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# AN ANALYTIC HOMOGENIZATION MODEL IN TRACTION AND BENDING FOR ORTHOTROPIC COMPOSITE PLATES WITH THE TYPE OF DOUBLE CORRUGATED CARDBOARD

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**Abstract.** In this paper, an analytic homogenization model in traction and flexion for the double corrugated cardboard plates is presented. The proposed analytical homogenization model allows modelling the 3D double corrugated cardboard with a 2D homogenized plate. This model is essentially based on the theory of stratification and then improved by using the theory of sandwich. This model was validated by comparing the results of Abaqus-3D and H-2D model using ″user's subroutine″ ″UGENS″. The homogenization model can be used not only for corrugated cardboard plates, but also for naval and aeronautic composite structures.

*Keywords*: Analytical homogenization, corrugated cardboard, orthotropic plates.

## 1. INTRODUCTION

Nowadays, the corrugated cardboard is widely used in the packaging industry, such as corrugated cardboard boxes, insert cardboard sheet in pallet systems. It is essential to predict the mechanical behavior of these systems in order to use such materials effectively. The numerical modeling of this kind of orthotropic composite plates by shell elements is too tedious and time consuming. Many homogenization models were obtained by analytical, numerical and experimental methods [1–5]. By using some FE models and commercial FE software, the mechanical behaviors of corrugated cardboard were studied by other authors [6,7], but it just limited with the single corrugated cardboard.

The corrugated cardboard is produced by a converting process in which three or more layers are laminated together. The flat layers are called liners and the corrugated cores are referred to as flutes (Fig. 1). Corrugated cardboard is one of the most used packaging materials to make boxes or interlayers for goods transport. The manufacturing process gives three characteristic directions: the machine direction (MD), the cross direction (CD), and the thickness direction (ZD).

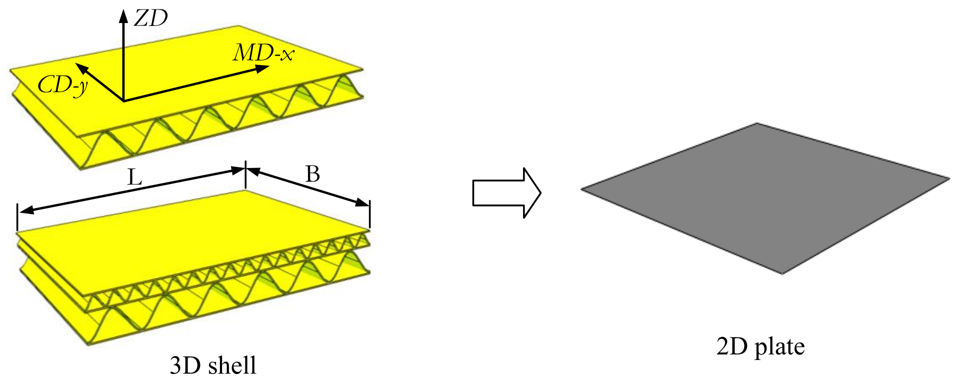


*Fig. 1.* Homogenization model

This paper presents an efficient homogenization model for the mechanical behavior of a corrugated cardboard composed of three or five layers (single or double flutes). The homogenization is carried out by calculating analytically the global rigidities of the corrugated cardboard and then this 3*D* structure is replaced by an equivalent homogenized 2D plate. The simulations in traction and flexion of Abaqus-3D and H-2D model of double corrugated cardboard will be studied in this article. This 2D homogenization model is very fast and has close results comparing to the 3D model using the Abaqus shell elements.

## 2. RECALL OF MINDLIN'S THEORY AND THEORY OF LAMINATED PLATES

For a model composite plate, the Mindlin theory is often used. It is assumed that a right segment and perpendicular to the mean surface remains straight but not perpendicular to the medium surface after deformation. This assumption allows to consider the transverse shear deformations. The membrane forces, bending moments and torsional and transverse shear forces are obtained by integration of the constraints on the thickness

$$\{N(x,y)\} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} dz, \tag{1}$$

$$\{M(x,y)\} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} dz, \tag{2}$$

$$\{T(x,y)\} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} dz. \tag{3}$$

If we consider a composite sheet consisting of several layers, the resulting forces defined above may be combined in layers

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}_k \left( \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_m + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) dz, \tag{4}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} z dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}_k \left( z \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_m + z^2 \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) dz, \tag{5}$$

$$\begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}_k \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} dz. \tag{6}$$

After the integration along the thickness, we obtain the overall stiffness matrix that links the generalized deformations with resultant forces

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ T_x \\ T_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & B_{33} & 0 & 0 \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & 0 & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 & D_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xm} \\ \varepsilon_{ym} \\ \gamma_{xym} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{7}$$

with

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} \left[ h^k - h^{k-1} \right] Q_{ij}^k = \sum_{k=1}^{n} Q_{ij}^k t^k, \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} \left[ \left( h^k \right)^2 - \left( h^{k-1} \right)^2 \right] Q_{ij}^k = \sum_{k=1}^{n} Q_{ij}^k t^k z^k, \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} \left[ \left( h^k \right)^3 - \left( h^{k-1} \right)^3 \right] Q_{ij}^k = \sum_{k=1}^{n} Q_{ij}^k \left[ t^k \left( z^k \right)^2 + \frac{\left( t^k \right)^3}{12} \right], \\ F_{ij} &= \sum_{k=1}^{n} \left[ h^k - h^{k-1} \right] C_{ij}^k = \sum_{k=1}^{n} C_{ij}^k t^k. \end{aligned} \tag{8}$$

The law of behavior above can be written in matrix form

$$\begin{bmatrix} \{N\} \\ \{M\} \\ \{T\} \end{bmatrix} = \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D] & [0] \\ [0] & [0] & [F] \end{bmatrix} \begin{bmatrix} \{\varepsilon_m\} \\ \{\kappa\} \\ \{\gamma_s\} \end{bmatrix}, \tag{9}$$

where $\{N\}$, $\{T\}$ and $\{M\}$ are the internal forces and moments; $[A]$, $[D]$, $[B]$ and $[F]$ are the stiffness matrices related to the membrane forces, the bending-torsion moments, the bending-torsion-membrane coupling effects and the transverse shear forces respectively;

$\{\varepsilon\}$ is the membrane strain vector, $\{\kappa\}$ is the curvature vector and $\{\gamma_s\}$ is the transverse shear strain vector.

## 3. CORRUGATED CARDBOARD HOMOGENIZED MODEL

A 3D geometrical modeling of the liners and the flutes of the corrugated cardboard is a very tedious and time-consuming task. In our homogenization model, a corrugated cardboard panel is replaced by a 2D plate. Instead of using a local constitutive law (relating the strains to the stresses) at each material point, the homogenization leads to global rigidities (relating the generalized strains to the resultant forces) for the equivalent homogeneous plate. The corrugated cardboard is more complex than a laminated plate because of the fluting cores and the cavities between the three liners. Consequently some global effective stiffnesses in the matrix (9) obtained by the theory of laminated plates should be modified [5,8].

Consider a double corrugated cardboard and using $a$, $b$, $c$, $d$, and $e$ to represent the lower liner, lower flute, intermediate liner, upper flute and the upper liner (Fig. 2). The geometry of each flute is defined by the following equations

$$\begin{cases} H^b(x) = \dfrac{h^b - t^b}{2} \sin\left(\dfrac{2\pi}{P^b} x\right) \\ \theta^b(x) = \tan^{-1}\left(\dfrac{dH^b(x)}{dx}\right) \end{cases} ; \quad \begin{cases} H^d(x) = \dfrac{h^d - t^d}{2} \sin\left(\dfrac{2\pi}{P^d} x\right) \\ \theta^d(x) = \tan^{-1}\left(\dfrac{dH^d(x)}{dx}\right) \end{cases} \tag{10}$$

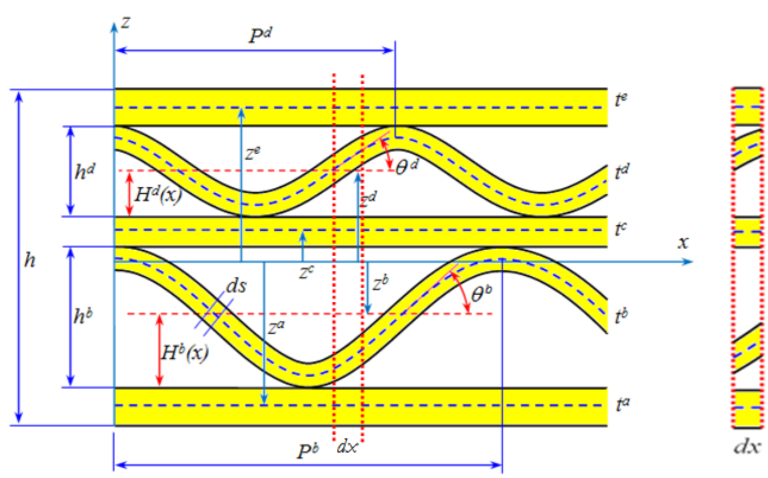


*Fig. 2.* Geometry of a double corrugated cardboard

To homogenize a panel corrugated double wall, we consider a representative volume element (RVE). This volume must be sufficiently small relative to the dimensions of the entire panel. Once the overall stiffness of each slice are obtained by integrating the thickness, the homogenization along $x$ is performed to calculate the average stiffness of all tranches over a period

$$[A] = \frac{1}{P}\int_0^P [A(x)]dx, [B] = \frac{1}{P}\int_0^P [B(x)]dx, [D] = \frac{1}{P}\int_0^P [D(x)]dx, [F] = \frac{1}{P}\int_0^P [F(x)]dx. \tag{11}$$

We note that, in Eq. (11), when the periods $P^d$ and $P^b$ of corrugated layers are different, it is necessary to take an RVE having the length $P$ that is the multiple of these periods.

### 3.1. Traction and bending stiffnesses related to $N_x, M_x, N_y, M_y$

Since the vertical position $(z)$ of a groove portion $(ds)$ is a function of $x$ and a thickness over its vertical section is a function of the angle of inclination of the groove $\theta_x$ (Fig. 2), Eq. (8) becomes

$$\begin{aligned}
A_{ij} &= Q^a_{ij}t^a + Q^b_{ij}\frac{t^b}{\cos\theta^b} + Q^c_{ij}t^c + Q^d_{ij}\frac{t^d}{\cos\theta^d} + Q^e_{ij}t^e, \quad t^b_v = \frac{t^b}{\cos\theta^b}, \quad t^d_v = \frac{t^d}{\cos\theta^d}, \\
B_{ij} &= Q^a_{ij}t^a z^a + Q^b_{ij}\frac{t^b}{\cos\theta^b}z^b + Q^c_{ij}t^c z^c + Q^d_{ij}\frac{t^d}{\cos\theta^d}z^d + Q^e_{ij}t^e z^e, \\
D_{ij} &= Q^a_{ij}\left[t^a(z^a)^2 + \frac{1}{12}(t^a)^3\right] + Q^b_{ij}\left[\frac{t^b}{\cos\theta^b}\left(z^b\right)^2 + \frac{1}{12}\left(\frac{t^b}{\cos\theta^b}\right)^3\right] + Q^c_{ij}\left[t^c(z^c)^2 + \frac{1}{12}(t^c)^3\right] \\
&\quad + Q^d_{ij}\left[\frac{t^d}{\cos\theta^d}\left(z^d\right)^2 + \frac{1}{12}\left(\frac{t^d}{\cos\theta^d}\right)^3\right] + Q^e_{ij}\left[t^e(z^e)^2 + \frac{1}{12}(t^e)^3\right],
\end{aligned} \tag{12}$$

with

$$\begin{aligned}
& h = t^a + h^b + t^c + h^d + t^e, \\
& z^a = -\frac{h}{2} + \frac{t^a}{2}, \quad z^e = \frac{h}{2} - \frac{t^e}{2}, \quad z^c = -\frac{h}{2} + t^a + h^b + \frac{t^c}{2}, \\
& z^b(x) = -\frac{h}{2} + t^a + \frac{h^b}{2} + \frac{1}{2}\left(h^b - t^b\right)\sin\left(\frac{2\pi}{P^b}x\right), \\
& \frac{dz^b}{dx} = \frac{\pi\left(h^b - t^b\right)}{P^b}\cos\left(\frac{2\pi}{P^b}x\right), \quad \theta^b(x) = \tan^{-1}\left(\frac{dz^b}{dx}\right), \\
& z^d(x) = \frac{h}{2} - t^e - \frac{h^d}{2} + \frac{1}{2}\left(h^d - t^d\right)\sin\left(\frac{2\pi}{P^d}x\right), \\
& \frac{dz^d}{dx} = \frac{\pi\left(h^d - t^d\right)}{P^d}\cos\left(\frac{2\pi}{P^d}x\right), \quad \theta^d(x) = \tan^{-1}\left(\frac{dz^d}{dx}\right).
\end{aligned}$$

For each of the two grooves, the homogenization on their period (along $x$) should be calculated numerically according to Eq. (11).

### 3.2. Implementation of the homogenized model in an FE analysis

For this problem, the corrugated cardboard is modelled by shell elements, the internal forces and moments are calculated incrementally. Their increments $dN$ and $dM$ have the following relationship with the increments of the membrane strains and bending-torsion curvature

$$\begin{bmatrix} \{dN\} \\ \{dM\} \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{bmatrix} \{d\varepsilon\} \\ \{d\kappa\} \end{bmatrix} = [H] \begin{bmatrix} \{d\varepsilon\} \\ \{d\kappa\} \end{bmatrix}, \tag{13}$$

where the stiffness matrix $[H]$ is determined by our homogenization model which is implemented into ABAQUS by programming a user subroutine UGENS.

In ABAQUS, the 4-nodal shell element S4R has 6 degrees of freedom per node (3 displacements and 3 rotations). In the UGENS subroutine, the generalized forces and strains are passed in using the arrays FORCE (6) and STRAN (6), and will be updated at the end of the subroutine; the known generalized strain and curvature increments $\{d\varepsilon\}$ and $\{d\kappa\}$ are passed in using the array DSTRAN (6). The main purpose of the UGENS is to calculate the stiffness matrix $[H]$ in Eq. (13) which is stored in the array DDNDDE (6, 6) at the end of the subroutine. This array will be used by ABAQUS to define the element stiffness matrix in the FE solution.

## 4. RESULTS AND DISCUSSION

To validate our H-model, we first discretize the five layers of corrugated cardboard by shell elements S4R of Abaqus to obtain the model Abaqus-3D; Then, we discretize the middle surface of corrugated cardboard by shell elements S4R of Abaqus combined with our H-model (using "user's subroutine" "UGENS") to obtain H-2D model. The confrontation of the results allow us to evaluate the efficiency and accuracy of our homogenization model.

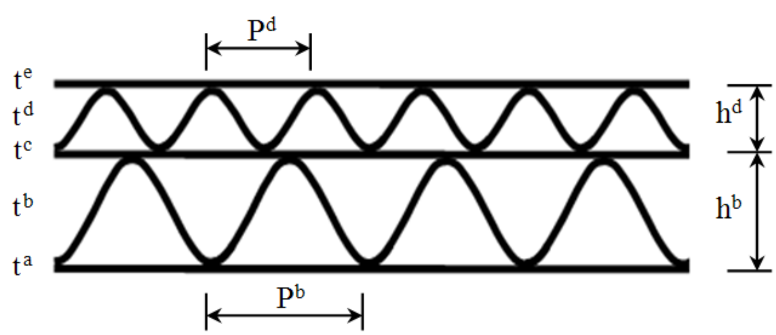


*Fig. 3.* Geometry of the CD section of corrugated cardboard

The calculations and comparisons are made on a corrugated panel having CD section illustrated in Fig. 3. Geometric data are: period (or step) and height of the lower groove $P^b = 9$ mm and $h^b = 5.2$ mm, those of the upper groove $P^d = 6$ mm and $h^d = 2.9$ mm, thicknesses $t^a = t^c = t^e = 0.25$ mm, $t^b = t^d = 0.26$ mm. The properties of materials are given in Tab. 1 [9]. The rigidities of 2D equivalent plate are calculated as shown in Tab. 2.

*Table 1.* Material properties of the five layers of the corrugated cardboard

| Layers | $E_1$ (MPa) | $E_2$ (MPa) | $E_3$ (MPa) | $G_{12}$ (MPa) | $G_{13}$ (MPa) | $G_{23}$ (MPa) | $\nu_{12}$ | $\nu_{13}$ | $\nu_{23}$ |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | 8250 | 2900 | 2900 | 1890 | 7 | 70 | 0.43 | 0.01 | 0.01 |
| $b, d$ | 4500 | 4500 | 3000 | 1500 | 3.5 | 35 | 0.40 | 0.01 | 0.01 |
| $c, e$ | 8180 | 3120 | 3120 | 1950 | 7 | 70 | 0.43 | 0.01 | 0.01 |

*Table 2.* Rigidities of the equivalent plate

| Rigidities | $A_{11}$ (N/mm) | $A_{12}$ (N/mm) | $A_{22}$ (N/mm) | $B_{11}$ (N) | $B_{12}$ (N) | $B_{22}$ (N) | $D_{11}$ (N.mm) | $D_{12}$ (N.mm) | $D_{22}$ (N.mm) |
|---|---|---|---|---|---|---|---|---|---|
| Values | 6606.2 | 1055.1 | 5989.8 | 2507.1 | 526.1 | 2914.5 | 75214.1 | 11870.5 | 49672.4 |

We use a corrugated panel having length $L = 162$ mm and width $B = 162$ mm. This panel is tested under different types of loading: traction and bending. For the simulation of the homogenized plate using our H-2D model, the middle surface is discretized into 2916 quadrilateral elements S4R and nodes 3025. But for the Abaqus simulation-3D, 52116 quadrilateral elements S4R and 48293 nodes are needed. Indeed, to fully describe the geometry of the groove, it takes at least 16 elements over a period of groove.

In both types of simulations (Abaqus-3D model and H-2D model), a rigid plate is bonded to the MD or CD section at the right end of the cardboard panel to better apply forces or moments (Fig. 4, 5). The calculations by our H-2D model are very fast while calculations by Abaqus-3D are much longer. The comparisons of results obtained by the two models and the percentages of error in H-2D model compared to Abaqus-3D results for the traction and bending are presented in Tab. 3, we note that the numerical results given by the two models are very close.

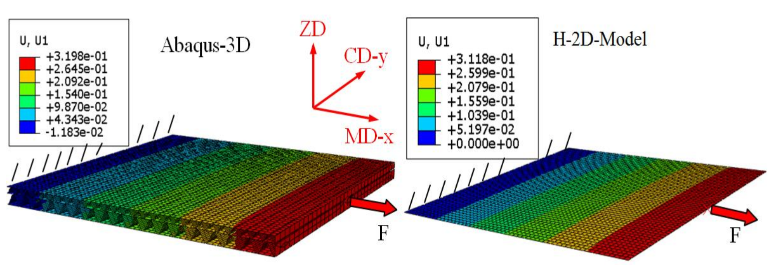


*Fig. 4.* Simulation of Abaqus-3D and H-model in traction for the MD section

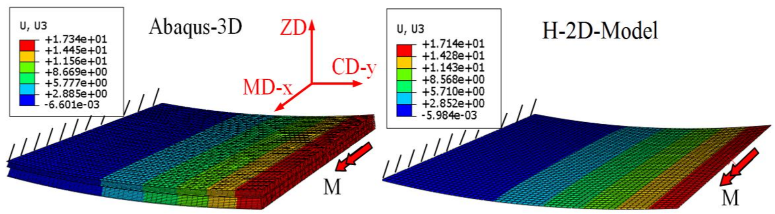


*Fig. 5.* Simulation of Abaqus-3D and H-model in bending for the CD section

*Table 3.* Comparison between Abaqus-3D and 2D-H-model for plate in traction and bending

| | | | Abaqus-3D | H-2D-Model | Error (%) |
|---|---|---|---|---|---|
| Traction $F = 2000$ N | MD | Displacement $U_1$ (mm) | 0.3110 | 0.3118 | +0.26 |
| | | CPU time (s) | 48.7 | 1 | 48.7 times |
| | CD | Displacement $U_2$ (mm) | 0.3433 | 0.3455 | +0.64 |
| | | CPU time (s) | 47.2 | 1 | 47.2 times |
| Bending $M = 10$ KN.mm | MD | Displacement $U_3$ (mm) | 11.21 | 11.14 | −0.62 |
| | | CPU time (s) | 39.7 | 1 | 39.7 times |
| | CD | Displacement $U_3$ (mm) | 17.34 | 17.14 | −1.15 |
| | | CPU time (s) | 39.1 | 1 | 39.1 times |

## 5. CONCLUSION

In this article, we have proposed an analytic homogenization model for the traction and flexion problem of double corrugated core cardboards. The comparison of the results obtained by the analytic formulas, by the Abaqus 3D simulations and by the Abaqus-Ugens 2D simulations has proved the validation of the present traction and flexion homogenization model. The present H-model allows us to largely reduce not only the time for the geometry creation and FE calculation, but also the computational hardware requirements for the large-scale numerical modelling of packaging systems composed of double corrugated core cardboards.

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