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EXTENSION OF DUAL EQUIVALENT LINEARIZATION TECHNIQUE TO FLUTTER ANALYSIS OF TWO DIMENSIONAL NONLINEAR AIRFOILS

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Abstract. This paper extends the dual equivalent linearization technique (ELT) to obtain flutter speeds of a two-dimensional airfoil with nonlinear stiffness and damping in pitch degree of freedom. Although the use of dual ELT has been investigated in some previous papers, this paper presents an extension of dual ELT using the global-local approach, in which the local equivalent linearization coefficients are averaged in the global sense. The numerical calculation shows that the extended dual ELT gives more accurate flutter speeds in comparison with the ones of classical ELT.

Keywords: Airfoil flutter, dual equivalent linearization technique, global-local approach, limit cycle oscillation, nonlinearity.

1. INTRODUCTION

The most dramatic physical phenomenon in the field of aeroelasticity is flutter, a dynamic instability which often leads to catastrophic structural failure. Classical theories of linear flutter have been presented for a long time [1, 2]. However, the real system exhibits the nonlinear behavior due to the control mechanisms or the connecting parts between wing, pylon, engine, external stores or the large deflection. Nonlinear airfoil flutter is a typical self-excited vibration with rich nonlinear dynamical behaviors, such as limit cycle oscillation (LCO), bifurcation and chaos [3]. Nonlinear aeroelasticity has been a subject of high interest the literature is now extensive [4, 5], in which many issues are still under active investigation. In the context of nonlinear aeroelasticity, a LCO is one of the simplest dynamic bifurcations but is a good general description for many nonlinear aeroelastic behaviors. The LCO may occur once the dynamic stability (flutter) boundary has been exceeded. The LCO of the airfoils with stiffness nonlinearities has been investigated by several methods such as harmonic balance method [6, 7], center manifold theory [8], point transformation method [9], perturbation-incremental method [10], precise integration method [11], numerical methods [12,13] and equivalent linearization technique (ELT) [14,15].

The ELT was widely applied to various nonlinear vibration problems because of its simplicity and effectiveness. The approximate solution obtained by the ELT gives some clear physical explanation of the nonlinear phenomenon. The most important procedure of ELT is to derive an equivalent linear system based on some certain equivalent conditions. In the field of dynamical stochastic nonlinear system, the ELT has a long history of development and a numerous versions of ELT have been proposed [16]. Recently, in a series of papers, Anh et al. [17–21] proposed and proved the effectiveness of a so-called dual ELT in nonlinear stochastic systems. The dual approach has been also successfully applied to the problem of tuned mass damper design [21–23]. In [20, 23], the global-local approach applied to the dual criterion has been presented. In this paper, using the global-local approach, we extend the dual ELT by averaging the equivalent linearization coefficients and apply successfully the extended ELT to the flutter LCO problem.

2. EQUATIONS OF MOTION

Fig. 1 shows a classical spring-supported airfoil section. The model represents an airfoil with a single bending and a single torsional mode. This two-dimensional model is a typical section of a three-dimensional wing, where the spring constants are adjusted to match the actual uncoupled free vibration frequencies of the wing, while the mass and geometric properties are taken as those of a typical section. The two-dimensional model can give the approximated critical flutter speed of the actual wing with the location of the typical section is generally taken in the neighborhood of 0.7 span from the root [2].

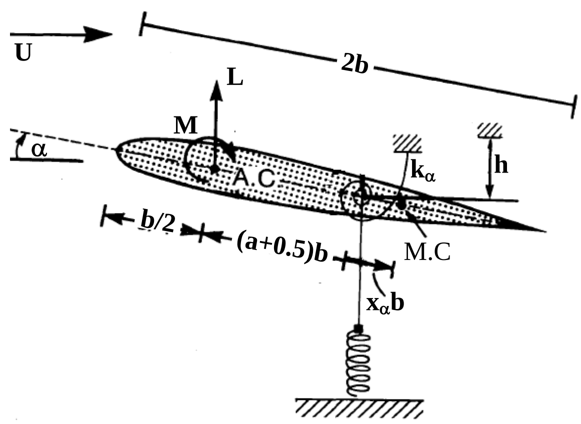

Fig. 1. Airfoil section supported by vertical and torsional springs

As shown in Fig. 1, the airfoil semi-chord is denoted as b. The elastic axis of the model is located at a distance ab from the mid-chord. The mass center (M.C) is located at a distance $x_\alpha b$ from the elastic axis. The plunge deflection (vertical displacement at the elastic axis) h is positive downward. The pitch angle (angle of twist) α is positive nose-up about the elastic axis. The equation of motion of the aeroelastic model has the form [24–28]

$$\begin{bmatrix} m_T & m_W x_\alpha b \\ m_W x_\alpha b & I_\alpha \end{bmatrix} \begin{bmatrix} \ddot{h} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\alpha \end{bmatrix} \begin{bmatrix} \dot{h} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\alpha \end{bmatrix} \begin{bmatrix} h \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ g\left(\alpha, \dot{\alpha}\right) \end{bmatrix} = \begin{bmatrix} -L \\ M \end{bmatrix}, \quad (1)$$

where m_w is the mass of the wing, m_T is the total mass, I_α is mass moment of inertia, k_h and k_α are the linear plunge and pitch stiffness, c_h and c_α are the linear damping coefficients, $g\left(\alpha, \dot{\alpha}\right)$ is a general nonlinear function of the pitch angle α and its derivative $\dot{\alpha}$. The aerodynamic moment M and lift L are assumed having the quasi-steady form

$$\begin{aligned} L &= \rho U^2 b s c_{l\alpha}\left(\alpha + \frac{\dot{h}}{U} + \left(\frac{1}{2} - a\right) b \frac{\dot{\alpha}}{U}\right), \\ M &= \rho U^2 b^2 s c_{m\alpha}\left(\alpha + \frac{\dot{h}}{U} + \left(\frac{1}{2} - a\right) b \frac{\dot{\alpha}}{U}\right), \end{aligned} \quad (2)$$

where U is the free stream velocity, ρ is the air density, s is the wing section span, $c_{l\alpha}$ and $c_{m\alpha}$ are the aerodynamic lift and moment coefficients per angle of attack. The equation of motion (1) can be transformed to the following state-space form

$$\begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ \ddot{h} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2\left(U\right) & -c_1\left(U\right) & -c_2\left(U\right) \\ -k_3 & -k_4\left(U\right) & -c_3\left(U\right) & -c_4\left(U\right) \end{bmatrix} \begin{bmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ p_1 \\ p_2 \end{bmatrix} g\left(\alpha, \dot{\alpha}\right), \quad (3)$$

where the coefficients are defined as

$$d = m_T I_\alpha - \left(m_W x_\alpha b\right)^2, k_1 = \frac{k_h I_\alpha}{d}, k_3 = \frac{-m_W x_\alpha b k_h}{d}, p_1 = \frac{-b m_W x_\alpha}{d}, p_2 = \frac{m_T}{d},$$

$$k_2 = \frac{\rho U^2 b s}{d}\left(m_W x_\alpha b^2 c_{m\alpha} + I_\alpha c_{l\alpha}\right), k_4 = \frac{-\rho U^2 b^2 s}{d}\left(m_T c_{m\alpha} + m_W x_\alpha c_{l\alpha}\right),$$

$$\begin{aligned} c_1 &= \frac{\rho b s\left(m_W x_\alpha b^2 c_{m\alpha} + c_{l\alpha} I_\alpha\right) U + c_h I_\alpha}{d}, \\ c_2 &= -\frac{\rho U b^2 s\left(m_W x_\alpha b^2 c_{m\alpha} + I_\alpha c_{l\alpha}\right)\left(a - \frac{1}{2}\right) + b m_W x_\alpha c_\alpha}{d}, \\ c_3 &= -\frac{\rho U b^2 s\left(m_T c_{m\alpha} + m_W x_\alpha c_{l\alpha}\right) + b m_W x_\alpha c_h}{d}, \\ c_4 &= \frac{\rho U b^3 s\left(m_T c_{m\alpha} + m_W x_\alpha c_{l\alpha}\right)\left(a - \frac{1}{2}\right) + m_T c_\alpha}{d}. \end{aligned}$$

3. EQUIVALENT LINEARIZATION TECHNIQUES

3.1. Classical ELT

Consider the following flutter LCO response

$$\alpha = A\sin\varphi, \quad \dot{\alpha} = B\cos\varphi, \quad (4)$$

where A and B respectively are the amplitudes of the pitch angle and its derivative, φ=ωt where ω is the LCO frequency. The nonlinear function g is linearized as

$$g\left(\alpha, \dot{\alpha}\right) = k_e\alpha + c_e\dot{\alpha} \quad (5)$$

where the equivalent stiffness k_e and equivalent damping c_e are found by a certain optimal criterion. There are many criteria for this purpose but the most extensively used criterion is the mean square error criterion which requires the following error be minimum

$$\int_0^{2\pi} \left(g\left(\alpha,\dot{\alpha}\right) - k_e\alpha - c_e\dot{\alpha}\right)^2 d\varphi \to \min_{k_e,c_e}. \tag{6}$$

The minimum conditions

$$\frac{\partial \int_0^{2\pi} \left(g\left(\alpha,\dot{\alpha}\right) - k_e\alpha - c_e\dot{\alpha}\right)^2 d\varphi}{\partial k_e} = 0, \quad \frac{\partial \int_0^{2\pi} \left(g\left(\alpha,\dot{\alpha}\right) - k_e\alpha - c_e\dot{\alpha}\right)^2 d\varphi}{\partial c_e} = 0,$$

yield

$$k_e = \frac{\int_0^{2\pi} g\alpha d\varphi}{\int_0^{2\pi} \alpha^2 d\varphi}, \quad c_e = \frac{\int_0^{2\pi} g\dot{\alpha} d\varphi}{\int_0^{2\pi} \dot{\alpha}^2 d\varphi}, \tag{7}$$

using (4) in (7) gives

$$k_e = \frac{1}{\pi A}\int_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\sin\varphi d\varphi, \quad c_e = \frac{1}{\pi B}\int_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\cos\varphi d\varphi \tag{8}$$

for a certain type of nonlinear function , the equivalent stiffness k_e and damping c_e in (8) are the functions of the pitch amplitude A and pitch velocity amplitude B.

3.2. Dual ELT and its extension

In [17–20], Anh et al. proposed a so-called dual criterion to obtain a new ELT. This type of ELT has showed its effectiveness in analyzing a numerous class of nonlinear stochastic systems. In [18], the authors stated that the classical ELT is based on replacing the original nonlinear system by an equivalent linear one. The dual approach does not only consider this "forward" replacement but also the "backward" replacement, in which the obtained equivalent linear system is replaced by another nonlinear one that belongs to the same class as the original nonlinear system. In [19], the authors also proposed the weighted dual criterion, which is used in the following dual criterion of this paper

$$(1-p)\int_0^{2\pi} \left(g\left(\alpha,\dot{\alpha}\right) - k_e\alpha - c_e\dot{\alpha}\right)^2 d\varphi + p\int_0^{2\pi} \left(\lambda g\left(\alpha,\dot{\alpha}\right) - k_e\alpha - c_e\dot{\alpha}\right)^2 d\varphi \to \min_{k_e,c_e,\lambda} \tag{9}$$

where the first term describes the conventional replacement and second term is its dual replacement, p is a given weighting coefficient. If $p = 0$ we obtain the classical ELT. If

$p = 1/2$ we obtain the dual ELT presented in [18]. Three minimum conditions of (9)

$$\frac{\partial\left((1-p)\int\limits_0^{2\pi}\left(g\left(\alpha,\dot{\alpha}\right)-k_e\alpha-c_e\dot{\alpha}\right)^2 d\varphi+p\int\limits_0^{2\pi}\left(\lambda g\left(\alpha,\dot{\alpha}\right)-k_e\alpha-c_e\dot{\alpha}\right)^2 d\varphi\right)}{\partial k_e}=0, \quad (10)$$

$$\frac{\partial\left((1-p)\int\limits_0^{2\pi}\left(g\left(\alpha,\dot{\alpha}\right)-k_e\alpha-c_e\dot{\alpha}\right)^2 d\varphi+p\int\limits_0^{2\pi}\left(\lambda g\left(\alpha,\dot{\alpha}\right)-k_e\alpha-c_e\dot{\alpha}\right)^2 d\varphi\right)}{\partial c_e}=0, \quad (11)$$

$$\frac{\partial\left((1-p)\int\limits_0^{2\pi}\left(g\left(\alpha,\dot{\alpha}\right)-k_e\alpha-c_e\dot{\alpha}\right)^2 d\varphi+p\int\limits_0^{2\pi}\left(\lambda g\left(\alpha,\dot{\alpha}\right)-k_e\alpha-c_e\dot{\alpha}\right)^2 d\varphi\right)}{\partial \lambda}=0, \quad (12)$$

yield

$$k_e=\frac{1-p}{1-p\mu}\frac{\int\limits_0^{2\pi}g\alpha d\varphi}{\int\limits_0^{2\pi}\alpha^2 d\varphi}, \quad c_e=\frac{1-p}{1-p\mu}\frac{\int\limits_0^{2\pi}g\dot{\alpha} d\varphi}{\int\limits_0^{2\pi}\dot{\alpha}^2 d\varphi}, \quad (13)$$

where p as stated above is a given value between 0 and 1/2 $\left(0\le p\le\frac{1}{2}\right)$, μ is a notation to simplify the formula and is given by

$$\mu=\frac{\left(\int\limits_0^{2\pi}g\alpha d\varphi\right)^2}{\left(\int\limits_0^{2\pi}g^2 d\varphi\right)\left(\int\limits_0^{2\pi}\alpha^2 d\varphi\right)}+\frac{\left(\int\limits_0^{2\pi}g\dot{\alpha} d\varphi\right)^2}{\left(\int\limits_0^{2\pi}g^2 d\varphi\right)\left(\int\limits_0^{2\pi}\dot{\alpha}^2 d\varphi\right)}. \quad (14)$$

Using (4) in (13) and (14) gives

$$\begin{aligned} k_e&=\frac{1-p}{\pi A\left(1-p\mu\right)}\int\limits_0^{2\pi}g\left(A\sin\varphi,B\cos\varphi\right)\sin\varphi d\varphi,\\ c_e&=\frac{1-p}{\pi B\left(1-p\mu\right)}\int\limits_0^{2\pi}g\left(A\sin\varphi,B\cos\varphi\right)\cos\varphi d\varphi, \end{aligned} \quad (15)$$

$$\mu = \frac{\left(\int\limits_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\sin\varphi d\varphi\right)^2 + \left(\int\limits_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\cos\varphi d\varphi\right)^2}{\pi\int\limits_0^{2\pi}\left(g\left(A\sin\varphi, B\cos\varphi\right)\right)^2 d\varphi}, \tag{16}$$

However, the equivalent stiffness and damping (15) obtained from the weighted dual criterion ELT (9) depend on the weighting coefficient p. In this sense, the equivalent stiffness k_e and damping c_e in (15) can be considered as local equivalent linearization coefficients. It allows the flexibility in varying the local parameter p. However, the main disadvantage of the local solution is that there is still no clear way to find the parameter p. Using the global-local approach presented in [20, 23], it is suggested that instead of finding a special value of p, one may consider its varying in the global domain of integration. Thus, the equivalent coefficients can be suggested as the mean values of all local equivalent coefficients as follows

$$\begin{aligned} k_{eg} &= 2\int\limits_0^{1/2} k_e dp = 2\int\limits_0^{1/2}\left[\frac{1-p}{\pi A\left(1-p\mu\right)}\int\limits_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\sin\varphi d\varphi\right]dp, \\ c_{eg} &= 2\int\limits_0^{1/2} c_e dp = 2\int\limits_0^{1/2}\left[\frac{1-p}{\pi B\left(1-p\mu\right)}\int\limits_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\cos\varphi d\varphi\right]dp, \end{aligned} \tag{17}$$

in which k_{eg} and c_{eg}, respectively, denote the equivalent stiffness and damping obtained by global-local approach. The integrals reduce to

$$\begin{aligned} k_{eg} &= \left(\frac{1}{\mu} + \frac{2\left(1-\mu\right)}{\mu^2}\ln\left(1-\frac{\mu}{2}\right)\right)\frac{\int\limits_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\sin\varphi d\varphi}{\pi A}, \\ c_{eg} &= \left(\frac{1}{\mu} + \frac{2\left(1-\mu\right)}{\mu^2}\ln\left(1-\frac{\mu}{2}\right)\right)\frac{\int\limits_0^{2\pi} g\left(A\sin\varphi, B\cos\varphi\right)\cos\varphi d\varphi}{\pi B}. \end{aligned} \tag{18}$$

In brief, the extended dual ELT gives the equivalent stiffness and damping as (18) where the notation μ is taken from (16).

4. FLUTTER SPEED PREDICTION AND STABILITY ANALYSIS

4.1. Flutter speed prediction

We will use the equivalent linear coefficients (8) and (18) for further flutter analysis. The linearized equation of (3) has form

$$\begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ \ddot{h} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 - p_1 k_e & -c_1 & -c_2 - p_1 c_e \\ -k_3 & -k_4 - p_2 k_e & -c_3 & -c_4 - p_2 c_e \end{bmatrix} \begin{bmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{bmatrix}. \tag{19}$$

The response of (19) is convergent when the flow velocity is smaller than a critical value (flutter speed). When the flow velocity is higher than flutter speed, the LCO will occur. We can assume the occurrence of following phenomenon. When the flow velocity is smaller than flutter speed, the linear system (19) has two pairs of conjugate poles with negative real parts, which means the convergence of the response. When the flow velocity reaches the flutter speed, a pair of conjugate poles becomes purely imaginary, which means the occurrence of LCO. Denote the poles of (19) at flutter speed as $\pm i\omega, -\zeta_1 \pm i\omega_1$, where i is the imaginary unit, ω is the LCO frequency as denoted above, ζ_1, ω_1 are positive real number. The characteristic polynomial of (19) will has form

$$\begin{aligned} P(\lambda) = \det &\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 - p_1 k_e & -c_1 & -c_2 - p_1 c_e \\ -k_3 & -k_4 - p_2 k_e & -c_3 & -c_4 - p_2 c_e \end{bmatrix} - \lambda I_4 \right) \\ &= (\lambda - i\omega)(\lambda + i\omega)(\lambda + \zeta_1 - i\omega_1)(\lambda + \zeta_1 + i\omega_1), \end{aligned} \tag{20}$$

in which the variable in the brackets are omitted for simplicity, I_4 is the 4×4 identity matrix. Equating the coefficients of the polynomials in (20) gives four algebraic equations to solve four variables ω, ζ_1, ω_1 and U. The solutions can be obtained in the following form

$$\zeta_1 = \frac{c_1 + c_4 + p_2 c_e}{2} \tag{21}$$

$$\omega_1^2 = p_2 k_e - (c_2 + p_1 c_e) c_3 + k_1 - \omega^2 + k_4 + c_1 (c_4 + p_2 c_e) - \zeta_1^2 \tag{22}$$

$$\omega^2 = \frac{-c_3 k_2 + k_1 (c_4 + p_2 c_e) - k_3 (c_2 + p_1 c_e) + k_4 c_1 + p_2 k_e c_1 - c_3 p_1 k_e}{c_1 + c_4 + p_2 c_e} \tag{23}$$

$$\begin{aligned} &k_1 k_4 - k_3 k_2 + k_1 p_2 k_e - k_3 p_1 k_e - \omega^2 p_2 k_e + \omega^2 (c_2 + p_1 c_e) c_3 \\ &\quad - \omega^2 k_1 + \omega^4 - \omega^2 k_4 - \omega^2 c_1 (c_4 + p_2 c_e) = 0 \end{aligned} \tag{24}$$

By substituting ω from (23) to (24) and by noting that

$$\omega = \frac{B}{A},$$

Eqs. (23) and (24) change to

$$\frac{B^2}{A^2} = \frac{-c_3 k_2 + k_1 (c_4 + p_2 c_e) - k_3 (c_2 + p_1 c_e) + k_4 c_1 + p_2 k_e c_1 - c_3 p_1 k_e}{c_1 + c_4 + p_2 c_e}, \tag{25}$$

$$\begin{aligned} &(k_1 + k_4 + c_1 c_4 - c_2 c_3 + p_2 k_e + p_2 c_e c_1 - p_1 c_e c_3)(c_1 + c_4 + p_2 c_e) \times \\ &(k_1 c_4 - c_3 k_2 - k_3 c_2 + k_4 c_1 + p_2 k_1 c_e + p_2 k_e c_1 - p_1 k_3 c_e - p_1 k_e c_3) \\ &-(k_1 c_4 - c_3 k_2 - k_3 c_2 + k_4 c_1 + p_2 k_1 c_e - p_1 k_3 c_e + p_2 k_e c_1 - c_3 p_1 k_e)^2 = \\ &(k_1 k_4 - k_3 k_2 + k_1 p_2 k_\alpha - k_3 p_1 k_\alpha)(c_1 + c_4 + p_2 c_e)^2 \end{aligned} \tag{26}$$

By noting that the equivalent stiffness k_e and equivalent damping c_e are the functions of A and B (as seen in (8) or (18)) and k_2, k_4, c_i ($i = 1, \dots, 4$) are the polynomial functions of flow velocity U, Eqs. (25) and (26) give the relation between the flutter speed and the pitch amplitude A or B. Moreover, Eq. (26) is a polynomial equation of U. After some manipulation, Eq. (26) can reduce to a quintic equation of U, whose lowest positive root is the flutter speed.

The relation between U and A can be obtained by the following procedure.

- Given a certain value to A.

- By solving the quintic equation (26), we can express U as the function of B. It is noted that the quintic equation is solved very fast and efficiently by the "roots" function in MATLAB.

- Substitute U as a function of B to (25), we obtain a nonlinear scalar equation with respect to B. It is noted that the nonlinear scalar equation is solved very fast and efficiently by the "fzero" function in MATLAB.

- After solving the nonlinear scalar equation to obtain B, U is determined because it is a function of B as stated above.

- At last, for a certain value of A, we can obtain the corresponding value of U. Then the relation between U and A can be drawn.

The relation between U and B can be obtained in the similar manner by changing the roles of A and B.

4.2. Stability condition

For a flutter speed, there can be one or more corresponding pitch amplitude. However, only the stable amplitude can occur in practice. The following condition [29] may be used to determine the stability of LCO in the presence of amplitude perturbations. For a limit cycle with amplitude A, the linearized system has two eigenvalues with zero real parts (as shown in (20)). Small perturbations in the limit cycle amplitude make changes in these two eigenvalues. Denote the change of the real part as $\Delta\sigma$. If a positive amplitude perturbation ($\Delta A > 0$) results in the negative real parts ($\Delta\sigma < 0$) of the aforementioned eigenvalues, the LCO is stable because the energy is dissipated until the amplitude decays to its unperturbed value. Similarly, a negative amplitude perturbation ($\Delta A < 0$) results in the positive real parts ($\Delta\sigma > 0$) of the eigenvalues also forms a stable LCO because the amplitude grows until the unperturbed LCO is again attained. In brief, the condition

$$\Delta\sigma.\Delta A < 0, \tag{27}$$

defines a stable LCO.

In calculation, the stability condition is checked by the following procedure. With a certain value of the pitch amplitude A, the flutter speed is obtained by the procedure presented in section 3.1. Then the flow velocity is fixed at the flutter speed while the pitch amplitude A is given a small perturbation ΔA. The amplitude of pitch velocity B is obtained by scalar equation . . . Then the new roots of the characteristic polynomial $P(\lambda)$ in (20) are obtained. Two purely imaginary roots now become two conjugate roots with nonzero real part, denoted as $\Delta\sigma$. If two perturbations $\Delta\sigma$ and ΔA satisfy the condition (27), the corresponding state of flutter is stable.

5. NUMERICAL SIMULATION

In this section, two numerical examples presented in [24] are performed to verify the proposed ELTs. In [24], the numerical examples consider the leading-edge and/or trailing-edge control surfaces. However, because the control problem is beyond the scope of this paper, only the open loop systems are analyzed. The nonlinear function $g(\alpha, \dot{\alpha})$ in both examples is expressed in polynomial form up to 5$^{\text{th}}$ order as follows

$$g = k_{\alpha 2}\alpha^2 + k_{\alpha 3}\alpha^3 + k_{\alpha 4}\alpha^4 + k_{\alpha 5}\alpha^5. \tag{28}$$

Substituting the nonlinear form (28) into the formula (16) of μ, and the equivalent stiffnesses (8) or (17) give the following formulas

$$\mu = \frac{\left(\dfrac{3A^2 k_{\alpha 3}}{4} + \dfrac{5A^4 k_{\alpha 5}}{8}\right)^2}{k_{\alpha 5}^2 \dfrac{63A^8}{128} + \left(2k_{\alpha 3}k_{\alpha 5} + k_{\alpha 4}^2\right)\dfrac{35A^6}{64} + \left(2k_{\alpha 2}k_{\alpha 4} + k_{\alpha 3}^2\right)\dfrac{5A^4}{8} + k_{\alpha 2}^2 \dfrac{3A^2}{4}}$$

$$k_e = \gamma\left(\frac{3A^2 k_{\alpha 3}}{4} + \frac{5A^4 k_{\alpha 5}}{8}\right)$$

where γ is the coefficient depending on the ELT used as follows

* For the classical ELT: $\gamma = 1$

* For the extended dual ELT: $\gamma = \dfrac{1}{\mu} + \dfrac{2(1-\mu)}{\mu^2}\ln\left(1 - \dfrac{\mu}{2}\right)$

5.1. Example 1

The numerical values of parameters of Example 1 are shown in Tab. 1.

Table 1. Numerical values of parameters of Example 1

A	b (m)	m_T (kg)	m_W (kg)	I_α (kg.m^2)	x_α	k_h (N/m)
−0.6847	0.135	12.387	2.049	0.0558	0.3314	2884.4
c_h (kg/s)	$c_{l\alpha}$	$c_{m\alpha}$	s (m)	ρ (kg/m^3)	c_α (kg.m^2/s)	k_α (N.m)
27.43	6.38	−1.16	0.6	1.225	0.036	6.833
$k_{\alpha 2}$ (N.m)	$k_{\alpha 3}$ (N.m)	$k_{\alpha 4}$ (N.m)	$k_{\alpha 5}$ (N.m)			
9.967	667.685	26.569	−5087.931			

The numerical solutions are obtained by the ode45 function in MATLAB. The following process is carried out to plot the numerical relation between the flutter speed and the pitch amplitude:

- First, an initial condition of pitch angle α is set. Other initial conditions $(h(0), \dot{h}(0), \dot{\alpha}(0))$ are set to zeros.

- Gradually increase the flow velocity U. For each flow velocity, the nonlinear differential equations (19) are solved from 0s to 120s. If the response converges, the flow velocity is smaller than the flutter speed. When the response starts to make a LCO, the flow velocity is the flutter speed and the amplitude of LCO is recorded.

- Increase the initial pitch angle $\alpha(0)$ and repeat the step 2. The initial pitch angle is increased until the pitch amplitude has no significant changes.

The relation between the initial pitch angle and the flutter pitch amplitude (calculated numerically) is shown in Fig. 2. Fig. 3 shows the flutter boundary obtained by numerical calculations and two ELTs. Tab. 2 shows the comparisons of some flutter speeds.

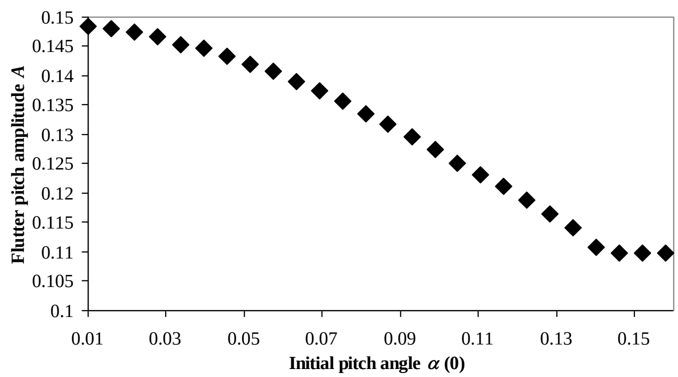

Fig. 2. Relation between initial condition and flutter pitch amplitude in Example 1

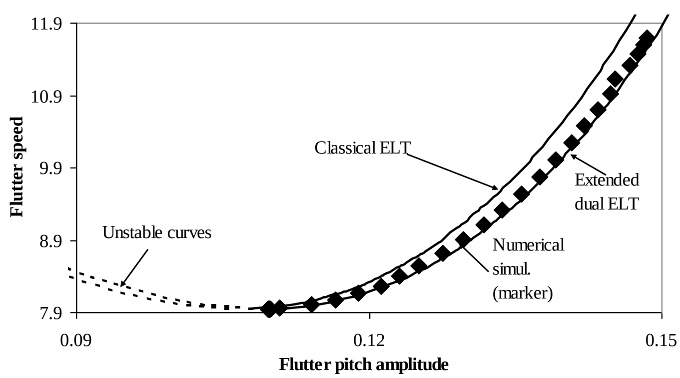

Fig. 3. Relation between flutter speed and flutter pitch amplitude in Example 1

5.2. Example 2

The numerical values of parameters of Example 2 are shown in Tab. 3.

The relation between the initial pitch angle and the flutter pitch amplitude (calculated numerically) is shown in Fig. 4. Fig. 5 shows the flutter boundary obtained by numerical calculations and two ELTs. Tab. 4 shows the comparisons of some flutter speeds.

Table 2. Comparisons of flutter speeds in Example 1

	Response at large amplitude (large nonlinearity)		Minimum flutter speed (m/s)
	Flutter pitch amplitude (rad)	Flutter speed (m/s)	
Numerical calculation (previous paper [24])	0.1485	11.6	7.9
Numerical calculation (this paper)		11.69	7.94
Classical ELT		12.2744 (5% larger than numerical value)	7.9484
Extended dual ELT		11.5305 (1.36% smaller than numerical value)	7.9472

Table 3. Numerical values of parameters of Example 2

A	b (m)	m_T (kg)	m_W (kg)	I_α (kg.m^2)	x_α	k_h (N/m)
-0.6719	0.1905	15.57	5.23	0.1419	0.5721	2844
c_h (kg/s)	$c_{l\alpha}$	$c_{m\alpha}$	s (m)	ρ (kg/m^3)	c_α (kg.m^2/s)	k_α (N.m)
27.43	6.757	-1.162	0.5945	1.225	0.0184	12.77
$k_{\alpha 2}$ (N.m)	$k_{\alpha 3}$ (N.m)	$k_{\alpha 4}$ (N.m)	$k_{\alpha 5}$ (N.m)			
53.47	1003	0	0			

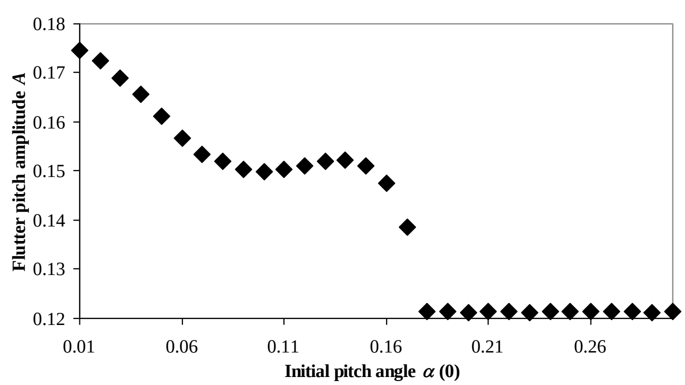

Fig. 4. Relation between initial condition and flutter pitch amplitude in Example 2

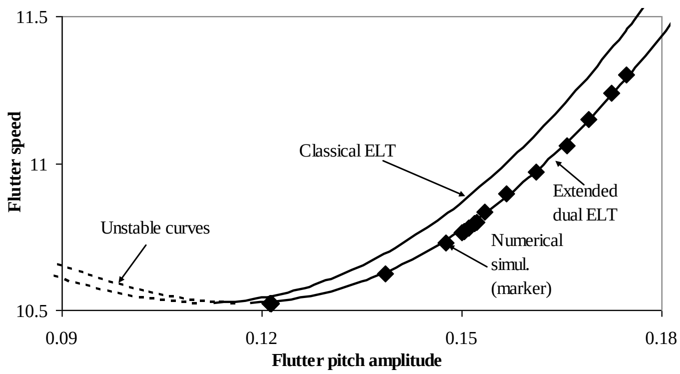

Fig. 5. Relation between flutter speed and flutter pitch amplitude in Example 2

Table 4. Comparisons of flutter speeds in Example 2

<table>
<tr><th rowspan="2"></th><th colspan="2">Response at large amplitude
(large nonlinearity)</th><th rowspan="2">Minimum flutter speed (m/s)</th></tr>
<tr><th>Flutter pitch amplitude (rad)</th><th>Flutter speed (m/s)</th></tr>
<tr><td>Numerical calculation (previous paper [24])</td><td rowspan="4">0.1746</td><td>11.4</td><td>10.6</td></tr>
<tr><td>Numerical calculation (this paper)</td><td>11.3</td><td>10.525</td></tr>
<tr><td>Classical ELT</td><td>11.4481 (1.31% larger than numerical value)</td><td>10.5249</td></tr>
<tr><td>Extended dual ELT</td><td>11.2852 (0.13% smaller than numerical value)</td><td>10.5248</td></tr>
</table>

5.3. Discussions on results

- As seen in Tab. 2 and Tab. 4, the numerical results in this paper are in good agreement with those in previous paper [24]. This guarantees the reliability of the numerical procedure in this paper.

- In Figs. 2 and 4, there is a jump phenomenon in the relation between the initial pitch angle and flutter pitch amplitude. When the initial angle is small, the flutter pitch amplitude suddenly jumps to a large value. Increase the initial angle more, the flutter pitch amplitude decreases and tends to a constant value. The jump phenomenon clearly shows the nonlinear effect.

- In Figs. 3 and 5, all the methods show the existence of a minimum flutter speed. The comparisons in the last column of Tabs. 2 and 4 show that both ELTs can predict the minimum flutter speed quite well. It is noted that the ELTs can find the minimum flutter speed quite simply by solving the quintic equation (24).

- The proposed extended dual ELT based on weighted dual criterion and global-local approach shows its accuracy in the region of large nonlinearity (large amplitude). As shown in Tabs. 3 and 5, in Examples 1 and 2, the errors of the extended dual ELT are about 1/3 and 1/10 of the error of the classical ELT, respectively.

6. CONCLUSIONS

The equivalent linearization technique (ELT) in this paper is improved by a so-called weighted dual criterion and global-local approach. The dual criterion consists of "forward" and "backward" replacements with a local weighting coefficient. The obtained local equivalent coefficients are then averaged to be the global one. The proposed ELT then is applied to predict the flutter limit cycle oscillation of an airfoil section with nonlinear pitch stiffness and nonlinear pitch damping. Two numerical examples of the nonlinear polynomial pitch stiffness have been carried out. The results show that the proposed ELT can reduce the error to about 1/3 (in Example 1) and 1/10 (in Example 2) of the error given by the classical ELT. It is expected that the extended dual ELT can be used as an alternative effective tool for flutter analysis of more complicate nonlinear systems.

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