35) must vanish, therefore we have

$$f(b_1)F(b_2) - f(b_2)F(b_1) = 0. \tag{37}$$

Using (36) into (37) and after some calculations we arrive at

$$\begin{gathered} i(t_{33}t_{44} - t_{34}t_{43})[\gamma;\beta] - (t_{33}t_{41} - t_{43}t_{31})[\beta] + i(t_{33}t_{42} - t_{43}t_{32})[\alpha;\beta] \\ -i(t_{34}t_{41} - t_{44}t_{31})[\gamma] - (t_{34}t_{42} - t_{44}t_{32})[\alpha;\gamma] + i(t_{31}t_{42} - t_{32}t_{41})[\alpha] = 0. \end{gathered} \tag{38}$$

With the help of (26) and (33), it is not difficult to verify that

$$\begin{aligned} [\gamma;\beta] =& c_{66}\Big\{ \big[c_{12}^2 - c_{22}(c_{11} - X)\big] b_1 b_2 + X(c_{11} - X) \Big\}\theta, \\ [\alpha;\beta] =& c_{66}(c_{11} - X)(b_1 + b_2)\theta, \ [\alpha;\gamma] = c_{66}(c_{11} - X - c_{12}b_1b_2)\theta, \\ [\alpha] =& (X - c_{11} - c_{66}b_1b_2)\theta, \ [\beta] = [\alpha;\gamma], \ [\gamma] = c_{22}c_{66}b_1b_2(b_1 + b_2)\theta, \end{aligned} \tag{39}$$

where $b_1 b_2 = \sqrt{P}$, $b_1 + b_2 = \sqrt{S + 2\sqrt{P}}$ and $\theta = (b_2 - b_1)/[(c_{12} + c_{66})b_1 b_2]$. After multiplying two sides of Eq. (38) by $[\bar{\gamma}][\bar{\alpha};\bar{\beta}]/\theta$ and taking into account (39), this equation becomes

$$A_0 + B_0 \text{ch}\varepsilon_1 \text{ch}\varepsilon_2 + C_0 \text{sh}\varepsilon_1 \text{sh}\varepsilon_2 + D_0 \text{ch}\varepsilon_1 \text{sh}\varepsilon_2 + E_0 \text{sh}\varepsilon_1 \text{ch}\varepsilon_2 = 0, \tag{40}$$

where A_0, B_0, C_0, D_0 and E_0 are given by

$$\begin{aligned} A_0 =& \ 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_1\bar{\gamma}_2(X - c_{11} - c_{66}\sqrt{P}) \\ &- c_{66}[\bar{\alpha};\bar{\beta}\bar{\gamma}]^{(+)}\Big\{ \big[c_{12}^2 - c_{22}(c_{11} - X)\big]\sqrt{P} + X(c_{11} - X) \Big\} \\ &- c_{66}\Big[\bar{\gamma}_1\bar{\gamma}_2[\bar{\alpha};\bar{\beta}]^{(+)} + \bar{\beta}_1\bar{\beta}_2[\bar{\gamma}]^{(+)} \Big](c_{11} - X - c_{12}\sqrt{P}), \\ B_0 =& \ -A_0 + c_{66}[\bar{\gamma}][\bar{\alpha};\bar{\beta}]\Big\{ \big[c_{12}^2 - c_{22}(c_{11} - X)\big]\sqrt{P} + X(c_{11} - X) \Big\}, \\ C_0 =& \ [\bar{\beta}^2;\bar{\gamma}^2]^{(+)}(X - c_{11} - c_{66}\sqrt{P}) \\ &- c_{66}[\bar{\alpha}\bar{\beta};\bar{\gamma}]^{(+)}\Big\{ \big[c_{12}^2 - c_{22}(c_{11} - X)\big]\sqrt{P} + X(c_{11} - X) \Big\} \\ &- c_{66}\Big([\bar{\alpha}\bar{\beta};\bar{\gamma}^2]^{(+)} + [\bar{\beta}^2;\bar{\gamma}]^{(+)} \Big)(c_{11} - X - c_{12}\sqrt{P}), \\ D_0 =& \ c_{66}\Big[\bar{\beta}_1\bar{\gamma}_2[\bar{\gamma}](X - c_{11}) + c_{22}\bar{\beta}_2\bar{\gamma}_1[\bar{\alpha};\bar{\beta}]\sqrt{P} \Big]\sqrt{S + 2\sqrt{P}}, \\ E_0 =& \ c_{66}\Big[\bar{\beta}_2\bar{\gamma}_1[\bar{\gamma}](c_{11} - X) - c_{22}\bar{\beta}_1\bar{\gamma}_2[\bar{\alpha};\bar{\beta}]\sqrt{P} \Big]\sqrt{S + 2\sqrt{P}}. \end{aligned} \tag{41}$$

Equation (40) is the desired secular equation. From (7), (10), (28), (30), (33) and (41), it is clear that Eq. (40) is totally explicit.

When $\varepsilon = 0$, Eq. (40) becomes $A_0 + B_0 = 0$, or equivalently

$$(c_{66} - X)\left[c_{12}^2 - c_{22}(c_{11} - X)\right] + X\sqrt{c_{22}c_{66}}\sqrt{(c_{11} - X)(c_{66} - X)} = 0, \tag{42}$$

according to the second of (41). This equation is the secular equation of Rayleigh waves propagating along the traction-free surface of a compressible orthotropic half-space (see Eq. (2.17) in [15]).

3.3. Two dimensionless forms of the secular equation

It is useful to convert the secular equation (40) into dimensionless form. To do that we use the following dimensionless parameters (see also [11])

$$\begin{gathered} x = \frac{X}{c_{66}},\ e_1 = \frac{c_{11}}{c_{66}},\ e_2 = \frac{c_{22}}{c_{66}},\ e_3 = \frac{c_{12}}{c_{66}},\ \bar{e}_1 = \frac{\bar{c}_{11}}{\bar{c}_{66}},\ \bar{e}_2 = \frac{\bar{c}_{66}}{\bar{c}_{22}},\ \bar{e}_3 = \frac{\bar{c}_{12}}{\bar{c}_{66}}, \\ r_\mu = \frac{\bar{c}_{66}}{c_{66}},\ r_v = \frac{c_2}{\bar{c}_2},\ c_2 = \sqrt{\frac{c_{66}}{\rho}},\ \bar{c}_2 = \sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}}. \end{gathered} \tag{43}$$

Dimensionless form 1:
By dividing two sides of Eq. (40) by $(c_{66})^5$ it converts to

$$A_1 + B_1 \mathrm{ch}\varepsilon_1 \mathrm{ch}\varepsilon_2 + C_1 \mathrm{sh}\varepsilon_1 \mathrm{sh}\varepsilon_2 + D_1 \mathrm{ch}\varepsilon_1 \mathrm{sh}\varepsilon_2 + E_1 \mathrm{sh}\varepsilon_1 \mathrm{ch}\varepsilon_2 = 0, \tag{44}$$

where

$$\begin{aligned} A_1 =& 2\bar{\beta}_1^*\bar{\beta}_2^*\bar{\gamma}_1^*\bar{\gamma}_2^*\big(x - e_1 - \sqrt{P}\big) \\ & - [\bar{\alpha};\bar{\beta}^*\bar{\gamma}^*]^{(+)}\big[(e_3^2 - e_1e_2 + e_2x)\sqrt{P} + x(e_1 - x)\big] \\ & - \Big(\bar{\gamma}_1^*\bar{\gamma}_2^*[\bar{\alpha};\bar{\beta}^*]^{(+)} + \bar{\beta}_1^*\bar{\beta}_2^*[\bar{\gamma}^*]^{(+)}\Big)\big(e_1 - x - e_3\sqrt{P}\big), \\ B_1 =& - A_1 + [\bar{\gamma}^*][\bar{\alpha};\bar{\beta}^*]\big[(e_3^2 - e_1e_2 + e_2x)\sqrt{P} + x(e_1 - x)\big], \\ C_1 =& \big[(\bar{\beta}^*)^2;(\bar{\gamma}^*)^2\big]^{(+)}\big(x - e_1 - \sqrt{P}\big) \\ & - [\bar{\alpha}\bar{\beta}^*;\bar{\gamma}^*]^{(+)}\big[(e_3^2 - e_1e_2 + e_2x)\sqrt{P} + x(e_1 - x)\big] \\ & - \Big([\bar{\alpha}\bar{\beta}^*;(\bar{\gamma}^*)^2]^{(+)} + [(\bar{\beta}^*)^2;\bar{\gamma}^*]^{(+)}\Big)\big(e_1 - x - e_3\sqrt{P}\big), \\ D_1 =& \Big[\bar{\beta}_1^*\bar{\gamma}_2^*[\bar{\gamma}^*](x - e_1) + e_2\bar{\beta}_2^*\bar{\gamma}_1^*[\bar{\alpha};\bar{\beta}^*]\sqrt{P}\Big]\sqrt{S + 2\sqrt{P}}, \\ E_1 =& \Big[\bar{\beta}_2^*\bar{\gamma}_1^*[\bar{\gamma}^*](e_1 - x) - e_2\bar{\beta}_1^*\bar{\gamma}_2^*[\bar{\alpha};\bar{\beta}^*]\sqrt{P}\Big]\sqrt{S + 2\sqrt{P}}, \end{aligned} \tag{45}$$

in which, the quantities $\bar{\alpha}_k$, $\bar{\beta}_k^*$ and $\bar{\gamma}_k^*$, S and P are given by

$$\begin{aligned} &\bar{\alpha}_k = \frac{\bar{b}_k^2 + r_v^2 x - \bar{e}_1}{(1 + \bar{e}_3)\bar{b}_k},\ \bar{\beta}_k^* = r_\mu(\bar{b}_k - \bar{\alpha}_k),\ \bar{\gamma}_k^* = r_\mu\big(\bar{e}_3 + \frac{\bar{\alpha}_k\bar{b}_k}{\bar{e}_2}\big),\ k = 1,2 \\ &S = \frac{e_2(e_1 - x) + 1 - x - (e_3 + 1)^2}{e_2},\ P = \frac{(e_1 - x)(1 - x)}{e_2}, \end{aligned} \tag{46}$$

$\bar{b}_1$, $\bar{b}_2$ are defined by (7) with $\bar{P}$ and $\bar{S}$ being expressed in terms of the dimensionless parameters as

$$\bar{S} = (\bar{e}_1 - r_v^2 x) + \bar{e}_2\big[1 - r_v^2 x - (\bar{e}_3 + 1)^2\big],\ \bar{P} = \bar{e}_2(\bar{e}_1 - r_v^2 x)(1 - r_v^2 x) \tag{47}$$

It is clear that the squared dimensionless velocity x of Rayleigh waves depends on nine dimensionless parameters: e_k, $\bar{e}_k$ $(k = 1, 2, 3)$, r_μ, r_v and ε.

As an example, we use the secular equation (44) to compute the squared dimensionless wave velocity x with $e_1 = 2.5, e_2 = 3, e_3 = 0.4, \bar{e}_1 = 3.1, \bar{e}_2 = 1, \bar{e}_3 = 0.5, r_\mu = 0.5, r_v = 2.8$. Fig. 1 shows the velocity curves of first six modes in the interval $\varepsilon \in [0\ 3]$.

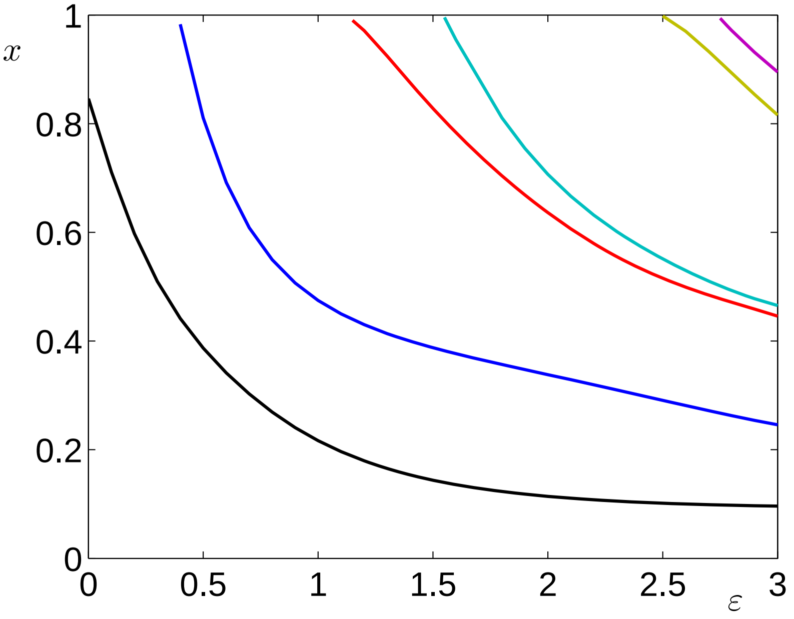

Fig. 1. Velocity curves of first six modes in the interval [0 3]. Here we take $e_1 = 2.5, e_2 = 3, e_3 = 0.4, \bar{e}_1 = 3.1, \bar{e}_2 = 1, \bar{e}_3 = 0.5, r_\mu = 0.5, r_v = 2.8$.

Dimensionless form 2:
Eq. (40) can be rewritten as follows

$$\begin{aligned}&(B_0 + C_0)\text{sh}^2\big[\frac{\varepsilon(\bar{b}_1 + \bar{b}_2)}{2}\big] + (B_0 - C_0)\text{sh}^2\big[\frac{\varepsilon(\bar{b}_1 - \bar{b}_2)}{2}\big]\\ &+ \frac{E_0 + D_0}{2}\text{sh}[\varepsilon(\bar{b}_1 + \bar{b}_2)] + \frac{E_0 - D_0}{2}\text{sh}[\varepsilon(\bar{b}_1 - \bar{b}_2)] + A_0 + B_0 = 0,\end{aligned} \tag{48}$$

Using (41) and the variables η and $\bar{\eta}$ given by $\eta = \sqrt{\dfrac{c_{66} - \rho c^2}{c_{11} - \rho c^2}}$, $\bar{\eta} = \sqrt{\dfrac{\bar{c}_{66} - \bar{\rho} c^2}{\bar{c}_{11} - \bar{\rho} c^2}}$, after some calculations we have

$$
\begin{aligned}
\frac{(B_0+C_0)}{c_{66}(c_{11}-X)} &= -\bar{c}_{66}[\bar{\alpha};\bar{\beta}]^2 \frac{\bar{f}(\bar{\eta})}{\bar{\eta}^2-1}\,\frac{r_\mu}{(\bar{b}_1+\bar{b}_2)^2}\Big\{(1+\eta e_2^{-1/2})\frac{\bar{f}(\bar{\eta})}{\bar{\eta}^2-1} \\
&\qquad -2r_\mu^{-1}(1-e_3e_2^{-1/2}\eta)(1-\bar{e}_3\bar{e}_2^{1/2}\bar{\eta})+r_\mu^{-2}(1+\bar{\eta}\bar{e}_2^{1/2})\frac{f(\eta)}{\eta^2-1}\Big\}, \\
\frac{(B_0-C_0)}{c_{66}(c_{11}-X)} &= \bar{c}_{66}[\bar{\alpha};\bar{\beta}]^2 \frac{\bar{f}(-\bar{\eta})}{\bar{\eta}^2-1}\,\frac{r_\mu}{(\bar{b}_1-\bar{b}_2)^2}\Big\{(1+\eta e_2^{-1/2})\frac{\bar{f}(-\bar{\eta})}{\bar{\eta}^2-1} \\
&\qquad -2r_\mu^{-1}(1-e_3e_2^{-1/2}\eta)(1+\bar{e}_3\bar{e}_2^{1/2}\bar{\eta})+r_\mu^{-2}(1-\bar{\eta}\bar{e}_2^{1/2})\frac{f(\eta)}{\eta^2-1}\Big\}, \qquad (49)\\
\frac{(E_0+D_0)}{c_{66}(c_{11}-X)} &= \bar{c}_{66}[\bar{\alpha};\bar{\beta}]^2 \frac{1}{\bar{b}_1+\bar{b}_2}\frac{\bar{f}(\bar{\eta})}{\bar{\eta}^2-1}(b_1+b_2)[e_2^{1/2}\eta+\bar{e}_2^{-1/2}\bar{\eta}], \\
\frac{(E_0-D_0)}{c_{66}(c_{11}-X)} &= -\bar{c}_{66}[\bar{\alpha};\bar{\beta}]^2 \frac{1}{\bar{b}_1-\bar{b}_2}\frac{\bar{f}(-\bar{\eta})}{\bar{\eta}^2-1}(b_1+b_2)[e_2^{1/2}\eta-\bar{e}_2^{-1/2}\bar{\eta}], \\
\frac{(A_0+B_0)}{c_{66}(c_{11}-X)} &= \bar{c}_{66}[\bar{\alpha};\bar{\beta}]^2\, r_\mu^{-1}\,\bar{e}_2^{-1/2}\,\bar{\eta}\,\frac{f(\eta)}{\eta^2-1},
\end{aligned}
$$

where

$$
f(\eta)=e_3^2e_2^{-1/2}\eta^3+e_1\eta^2+[e_2(e_1-1)-e_3^2]\eta e_2^{-1/2}-1, \qquad (50)
$$

with e_1, e_2, e_3, r_μ are defined by (43), $\bar{f}(\bar{\eta})$ is given by the first of (50) in which e_1, e_2 and e_3 are replaced by $\bar{e}_1$, $\bar{e}_2^* = 1/\bar{e}_2$ and $\bar{e}_3$, respectively.

After dividing two sides of Eq. (48) by $-r_\mu\bar{c}_{66}c_{66}(c_{11}-X)[\bar{\alpha};\bar{\beta}]^2/2$ and taking into account (49), this equation becomes

$$
\begin{aligned}
A(\eta,\bar{\eta})\frac{\mathrm{sh}^2\big[\frac{\varepsilon(\bar{b}_1+\bar{b}_2)}{2}\big]}{(\bar{b}_1+\bar{b}_2)^2} - A(\eta,-\bar{\eta})\frac{\mathrm{sh}^2\big[\frac{\varepsilon(\bar{b}_1-\bar{b}_2)}{2}\big]}{(\bar{b}_1-\bar{b}_2)^2} + B(\eta,\bar{\eta})\frac{\mathrm{sh}[\varepsilon(\bar{b}_1+\bar{b}_2)]}{\bar{b}_1+\bar{b}_2} \\
- B(\eta,-\bar{\eta})\frac{\mathrm{sh}[\varepsilon(\bar{b}_1-\bar{b}_2)]}{\bar{b}_1-\bar{b}_2} + C(\eta,\bar{\eta}) = 0, \qquad (51)
\end{aligned}
$$

where

$$
\begin{aligned}
A(\eta,\bar{\eta}) &= 2\frac{\bar{f}(\bar{\eta})}{1-\bar{\eta}^2}\Big\{(1+\eta e_2^{-1/2})\frac{\bar{f}(\bar{\eta})}{1-\bar{\eta}^2}+2r(1-e_3e_2^{-1/2}\eta)(1-\bar{e}_3\bar{e}_2^{*-1/2}\bar{\eta}) \\
&\qquad + r^2(1+\bar{\eta}\bar{e}_2^{*-1/2})\frac{f(\eta)}{1-\eta^2}\Big\}, \\
B(\eta,\bar{\eta}) &= \frac{r\bar{f}(\bar{\eta})}{1-\bar{\eta}^2}(b_1+b_2)[e_2^{1/2}\eta+\bar{e}_2^{*1/2}\bar{\eta}], \qquad (52)\\
C(\eta,\bar{\eta}) &= 2r^2\,\bar{e}_2^{*1/2}\,\bar{\eta}\,\frac{f(\eta)}{1-\eta^2}\ ,
\end{aligned}
$$

with $r = r_\mu^{-1}$.

By comparing Eq. (51) with the secular equation derived by Sotiropolous, Eq (16) in Ref. [14], we discover some misprints in this secular equation,. In particular

(i) In the expression for $A(\eta,\eta^*)$ (Eq. (17) in Ref. [14]): $2r(1-c_2c_3^{-1/2})(1-c_2^*c_3^{*-1/2})$ must be replaced by $2r(1-c_2c_3^{-1/2}\eta)(1-c_2^*c_3^{*-1/2}\eta^*)$.

(ii) In the expression for $C(\eta,\eta^*)$ (Eq. (19) in Ref. [14]): $c_3^{*-1/2}$ must be replaced by $c_3^{*1/2}$.

The same misprints have been occurred in the the secular equation (8) in Ref. [17] obtained by Sotiropolous and Tougelidis.

3.4. Isotropic case

When the layer and the substrate are both isotropic

$$c_{11}=c_{22}=\lambda+2\mu,\ c_{12}=\lambda,\ c_{66}=\mu,\ \bar{c}_{11}=\bar{c}_{22}=\bar{\lambda}+2\bar{\mu},\ \bar{c}_{12}=\bar{\lambda},\ \bar{c}_{66}=\bar{\mu}. \tag{53}$$

With the help of (53) and Eqs. (7), (10), (26) and (33), one can see that

$$\begin{gathered}
b_1=\sqrt{1-\gamma x},\ b_2=\sqrt{1-x},\ \alpha_1=b_1,\ \alpha_2=1/b_2\ ,\\
\bar{b}_1=\sqrt{1-\bar{\gamma}\bar{x}},\ \bar{b}_2=\sqrt{1-\bar{x}},\ \bar{\alpha}_1=-\bar{b}_1,\ \bar{\alpha}_2=-1/\bar{b}_2\ ,\\
\beta_1=-2\rho\, c_2^2\, b_1,\ \beta_2=-\rho\, c_2^2\,(2-x)/b_2,\ \gamma_1=-\rho\, c_2^2\,(2-x),\ \gamma_2=-2\,\rho\, c_2^2\ ,\\
\bar{\beta}_1=2\bar{\rho}\,\bar{c}_2^2\,\bar{b}_1,\ \bar{\beta}_2=\bar{\rho}\,\bar{c}_2^2\,(2-\bar{x})/\bar{b}_2,\ \bar{\gamma}_1=-\bar{\rho}\,\bar{c}_2^2\,(2-\bar{x}),\ \bar{\gamma}_2=-2\,\bar{\rho}\,\bar{c}_2^2\ ,
\end{gathered}\tag{54}$$

where

$$\begin{aligned}
&x=c^2/c_2^2, \quad c_2=\sqrt{\mu/\rho}, \quad \gamma=\mu/(\lambda+2\mu),\\
&\bar{x}=c^2/\bar{c}_2^2, \quad ,\ \bar{c}_2=\sqrt{\bar{\mu}/\bar{\rho}}, \quad \bar{\gamma}=\bar{\mu}/(\bar{\lambda}+2\bar{\mu}).
\end{aligned}\tag{55}$$

Introducing (54) into (41) we obtain the explicit secular equation for the isotropic case, namely

$$A_0+B_0\mathrm{ch}\varepsilon_1\mathrm{ch}\varepsilon_2+C_0\mathrm{sh}\varepsilon_1\mathrm{sh}\varepsilon_2+D_0\mathrm{ch}\varepsilon_1\mathrm{sh}\varepsilon_2+E_0\mathrm{sh}\varepsilon_1\mathrm{ch}\varepsilon_2=0, \tag{56}$$

in which A_0, B_0, C_0, D_0 and E_0 are given by

$$\begin{aligned}
A_0 =&\, 4\bar{b}_1\bar{b}_2(2-\bar{x})\Big\{2(2-\bar{x})(b_1b_2-1)+\big[4b_1b_2-(2-x)^2\big]r_\mu^{-2}\\
&-(4-\bar{x})(2b_1b_2+x-2)r_\mu^{-1}\Big\},\\
B_0 =&\, -A_0-\bar{b}_1\bar{b}_2\bar{x}^2\big[4b_1b_2-(2-x)^2\big]r_\mu^{-2}\ ,\\
C_0 =&\, 4\bar{b}_1^2\bar{b}_2^2\Big\{4b_1b_2(1-r_\mu^{-1})^2-\big[2-(2-x)r_\mu^{-1}\big]^2\Big\}+(2-\bar{x})^2\Big\{(2-\bar{x})^2(b_1b_2-1)\\
&-2(2-\bar{x})(2b_1b_2+x-2)r_\mu^{-1}+\big[4b_1b_2-(2-x)^2\big]r_\mu^{-2}\Big\},\\
D_0 =&\, \bar{b}_1\bar{x}x\big[b_2(2-\bar{x})^2-4b_1\bar{b}_2^2\big]r_\mu^{-1},\ E_0=\bar{b}_2\bar{x}x\big[b_1(2-\bar{x})^2-4b_2\bar{b}_1^2\big]r_\mu^{-1}\ ,
\end{aligned}\tag{57}$$

where $r_\mu=\mu/\bar{\mu}$, $r_v=c_2/\bar{c}_2$ and $\bar{x}=r_v^2x$. It is clear that for this isotropic case, the squared dimensionless velocity of Rayleigh waves x depends on five dimensionless parameters, say γ, $\bar{\gamma}$, r_μ, r_v and ε.

By multiplying two sides of Eq. (56) by $k^8/(-\bar{b}_1\bar{b}_2)$ we arrive immediately at the well-known secular equation of Rayleigh waves for the isotropic case, Eq. (3.113), p.117 in Ref. [13], that is derived by Ben-Menahem & Singh and is written in other notations.

4. CONCLUSIONS

This paper introduces a technique by which the transfer matrix in explicit form of an orthotropic layer can be easily obtained. This transfer matrix is applicable for both the wave propagation problem and the reflection/transmission problem. The obtained transfer matrix is employed to derive the explicit secular equations of Rayleigh waves propagating in an orthotropic half-space coated by an orthotropic layer of arbitrary thickness. The obtained secular equation recovers the one for the isotropic case as a special case. The converting of the obtained secular equation to the secular equation (16) in Ref. [14] reveals some misprints of the latter.

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