THE DISPERSION OF RAYLEIGH WAVES IN ORTHOTROPIC LAYERED HALF-SPACE USING MATRIX METHOD

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Abstract. In this paper, the secular equation of Rayleigh surface waves propagating in an orthotropic layered half-space is derived by the matrix method. All the layers and the half-space are assumed to have identical principle axes. The explicit form of the matrizant for each layer is obtained by the Sylvester's theorem. The derived secular equation takes only real values and depends only on the dimensionless variables and dimensionless material parameters. Hence, it is convenient in numerical calculation.

Keywords: Rayleigh waves, matrix method, Sylvester's theorem, orthotropy.

1. INTRODUCTION

The study of Rayleigh surface waves propagation in layered half-space is of considerable interest in the field of seismology since the Earth's surface could be considered to consist of several layers overlying a half-space. The first systematic and efficient method to find the secular equation is the propagator matrix method proposed by Thomson [1] and Haskell [2]. This method expresses the relation between the displacements and the stresses at two faces of a layer by a matrix called the layer propagator matrix. The product of these matrices is used to find relation of the displacements and stresses at the free surface and at the top of the half-space to formulate the secular equation using boundary conditions. This method has been modified to be more efficient and stable such as in Knopoff [3], Dunkin [4], Kennett [5] and Chen [6] and has been used widely recently.

For the homogeneous isotropic layered half-space, the layer propagator matrix has an explicit form because the vertical wave numbers in the layer are simple and always either real or pure imaginary. The situation become more complicated for the homogeneous anisotropic layers in which the vertical wave number could be complex. This make the propagator matrix could be complex leading to difficulties in numerical calculation and it is also not easy to have the layer propagator matrix in explicit form (see Crampin [7], Crampin and Taylor [8]). This also makes difficulties in handling the decay

condition in the half-space. For the orthotropic layers, an explicit expression of the transfer matrix is given in Solyanik [9] and its more convenient in numerical calculation form is then given in Rokhlin and Wang [10]. However, the elements of these transfer matrices are not always real.

In this paper, a more advanced formulation of the Thomson-Haskell method is used to obtain the matrizant of a layer, which is analogous to the layer propagator matrix, first. It is written in explicit form by the use of Sylvester's theorem. For the purpose of finding the secular equation, an equivalent dimensionless and real form of the matrizant is derived. Then, the difficulty in the decay condition will be overcome by using the Vieta's formulas for the propagation equation of the half-space. Finally, the secular equation in convenient form for numerical calculation is obtained and an numerical example in finding the dispersive curves of Rayleigh surface waves is presented.

2. FORMULATION OF THE PROBLEM

Consider a stack of n homogeneous orthotropic layers overlying on a homogeneous orthotropic half-space. By $h^{(k)}, \rho^{(k)}, c_{ij}^{(k)}$ $(i, j = \overline{1, 6})$ we denote the thickness, the density, and the material constants of the kth layer $(k = \overline{0, n})$. The underlying half-space is labeled as the zeroth layer $(k = 0)$. All the layers are assumed to have identical principal axes and the waves propagates along x_1-axis which is one of the three principal axes (see Fig. 1).
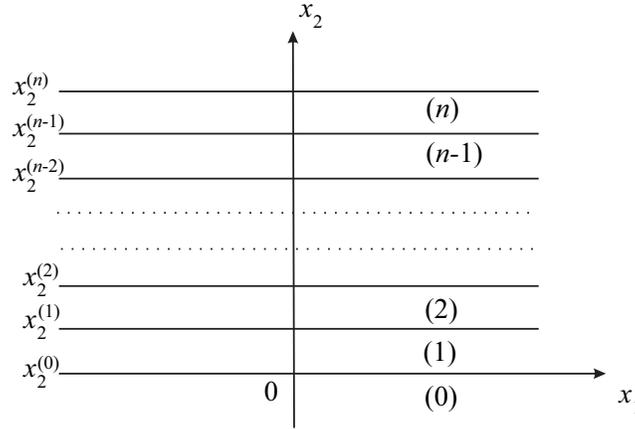

Fig. 1. Model of n orthotropic layers overlying an orthotropic half-space.
The half-space is labeled as layer (0)

Consider plane waves propagating in a layer whose material parameters are denoted by (ρ, c_{ij}) for the moment such that

$$
\begin{aligned}
u_1 &= -i y_2 \left(x_2; \omega, k \right) e^{i(\omega t - k x_1)}, \\
u_2 &= y_1 \left(x_2; \omega, k \right) e^{i(\omega t - k x_1)}, \\
u_3 &= 0,
\end{aligned}
\tag{1}
$$

where ω, k, t are the angular frequency, horizontal wave number and the time, respectively.

Since the layer is made of orthotropic elastic material, then the strain-stress relations are (see Ting [11])

$$\sigma_{11} = c_{11}u_{1,1} + c_{12}u_{2,2},$$
$$\sigma_{22} = c_{12}u_{1,1} + c_{22}u_{2,2}, \tag{2}$$
$$\sigma_{12} = c_{66}\left(u_{1,2} + u_{2,1}\right),$$

where σ_{ij} are the stresses of the layer, commas indicate differentiation with respect to spatial variable x_k. In the absence of body forces, equations of motion are

$$\sigma_{11,1} + \sigma_{12,2} = \rho\ddot{u}_1,$$
$$\sigma_{12,1} + \sigma_{22,2} = \rho\ddot{u}_2. \tag{3}$$

Let

$$\sigma_{22} = y_3 e^{i(\omega t - kx_1)},$$
$$\sigma_{12} = -iy_4 e^{i(\omega t - kx_1)}, \tag{4}$$

then substituting (1) into (2) we obtain

$$y_3 = c_{22}\frac{dy_1}{dx_2} - kc_{12}y_2,$$
$$y_4 = c_{66}\left(\frac{dy_1}{dx_2} + ky_1\right), \tag{5}$$

and

$$k\left(c_{12}\frac{dy_1}{dx_2} - kc_{11}y_2\right) + \frac{dy_4}{dx_2} = -\rho\omega^2 y_2,$$
$$-ky_4 + \frac{dy_3}{dx_2} = -\rho\omega^2 y_1. \tag{6}$$

Eqs. (5) and (6) can be written in the matrix form as follows

$$\frac{d\mathbf{y}}{dx_2} = \mathbf{U}\mathbf{y}, \tag{7}$$

where $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$ (the symbol "T" indicates the transpose) and

$$\mathbf{U} = \begin{bmatrix} 0 & k\dfrac{c_{12}}{c_{22}} & \dfrac{1}{c_{22}} & 0 \\ -k & 0 & 0 & \dfrac{1}{c_{66}} \\ -\rho\omega^2 & 0 & 0 & \dfrac{c_{66}}{k} \\ 0 & k^2\left(c_{11} - \dfrac{c_{12}^2}{c_{22}}\right) - \rho\omega^2 & -k\dfrac{c_{12}}{c_{22}} & 0 \end{bmatrix}. \tag{8}$$

The *propagation equation* of the layer, which is the characteristic equation determining the eigenvalues $\lambda = kb$ of the matrix $\mathbf{U}$, is

$$c_{22}c_{66}b^4 + \left[(c_{12} + c_{66})^2 + c_{22}\left(X - c_{11}\right) + c_{66}\left(X - c_{66}\right)\right]b^2 + (c_{11} - X)\left(c_{66} - X\right) = 0, \tag{9}$$

where $X = \rho c^2$. This is a quadratic equation in b^2 with solution b_1^2 and b_2^2 satisfying Vieta's formulas

$$
\begin{aligned}
b_1^2 + b_2^2 &= -\frac{(c_{12} + c_{66})^2 + c_{22}(X - c_{11}) + c_{66}(X - c_{66})}{c_{22}c_{66}} = S, \\
b_1^2 \cdot b_2^2 &= \frac{(c_{11} - X)(c_{66} - X)}{c_{22}c_{66}} = P.
\end{aligned}
\tag{10}
$$

Eq. (9) has four roots, namely $\pm\sqrt{b_1^2}$ and $\pm\sqrt{b_2^2}$. When $S^2 - 4P > 0$ the four roots are either real or pure imaginary. Otherwise, they are complex conjugate in pairs and note that

$$
b_1^2 = (b_2^2)^*,
\tag{11}
$$

in which the asterisk denotes the complex conjugate. The eigenvector of matrix $\mathbf{U}$ corresponding to eigenvalue $\lambda = kb$ is

$$
\mathbf{v} =
\begin{bmatrix}
b(c_{12} + c_{66}) \\
b^2 c_{22} - c_{66} + X \\
-k\left[c_{12}X - (b^2 c_{22} + c_{12})c_{66}\right] \\
kc_{66}b(b^2 c_{22} + c_{12} + X)
\end{bmatrix}.
\tag{12}
$$

Consider the Rayleigh surface waves propagating in layered half-space (Fig. 1) along the x_1-axis. The governing equations of the motion of Rayleigh waves in this model are

$$
\frac{d\mathbf{y}}{dx_2} = \mathbf{U}(x_2)\mathbf{y},
\tag{13}
$$

in which the coefficient matrix is a function of x_2 defined as

$$
\begin{aligned}
\mathbf{U}(x_2) &= \mathbf{U}^{(i)}, \quad \left(x_2 \in \left[x_2^{(i-1)}, x_2^{(i)}\right], i = \overline{1, n}\right), \\
\mathbf{U}(x_2) &= \mathbf{U}^{(0)}, \quad (x_2 \in [-\infty, 0]),
\end{aligned}
\tag{14}
$$

where $\mathbf{U}^{(i)}$ is of form (8) with all material parameters are replaced by those of ith layer.

The boundary conditions are the traction free at the top surface, i.e.

$$
y_3(H; \omega, k) = y_4(H; \omega, k) = 0,
\tag{15}
$$

where $H = \sum h_i$ is the total thickness of the layers, and the continuity of the displacements and stresses at each interface, i.e.

$$
\mathbf{y}^{(i)}(x_2^{(i)}; \omega, k) = \mathbf{y}^{(i+1)}(x_2^{(i)}; \omega, k), (i = \overline{0, n-1})
\tag{16}
$$

where $\mathbf{y}^{(i)}(x_2; \omega, k) = \mathbf{y}(x_2; \omega, k)$ at $x_2 = [x_2^{(i-1)}, x_2^{(i)}], (i = \overline{1, n})$ and the decay condition that

$$
\mathbf{y}(x_2; \omega, k) \to 0 \quad \text{as} \quad x_2 \to -\infty.
\tag{17}
$$

The differential equations (13) along with the boundary conditions (15), (16) and (17) will be used to determine the secular equation of the Rayleigh surface waves.

3. THE MATRIX METHOD

Consider the differential equations (13). A formal solution to this equation is

$$\mathbf{y}(x_2) = \mathbf{\Omega}(x_2, x_2') \mathbf{y}(x_2'), \tag{18}$$

where $\mathbf{y}(x_2')$ is the given initial value of $\mathbf{y}$ at $x_2 = x_2'$ and $\mathbf{\Omega}(x_2, x_2')$ is called the matrizant of $\mathbf{U}$. When $\mathbf{U}$ is a constant matrix, the matrizant is (see Frazer et. al [12])

$$\mathbf{\Omega}(x_2, x_2') = \exp\left[(x_2 - x_2')\mathbf{U}\right]. \tag{19}$$

Let n be the rank of $\mathbf{U}$ and λ_i $(i = 1, 2, \ldots, n)$ be the eigenvalues of $\mathbf{U}$, by Sylvester's theorem (e.g. see Frazer et. al [12]), we have

$$\mathbf{E}(h) = \exp(h\mathbf{U}) = \sum_{i=1}^{n} e^{\lambda_i h} \prod_{j \neq i} \frac{\lambda_j \mathbf{I} - \mathbf{U}}{\lambda_j - \lambda_i}. \tag{20}$$

Applying this result to our layered model, for example to the 1st layer, we have

$$\mathbf{y}\left(x_2^{(1)}\right) = \mathbf{E}^{(1)}(h_1) \mathbf{y}\left(x_2^{(0)}\right) = \mathbf{E}^{(1)}(h_1) \mathbf{y}(0), \tag{21}$$

where $\mathbf{E}^{(1)}(h_1) = \exp\left(h_1 \mathbf{U}^{(1)}\right)$ computed from (20) for matrix $\mathbf{U}^{(1)}$. Using this procedure for other layers, we have

$$\begin{aligned}
\mathbf{y}(H) = \mathbf{y}\left(x_2^{(n)}\right) &= \mathbf{E}^{(n)}(h_n) \mathbf{y}\left(x_2^{(n-1)}\right) \\
&= \mathbf{E}^{(n)}(h_n) \mathbf{E}^{(n-1)}(h_{n-1}) \mathbf{y}\left(x_2^{(n-2)}\right) = \cdots = \\
&= \mathbf{E}^{(n)}(h_n) \mathbf{E}^{(n-1)}(h_{n-1}) \ldots \mathbf{E}^{(2)}(h_2) \mathbf{E}^{(1)}(h_1) \mathbf{y}(0),
\end{aligned} \tag{22}$$

where $\mathbf{E}^{(i)}(h_i) = \exp\left(h_i \mathbf{U}^{(i)}\right)$, $(i = \overline{1, n})$.

To obtain $\mathbf{y}(0)$, we consider the system of waves propagating in the half-space. The wave system in the half-space is controlled by Eq. (7) with $\mathbf{U}$ replaced by $\mathbf{U}^{(0)}$. In general, Eq. (7) has four fundamental solutions of form

$$\mathbf{y}_j^{(0)}(x_2; \omega, k) = \mathbf{v}_j^{(0)} \exp\left(\lambda_j^{(0)} x_2\right), \quad (j = \overline{1, 4}) \tag{23}$$

where $\lambda_j^{(0)}$ are the eigenvalues of matrix $\mathbf{U}^{(0)}$ determined from (9) and $\mathbf{v}_j^{(0)}$ are the corresponding eigenvectors computed from (12). For the sake of simplicity, hereafter we denote $\mathbf{y}^{(i)}(x_2; \omega, k)$ by $\mathbf{y}^{(i)}(x_2)$.

Due to the decay condition, there are only two solutions in the half-space with the eigenvalue having positive real part. These two eigenvalues are $kb_1^{(0)}$, $kb_2^{(0)}$ and determined from (9). Hence, the displacement field in the half-space is

$$\mathbf{y}^{(0)}(x_2) = C_1 \mathbf{v}_1^{(0)} e^{kb_1^{(0)} x_2} + C_2 \mathbf{v}_2^{(0)} e^{kb_2^{(0)} x_2}, \tag{24}$$

where C_1 and C_2 are constant. At $x_2 = 0$, we have

$$\mathbf{y}(0) = \mathbf{y}^{(0)}(0) = C_1 \mathbf{v}_1^{(0)} + C_2 \mathbf{v}_2^{(0)}. \tag{25}$$

Denote

$$\mathbf{E}\left(H\right) = \mathbf{E}^{(n)}\left(h_n\right)\mathbf{E}^{(n-1)}\left(h_{n-1}\right)\dots\mathbf{E}^{(2)}\left(h_2\right)\mathbf{E}^{(1)}\left(h_1\right),\tag{26}$$

we have from Eq. (22) that

$$\mathbf{y}\left(H\right) = C_1\mathbf{E}\left(H\right)\mathbf{v}_1^{(0)} + C_3\mathbf{E}\left(H\right)\mathbf{v}_2^{(0)}.\tag{27}$$

The traction free condition (15) gives

$$\begin{aligned} y_3\left(H\right) &= 0,\\ y_4\left(H\right) &= 0. \end{aligned}\tag{28}$$

These are two homogeneous equations of C_1 and C_2. Hence, to obtain the nontrivial solution, we must have

$$\Delta\left(\omega, k\right) = \left[E_{3i}\left(H\right)\mathbf{v}_1^{(0)}(i)\right]\left[E_{4i}\left(H\right)\mathbf{v}_2^{(0)}(i)\right] - \left[E_{3i}\left(H\right)\mathbf{v}_2^{(0)}(i)\right]\left[E_{4i}\left(H\right)\mathbf{v}_1^{(0)}(i)\right] = 0,\tag{29}$$

in which the Einstein summation for i from 1 to 4 is applied. This is the secular equation of Rayleigh waves and it could be used to obtain the dispersion curves numerically such as by some numerical integral method (e.g see Takeuchi and Saito [13]). The numerical approach using Eq. (29) has some intrinsic difficulties but a problem could be arisen here is how to choose two out of four solutions in the half-space satisfying the decay condition because the eigenvalues of the propagation equation of the half-space are in general complex. In the following, the secular equation will be written in a new form to avoid the above difficulties.

Eq. (29) could be expressed as

$$\Delta\left(\omega, k\right) = \mathbf{M}\cdot\mathbf{V} = 0,\tag{30}$$

where $\mathbf{M}$ and $\mathbf{V}$ are two vectors of six components defined as

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1^{(0)}\left(1\right)\mathbf{v}_2^{(0)}\left(2\right) - \mathbf{v}_2^{(0)}\left(1\right)\mathbf{v}_1^{(0)}\left(2\right)\\ \mathbf{v}_1^{(0)}\left(1\right)\mathbf{v}_2^{(0)}\left(3\right) - \mathbf{v}_2^{(0)}\left(1\right)\mathbf{v}_1^{(0)}\left(3\right)\\ \mathbf{v}_1^{(0)}\left(1\right)\mathbf{v}_2^{(0)}\left(4\right) - \mathbf{v}_2^{(0)}\left(1\right)\mathbf{v}_1^{(0)}\left(4\right)\\ \mathbf{v}_1^{(0)}\left(2\right)\mathbf{v}_2^{(0)}\left(3\right) - \mathbf{v}_2^{(0)}\left(2\right)\mathbf{v}_1^{(0)}\left(3\right)\\ \mathbf{v}_1^{(0)}\left(2\right)\mathbf{v}_2^{(0)}\left(4\right) - \mathbf{v}_2^{(0)}\left(2\right)\mathbf{v}_1^{(0)}\left(4\right)\\ \mathbf{v}_1^{(0)}\left(3\right)\mathbf{v}_2^{(0)}\left(4\right) - \mathbf{v}_2^{(0)}\left(3\right)\mathbf{v}_1^{(0)}\left(4\right) \end{bmatrix} \text{ and } \mathbf{M} = \begin{bmatrix} E_{31}(H)E_{42}(H) - E_{41}(H)E_{32}(H)\\ E_{31}(H)E_{43}(H) - E_{41}(H)E_{33}(H)\\ E_{31}(H)E_{44}(H) - E_{41}(H)E_{34}(H)\\ E_{32}(H)E_{43}(H) - E_{42}(H)E_{33}(H)\\ E_{32}(H)E_{44}(H) - E_{42}(H)E_{34}(H)\\ E_{33}(H)E_{44}(H) - E_{43}(H)E_{34}(H) \end{bmatrix}.\tag{31}$$

Note that $b_1^{(0)} = \left(b_2^{(0)}\right)^*$. Therefore, $\mathbf{v}_1^{(0)} = \left(\mathbf{v}_2^{(0)}\right)^*$ and we can write vector $\mathbf{V}$ as

$$\mathbf{V} = \left(b_1^{(0)} - b_2^{(0)}\right)k\left(c_{66}^{(0)}\right)^3\left[\frac{\bar{V}_1}{kc_{66}^{(0)}}, \bar{V}_2, \bar{V}_3, \bar{V}_4, \bar{V}_5, kc_{66}^{(0)}\bar{V}_6\right]^T,\tag{32}$$

where $\bar{\mathbf{V}} = [\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4, \bar{V}_5, \bar{V}_6]^T$ is a dimensionless vector whose components are

$$
\begin{aligned}
\bar{V}_1 &= \left(e_3^{(0)} + 1\right)\left[-e_2^{(0)}\sqrt{P^{(0)}} - 1 + x^{(0)}\right], \\
\bar{V}_2 &= -\left(e_3^{(0)} + 1\right)\left(e_3^{(0)}x^{(0)} + e_2^{(0)}\sqrt{P^{(0)}} - e_3^{(0)}\right), \\
\bar{V}_3 &= -\left(e_3^{(0)} + 1\right)e_2^{(0)}\sqrt{P^{(0)}}\sqrt{S^{(0)} + 2\sqrt{P^{(0)}}}, \\
\bar{V}_4 &= -\left(e_3^{(0)} + 1\right)e_2^{(0)}\left(x^{(0)} - 1\right)\sqrt{S^{(0)} + 2\sqrt{P^{(0)}}}, \\
\bar{V}_5 &= -\left(e_2^{(0)}\right)^2 P^{(0)} + e_2^{(0)}\left(1 - x^{(0)}\right)S^{(0)} + e_2^{(0)}\left(e_3^{(0)} + 1\right)\sqrt{P^{(0)}} + \\
&\quad + \left(e_3^{(0)} + x^{(0)}\right)\left(1 - x^{(0)}\right), \\
\bar{V}_6 &= e_2^{(0)}e_3^{(0)}\left(x^{(0)} - 1\right)S^{(0)} - \left(e_2^{(0)}\right)^2 P^{(0)} + e_2^{(0)}\left(e_3^{(0)} + 1\right)x^{(0)}\sqrt{P^{(0)}} + \\
&\quad + e_3^{(0)}\left(e_3^{(0)} + x^{(0)}\right)\left(x^{(0)} - 1\right).
\end{aligned}
\tag{33}
$$

In this equation, we use the dimensionless quantities defined as

$$
e_1^{(i)} = \frac{c_{11}^{(i)}}{c_{66}^{(i)}}, \quad e_2^{(i)} = \frac{c_{22}^{(i)}}{c_{66}^{(i)}}, \quad e_3^{(i)} = \frac{c_{12}^{(i)}}{c_{66}^{(i)}}, \quad x^{(i)} = \frac{X^{(i)}}{c_{66}^{(i)}}, \quad (i = \overline{0, n})
\tag{34}
$$

and $S^{(0)}$ and $P^{(0)}$ are determined from (10) with parameters of the half-space. Note that for Rayleigh waves existence in the half-space, i.e. $b_1^{(0)}$ and $b_2^{(0)}$ having real part, then (see Vinh and Ogden [14])

$$
0 < X^{(0)} < \min\left\{c_{66}^{(0)}, c_{11}^{(0)}\right\},
\tag{35}
$$

which makes $P^{(0)}$ positive.

To calculate vector $\mathbf{M}$ in Eq. (31), we need an explicit form of layer matrizant expressed in Eq. (20). The eigenvalues of matrix $\mathbf{U}^{(i)}$ are roots of Eq. (9) and satisfy Eq. (10) for the ith layer. After some algebraical manipulation, we obtain the matrizant of the ith layer as

$$
\mathbf{E}^{(i)}(h_i) = \alpha^{(i)}\begin{pmatrix} \mathbf{E}_1^{(i)} & \dfrac{1}{kc_{66}^{(0)}}\mathbf{E}_2^{(i)} \\ kc_{66}^{(0)}\mathbf{E}_3^{(i)} & \mathbf{E}_4^{(i)} \end{pmatrix},
\tag{36}
$$

where

$$
\alpha^{(i)} = \begin{cases} \dfrac{1}{\sqrt{P^{(i)} - (S^{(i)})^2/4}}, & \text{if } (S^{(i)})^2 - 4P^{(i)} < 0 \\[4mm] \dfrac{1}{\sqrt{(S^{(i)})^2 - 4P^{(i)}}}, & \text{if } (S^{(i)})^2 - 4P^{(i)} > 0 \end{cases}
\tag{37}
$$

The matrices $\mathbf{E}_j^{(i)}$ $\left(j = \overline{1,4}\right)$ are two-by-two matrices whose entries are dimensionless and equal

$$E_1^{(i)}(1,1) = m_i + p_i a_i,$$

$$E_1^{(i)}(1,2) = \left[n_i e_3^{(i)} + q_i \left(b_i e_3^{(i)} - d_i\right)\right] e_2^{(i)},$$

$$E_1^{(i)}(2,1) = -n_i + q_i (d_i - a_i),$$

$$E_1^{(i)}(2,2) = m_i + q_i b_i,$$

$$E_2^{(i)}(1,1) = \frac{\left[n_i + q_i \left(a_i - c_i e_3^{(i)}\right)\right] e_2^{(i)}}{r_\mu^{(i)}},$$

$$E_2^{(i)}(1,2) = \frac{p_i c_i}{r_\mu^{(i)}},$$

$$E_2^{(i)}(2,1) = -E_2^{(i)}(1,2),$$

$$E_2^{(i)}(2,2) = \frac{n_i + q_i (b_i - c_i)}{r_\mu^{(i)}}, \tag{38}$$

$$E_3^{(i)}(1,1) = \left[-n_i x^{(i)} + q_i \left(d_i - a_i x^{(i)}\right)\right] r_\mu^{(i)},$$

$$E_3^{(i)}(1,2) = -p_i d_i r_\mu^{(i)},$$

$$E_3^{(i)}(2,1) = -E_3^{(i)}(1,2),$$

$$E_3^{(i)}(2,2) = \left[(n_i + q_i b_i)\left(e_1^{(i)} - e_2^{(i)}\left(e_3^{(i)}\right)^2 - x^{(i)}\right) + e_2^{(i)} e_3^{(i)} q_i d_i\right] r_\mu^{(i)},$$

$$E_4^{(i)}(1,1) = E_1^{(i)}(1,1),$$

$$E_4^{(i)}(1,2) = -E_1^{(i)}(2,1),$$

$$E_4^{(i)}(2,1) = -E_1^{(i)}(1,2),$$

$$E_4^{(i)}(2,2) = E_1^{(i)}(2,2),$$

where

$$r_\nu^{(i)} = \frac{c_{66}^{(0)}}{c_{66}^{(i)}} \frac{\rho^{(i)}}{\rho^{(0)}}, \quad r_\mu^{(i)} = \frac{c_{66}^{(i)}}{c_{66}^{(0)}}, \tag{39}$$

and

$$a_i = -e_2^{(i)}\left(e_3^{(i)} + x^{(i)}\right), \quad b_i = e_1^{(i)} - x^{(i)} - e_2^{(i)} e_3^{(i)}\left(e_3^{(i)} + 1\right),$$

$$c_i = e_2^{(i)}\left(e_3^{(i)} + 1\right), \quad d_i = x^{(i)} - e_1^{(i)} - e_3^{(i)} a_i, \tag{40}$$

and m_i, n_i, p_i, q_i are defined as the follows.

(1) If $(S^{(i)})^2 - 4P^{(i)} < 0$,

$$m_i = -\sinh\left(\varepsilon \bar{h}_i f_i\right) \sin\left(\varepsilon \bar{h}_i g_i\right) F_i + \cosh\left(\varepsilon \bar{h}_i f_i\right) \cos\left(\varepsilon \bar{h}_i g_i\right) G_i,$$

$$n_i = \frac{1}{\sqrt{P_i}}[\sinh\left(\varepsilon \bar{h}_i f_i\right) \cos\left(\varepsilon \bar{h}_i g_i\right) \left(f_i G_i + g_i F_i\right) +$$

$$- \cosh\left(\varepsilon \bar{h}_i f_i\right) \sin\left(\varepsilon \bar{h}_i g_i\right) \left(f_i F_i - g_i G_i\right)], \tag{41}$$

$$p_i = \sinh\left(\varepsilon \bar{h}_i f_i\right) \sin\left(\varepsilon \bar{h}_i g_i\right),$$

$$q_i = \frac{1}{\sqrt{P_i}}\left[\cosh\left(\varepsilon \bar{h}_i f_i\right) \sin\left(\varepsilon \bar{h}_i g_i\right) f_i - \sinh\left(\varepsilon \bar{h}_i f_i\right) \cos\left(\varepsilon \bar{h}_i g_i\right) g_i\right]$$

where

$$F_i = \frac{S^{(i)}}{2}, \quad G_i = \sqrt{P^{(i)} - F_i^2},$$

$$f_i = \frac{1}{\sqrt{2}}\sqrt{\sqrt{F_i^2 + G_i^2} + F_i}, \quad g_i = \frac{1}{\sqrt{2}}\sqrt{\sqrt{F_i^2 + G_i^2} - F_i} \tag{42}$$

and $\epsilon = kh$, $\bar{h}_i = h_i/H$ $(i = \overline{1,n})$.

(2) If $(S^{(i)})^2 - 4P^{(i)} > 0$

$$m_i = \cosh\left(\varepsilon \bar{h}_i b_3^{(i)}\right) \left(b_1^{(i)}\right)^2 - \cosh\left(\varepsilon \bar{h}_i b_1^{(i)}\right) \left(b_3^{(i)}\right)^2,$$

$$n_i = \frac{\sinh\left(\varepsilon \bar{h}_i b_3^{(i)}\right)}{b_3^{(i)}} \left(b_1^{(i)}\right)^2 - \frac{\sinh\left(\varepsilon \bar{h}_i b_1^{(i)}\right)}{b_1^{(i)}} \left(b_3^{(i)}\right)^2,$$

$$p_i = \cosh\left(\varepsilon \bar{h}_i b_1^{(i)}\right) - \cosh\left(\varepsilon \bar{h}_i b_3^{(i)}\right), \tag{43}$$

$$q_i = \frac{\sinh\left(\varepsilon \bar{h}_i b_1^{(i)}\right)}{b_1^{(i)}} - \frac{\sinh\left(\varepsilon \bar{h}_i b_3^{(i)}\right)}{b_3^{(i)}},$$

where

$$b_1^{(i)} = \sqrt{\frac{S^{(i)} + \sqrt{(S^{(i)})^2 - 4P^{(i)}}}{2}}, \quad b_3^{(i)} = \sqrt{\frac{S^{(i)} - \sqrt{(S^{(i)})^2 - 4P^{(i)}}}{2}}. \tag{44}$$

Note that in both cases of $(S^{(i)})^2 - 4P^{(i)}$ positive and negative, quantities m_i, n_i, p_i, q_i are always real that makes matrices $\mathbf{E}_j^{(i)}$ $(j = \overline{1,4})$ real. And due to the special form of matrices $\mathbf{E}^{(i)}$ (in Eq. (36)), their product denoted by $\mathbf{E}$ in Eq. (26) has the same form. This could be proved easily by direct matrix multiplication. In particular,

$$\mathbf{E}(H) = \left(\prod_{i=1}^{n} \alpha^{(i)}\right) \begin{pmatrix} \mathbf{E}_1 & \dfrac{1}{kc_{66}^{(0)}}\mathbf{E}_2 \\ kc_{66}^{(0)}\mathbf{E}_3 & \mathbf{E}_4 \end{pmatrix} \tag{45}$$

where

$$\bar{\mathbf{E}}(H) := \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_3 & \mathbf{E}_4 \end{pmatrix} = \prod_{i=1}^{n} \begin{pmatrix} \mathbf{E}_1^{(i)} & \mathbf{E}_2^{(i)} \\ \mathbf{E}_3^{(i)} & \mathbf{E}_4^{(i)} \end{pmatrix}. \tag{46}$$

Note that this matrix is dimensionless and real and it will be used to find the secular equation instead of the matrizant $\mathbf{E}(H)$ which has dimension and could be complex due to the factors $\left(\prod_{i=1}^{n} \alpha^{(i)}\right)$ and $\left(kc_{66}^{(0)}\right)$ in Eq. (45).

Inspecting Eq. (31) and Eq. (45), we see that

$$\mathbf{M} = \left(\prod_{i=1}^{n} \alpha^{(i)}\right)^2 kc_{66}^{(0)} \left[kc_{66}^{(0)} \bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4, \bar{M}_5, \frac{\bar{M}_6}{kc_{66}^{(0)}}\right], \tag{47}$$

where the dimensionless vector $\bar{\mathbf{M}} = [\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4, \bar{M}_5, \bar{M}_6]^T$ is defined similarly to vector $\mathbf{M}$ in Eq. (31) in which the matrix $\mathbf{E}$ is replaced by the dimensionless matrix $\bar{\mathbf{E}}$.

Substituting vectors $\mathbf{M}$ and $\mathbf{V}$ from Eq. (47) and Eq. (32) into the secular equation, Eq. (30), it can be written as

$$\left(b_1^{(0)} - b_2^{(0)}\right) k^2 (c_{66}^{(0)})^4 \left(\prod_{i=1}^{n} \alpha^{(i)}\right)^2 \bar{\mathbf{M}}\bar{\mathbf{V}} = 0. \tag{48}$$

Since $b_1^{(0)} - b_2^{(0)} \neq 0$ and $\alpha^{(i)} \neq 0$, this equation could be expressed as

$$\bar{\mathbf{M}}\bar{\mathbf{V}} = 0. \tag{49}$$

Note that this is a real equation of dimensionless frequency ϵ and velocity $x^{(0)}$, and dimensionless material parameters of layers and half-space.

4. NUMERICAL CALCULATION

For a given set of material parameters of layers and half-space, all the dimensionless parameters are computed by Eqs. (34), (39). Our aim is to compute the phase velocity of Rayleigh waves in the form $x^{(0)}$ for a given wave number expressed in form of ϵ. First, we compute vector $\bar{\mathbf{V}}$ in the secular equation, Eq. (49), by Eqs. (33). To calculate vector $\bar{\mathbf{M}}$, we compute all the matrizant of layers in dimensionless form by Eq. (26), then use Eq. (46) to compute their product to obtain the matrix $\bar{\mathbf{E}}$. The vector $\bar{\mathbf{M}}$ is then computed by Eq. (31) in which the matrix $\mathbf{E}$ is replaced by $\bar{\mathbf{E}}$. Finally, the phase velocity $x^{(0)}$ is found by setting the scalar product of vectors $\bar{\mathbf{V}}$ and $\bar{\mathbf{M}}$ equal zero.

Table 1. Properties of Materials

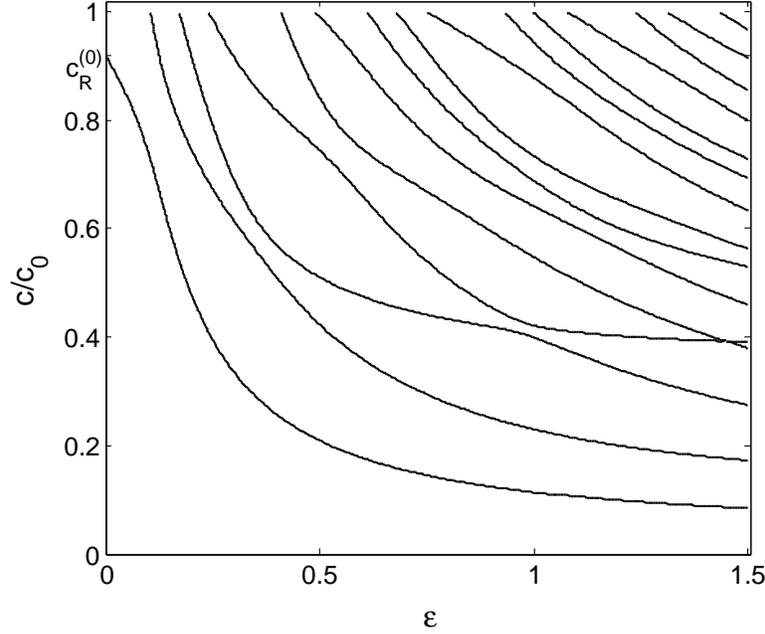| Material	E_1(GPa)	E_2(GPa)	G_{12}(GPa)	ν_{12}	ρ(g/cm^2)
Graphite/epoxy	30.00	0.750	0.375	0.2500	1.9
Carbon/epoxy	142.17	9.255	4.795	0.334	1.9
Glass/epoxy	38.49	9.367	3.414	0.2912	2.66
Steel	77.4	77.4	29.025	0.3333	7.9

Fig. 2. Phase velocity dispersion of some first modes for orthotropic
layered half-space given in Tab. 1.

To illustrate the method, we consider a model of orthotropic layered half-space consisting of three layers overlying a half-space with material properties given in Tab. 1 (see Liu and Xi, 2002, Chapter 3 [15]). All layers are assumed to have the same thickness. The dispersion curve of this model is shown in Fig. 2 with several first modes. The x-axis is the dimensionless frequency $\epsilon = kH$ where H is the sum of thicknesses of layers. The y-axis is the ratio of the phase velocity to the shear wave velocity of the half-space in the principal direction x_1-axis denoted by $c_0 = \sqrt{c_{66}^{(0)}/\rho^{(0)}}$. It can be seen from the figure that the threshold velocity of higher modes is c_0 as the result of condition given in (35).

Note that, when frequency gets smaller, the phase velocity approaches to the Rayleigh velocity of the half-space denoted by $c_R^{(0)}$. For the given material parameters of the half-space, this velocity could be computed by a formula given in Vinh and Ogden [14] as

$$\bar{c}_R^{(0)} = \frac{c_R^{(0)}}{c_0} = 0.9194. \tag{50}$$

This result is consistent with the our numerical calculation.

5. CONCLUSION

By using the Sylvester's theorem, the matrizant of a layer is expressed explicitly. Its equivalent form for the problem of finding dispersive curve of Rayleigh surface waves is then obtained. The equivalent form of the matrizant is always real and expressed in terms

of the dimensionless variables and parameters. The problem of the vertical wave number in the half-space being complex due to the anisotropy is handled by using the Vieta's formulas for the propagation equation to obtain the secular equation of Rayleigh waves in convenient form to numerical calculation. A numerical illustration for the dispersive curves of Rayleigh waves in an orthotropic layered half-space is presented.

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