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AN EFFICIENT NUMERICAL PROCEDURE FOR CALCULATING PERIODIC VIBRATIONS OF ELASTIC MECHANISMS

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Abstract. This paper proposes a numerical procedure based on the well-known Newmark integration method to determine initial conditions for the periodic solution of a system of linear differential equations with time-periodic coefficients. Based on this, steady-state periodic vibrations of mechanisms with elastic elements governed by linearized differential equations with time-periodic coefficients can be conveniently calculated. The proposed procedure is demonstrated by a dynamic model of a planar four-bar mechanism with the flexible coupler.

Keywords: Steady-state vibration, elastic mechanism, Newmark integration method, mode superposition method, dynamic stability.

1. INTRODUCTION

In high-speed machines, the motion of the transmission mechanisms is often composed of a combination of rigid body motion and elastic deformation [1, 2]. A review on the vibration and stability behavior of mechanisms with elastic links represents an update to earlier literatures surveys on this subject [3–5]. Many researchers have tried to represent the vibration of such mechanisms in a more and more realistic form. Up to now, the following models are used for modeling of flexible links of mechanisms: continuum models [6,7], lumped parameter models [8,9], finite element models [10–14].

In general, the mathematical formulation of this vibration problem is quite a complicated nonlinear differential equation, for which an exact solution is practically impossible. It is possible to calculate the transient solutions by the numerical methods. The linearized equations of motion of an elastic mechanism that performs the steady-state motion can then be expressed approximately by a set of n linear differential equations having time-periodic coefficients

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{C}(t)\dot{\mathbf{q}} + \mathbf{K}(t)\mathbf{q} = \mathbf{f}(t), \tag{1}$$

where $n \times n$ matrices $\mathbf{M}(t)$, $\mathbf{C}(t)$, $\mathbf{K}(t)$ and excitation force vector $\mathbf{f}(t)$ in Eq. (2) are time-periodic with the least period T [8–12]. For stability analysis, the homogeneous differential equation of Eq. (1) is considered

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{C}(t)\dot{\mathbf{q}} + \mathbf{K}(t)\mathbf{q} = \mathbf{0}. \tag{2}$$

This equation can also be represented in the form of Hill's or Mathieu's type equation as mentioned in [13, 15, 16]. Because the periodic vibrations are a commonly observed phenomenon of mechanisms in the steady-state motion, a number of methods and algorithms were developed for calculating periodic vibrations and dynamic stability analysis [15–29]. Periodic solutions of Eq. (1) can be found directly by other specialized techniques such as the harmonic balance method, the method of conventional oscillator, the WKB method, etc. [24–27]. The T-periodic solution can also be obtained directly and more conveniently by choosing an appropriate set of initial conditions for the vector of variable **q,** and then solving Eq. (1) within interval $[0\,,\,T]$ under these conditions using a numerical integration methods. For the last approach, an efficient numerical procedure was developed to estimate the initial conditions for the T-periodic solution based on the Runge-Kutta method, and tested by a number of applied problems [28, 29]. These studies indicated that the agreement between the experimental and calculating results with Runge-Kutta method is closer than the results calculated by the harmonic balance method and WKB method. To investigate the dynamic stability of elastic mechanisms, we can use Hill method [15] or numerical methods [28, 29].

In this paper, a numerical procedure based on the well-known Newmark integration method is developed to calculate steady-state periodic vibrations of elastic mechanisms governed by linearized differential equations with time-periodic coefficients. The proposed procedure is then demonstrated by a dynamic model of a four-bar mechanism with the flexible coupler.

2. NEWMARK PROCEDURE FOR FINDING INITIAL CONDITIONS OF PERIODIC VIBRATION OF LINEAR SYSTEMS

The procedure presented below for finding the T-periodic solution of Eq. (1) is based on the Newmark direct integration method. Firstly, the interval $[0\,,\,T]$ is now divided into m equal subintervals with the step-size $h = t_i - t_{i-1} = T/m$. We use notations $\mathbf{q}_i = \mathbf{q}(t_i)$ and $\mathbf{q}_{i+1} = \mathbf{q}(t_{i+1})$ to represent the solution of Eq. (1) at discrete times t_i and t_{i+1}, respectively. The T-periodic solution must satisfy the following conditions

$$\mathbf{q}(0) = \mathbf{q}(T),\;\; \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}(T),\;\; \ddot{\mathbf{q}}(0) = \ddot{\mathbf{q}}(T). \tag{3}$$

Based on the single-step integration method proposed by Newmark, we obtain the following approximation formulas [30, 31]

$$\mathbf{q}_{i+1} = \mathbf{q}_i + h\dot{\mathbf{q}}_i + h^2\left(\frac{1}{2} - \beta\right)\ddot{\mathbf{q}}_i + \beta h^2\ddot{\mathbf{q}}_{i+1}, \tag{4}$$

$$\dot{\mathbf{q}}_{i+1} = \dot{\mathbf{q}}_i + (1-\gamma)\,h\ddot{\mathbf{q}}_i + \gamma h\ddot{\mathbf{q}}_{i+1}, \tag{5}$$

where $\mathbf{q}_i,\ \dot{\mathbf{q}}_i,\ \ddot{\mathbf{q}}_i$ are approximations to the displacement, velocity and acceleration vectors at time t_i, β and γ are the constant parameters that define the method. The linear acceleration method, in which a linear variation of the acceleration in the time interval $[t_i\,,\,t_{i+1}]$ is assumed corresponds to the case $\gamma = 1/4$ and $\beta = 1/6$. The average acceleration method is defined by choosing $\gamma = 1/2, \beta = 1/4$. This case corresponds to the assumption that the acceleration is constant over the time interval $[t_i\,,\,t_{i+1}]$ and equal to $\frac{1}{2}\left(\ddot{\mathbf{q}}_i + \ddot{\mathbf{q}}_{i+1}\right)$ [31].

From Eq. (1) we have the following iterative computational scheme at time t_{i+1}

$$\mathbf{M}_{i+1}\ddot{\mathbf{q}}_{i+1} + \mathbf{C}_{i+1}\dot{\mathbf{q}}_{i+1} + \mathbf{K}_{i+1}\mathbf{q}_{i+1} = \mathbf{f}_{i+1}, \tag{6}$$

where $\mathbf{M}_{i+1} = \mathbf{M}\left(t_{i+1}\right), \mathbf{C}_{i+1} = \mathbf{C}\left(t_{i+1}\right), \mathbf{K}_{i+1} = \mathbf{K}\left(t_{i+1}\right)$ and $\mathbf{f}_{i+1} = \mathbf{f}\left(t_{i+1}\right)$.

In the next step, substitution of Eqs. (4) and (5) into Eq. (6) yields

$$\left(\mathbf{M}_{i+1}+\gamma h\mathbf{C}_{i+1}+\beta h^2\mathbf{K}_{i+1}\right)\ddot{\mathbf{q}}_{i+1}=\mathbf{f}_{i+1}-\mathbf{C}_{i+1}[\dot{\mathbf{q}}_i+(1-\gamma)h\ddot{\mathbf{q}}_i]-\mathbf{K}_{i+1}\left[\mathbf{q}_i+h\dot{\mathbf{q}}_i+h^2\left(\frac{1}{2}-\beta\right)\ddot{\mathbf{q}}_i\right]. \tag{7}$$

The use of Eqs. (4) and (5) leads to the prediction formulas for velocities and displacements at time t_{i+1}

$$\mathbf{q}^*_{i+1} = \mathbf{q}_i + h\dot{\mathbf{q}}_i + h^2\left(\frac{1}{2} - \beta\right)\ddot{\mathbf{q}}_i,\ \dot{\mathbf{q}}^*_{i+1} = \dot{\mathbf{q}}_i + (1-\gamma)\,h\ddot{\mathbf{q}}_i. \tag{8}$$

Eq. (8) can be expressed in the matrix form as

$$\begin{bmatrix} \mathbf{q}^*_{i+1} \\ \dot{\mathbf{q}}^*_{i+1} \end{bmatrix} = \mathbf{D}\begin{bmatrix} \mathbf{q}_i \\ \dot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_i \end{bmatrix}, \tag{9}$$

with

$$\mathbf{D} = \begin{bmatrix} \mathbf{I} & h\mathbf{I} & h^2\left(0.5-\beta\right)\mathbf{I} \\ \mathbf{0} & \mathbf{I} & \left(1-\gamma\right)h\mathbf{I} \end{bmatrix}, \tag{10}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix, $\mathbf{0}$ represents the $n \times n$ matrix of zeros. Eq. (7) can then be rewritten in the matrix form as

$$\ddot{\mathbf{q}}_{i+1} = \left(\mathbf{S}_{i+1}\right)^{-1}\mathbf{f}_{i+1} - \left(\mathbf{S}_{i+1}\right)^{-1}\mathbf{H}_{i+1}\begin{bmatrix} \mathbf{q}^*_{i+1} \\ \dot{\mathbf{q}}^*_{i+1} \end{bmatrix}, \tag{11}$$

where matrices $\mathbf{S}_{i+1}$ and $\mathbf{H}_{i+1}$ are defined by

$$\mathbf{S}_{i+1} = \mathbf{M}_{i+1} + \gamma h\mathbf{C}_{i+1} + h^2\beta\,\mathbf{K}_{i+1}, \tag{12}$$

$$\mathbf{H}_{i+1} = \begin{bmatrix} \mathbf{K}_{i+1} & \mathbf{C}_{i+1} \end{bmatrix}. \tag{13}$$

By substituting relationships (9) into (11) we find

$$\ddot{\mathbf{q}}_{i+1} = \left(\mathbf{S}_{i+1}\right)^{-1}\mathbf{f}_{i+1} - \left(\mathbf{S}_{i+1}\right)^{-1}\mathbf{H}_{i+1}\mathbf{D}\begin{bmatrix} \mathbf{q}_i \\ \dot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_i \end{bmatrix}. \tag{14}$$

From Eqs. (4), (5) and (8) we get the following matrix relationship

$$\begin{bmatrix} \mathbf{q}_{i+1} \\ \dot{\mathbf{q}}_{i+1} \\ \ddot{\mathbf{q}}_{i+1} \end{bmatrix} = \mathbf{T}\begin{bmatrix} \mathbf{q}^*_{i+1} \\ \dot{\mathbf{q}}^*_{i+1} \\ \ddot{\mathbf{q}}_{i+1} \end{bmatrix}, \tag{15}$$

where matrix $\mathbf{T}$ is expressed in the block matrix form as

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{I}\beta h^2 \\ \mathbf{0} & \mathbf{I} & \mathbf{I}\gamma h \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{16}$$

The combination of Eqs. (15), (9) and (14) yields a new computational scheme for determining the solution of Eq. (1) at the time t_{i+1} in the form

$$\begin{bmatrix} \mathbf{q}_{i+1} \\ \dot{\mathbf{q}}_{i+1} \\ \ddot{\mathbf{q}}_{i+1} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{D} \\ -\left(\mathbf{S}_{i+1}\right)^{-1} \mathbf{H}_{i+1}\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{q}_i \\ \dot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_i \end{bmatrix} + \mathbf{T} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \left(\mathbf{S}_{i+1}\right)^{-1} \mathbf{f}_{i+1} \end{bmatrix}. \tag{17}$$

In this equation, the iterative computation is eliminated by introducing the direct solution for each time step. Note that the matrices $\mathbf{T}$ and $\mathbf{D}$ are matrices of constants.

By setting

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{q}_i \\ \dot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_i \end{bmatrix}, \quad \mathbf{A}_{i+1} = \mathbf{T} \begin{bmatrix} \mathbf{D} \\ -\left(\mathbf{S}_{i+1}\right)^{-1} \mathbf{H}_{i+1}\mathbf{D} \end{bmatrix}, \quad \mathbf{b}_{i+1} = \mathbf{T} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \left(\mathbf{S}_{i+1}\right)^{-1} \mathbf{f}_{i+1} \end{bmatrix}. \tag{18}$$

Eq. (17) can then be rewritten in the following form

$$\mathbf{x}_i = \mathbf{A}_i \mathbf{x}_{i-1} + \mathbf{b}_i \ \ (i = 1, 2, \ldots, m). \tag{19}$$

Expansion of Eq. (19) for $i = 1$ to m yields

$$\begin{aligned} &\mathbf{x}_1 = \mathbf{A}_1 \mathbf{x}_0 + \mathbf{c}_1 \\ &\mathbf{x}_2 = \mathbf{A}_2 \mathbf{A}_1 \mathbf{x}_0 + \mathbf{c}_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\mathbf{x}_m = \left(\prod_{i=m}^{1} \mathbf{A}_i \right) \mathbf{x}_0 + \mathbf{c}_m \end{aligned} \tag{20}$$

where $\mathbf{c}_0 = \mathbf{0}, \ \mathbf{c}_1 = \mathbf{A}_1 \mathbf{c}_0 + \mathbf{b}_1, \ \mathbf{c}_2 = \mathbf{A}_2 \mathbf{c}_1 + \mathbf{b}_2, \ldots, \mathbf{c}_m = \mathbf{A}_m \mathbf{c}_{m-1} + \mathbf{b}_m$.

Using the condition of periodicity according to Eq. (3), the last equation of Eq. (20) yields a set of the linear algebraic equations

$$\left(\mathbf{I} - \prod_{i=m}^{1} \mathbf{A}_i \right) \mathbf{x}_0 = \mathbf{c}_m. \tag{21}$$

The solution of Eq. (21) gives us the initial value for the periodic solution of Eq. (1). Finally, the periodic solution of Eq. (1) with the obtained initial value can be calculated without difficulties using the computational scheme in Eq. (17).

3. PERIODIC TRANSVERSE VIBRATION OF THE FLEXIBLE COUPLER OF A PLANAR FOUR-BAR MECHANISM

One of the most challenging problems in dynamics of machines is the calculation of relative vibrations of elastic members. There are some important cases in which deformation plays an important role in the dynamic analysis. It happens, for instance, in lightweight high-speed mechanisms [3,4]. The occurrence of large base motion of a mechanism, in which all bodies are assumed to be rigid, can cause small relative vibrations of flexible links. Conversely, small vibrations of flexible links lead to deviations of its motion from desired base motion, i.e. to dynamic errors.

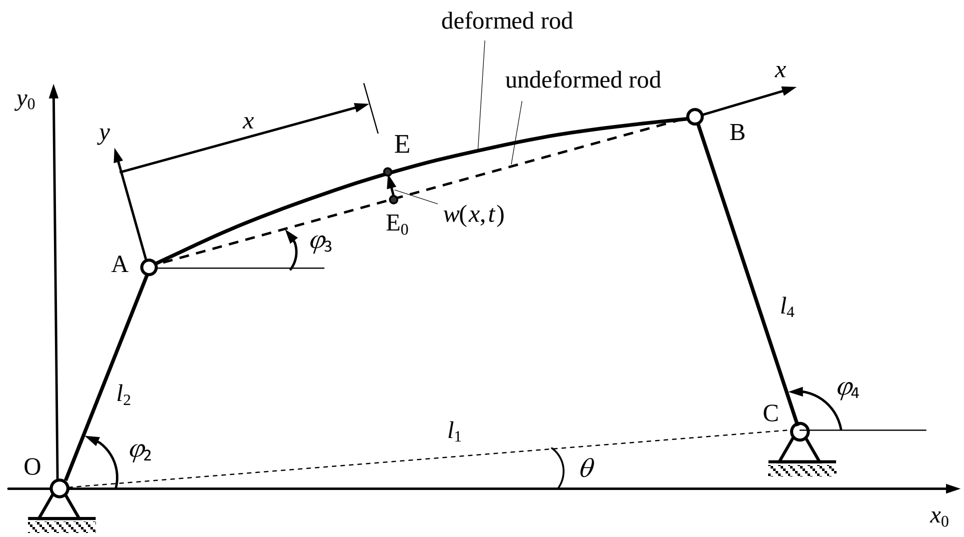

Fig. 1. Kinematic schema of a planar 4R-fourbar mechanism with flexible coupler

Planar four-bar mechanisms are widely used in reciprocating machines. Steady-state vibrations, dynamic stability analysis and vibration control of the flexible planar four-bar mechanism were the objective of a number of studies, e.g. [6, 7]. In this example, a planar revolute-jointed 4-link mechanism with the flexible coupler is considered to investigate the problem of relative transverse vibration of the coupler that becomes a serious factor at high speed. The kinematic diagram of the mechanism is depicted in Fig. 1. The origin of the ground-fixed coordinate frame $\{x_0, y_0\}$ coincides with joint 0 of input link 2. Assuming that the geometrical axis of the coupler (link 3) without deformation is a segment of the straight line that is chosen for x-axis. Neglecting the longitudinal oscillation, the objective of the investigation is to derive analytically the governing equation of the relative transverse vibration of the coupler in the direction of y-axis, and to apply the numerical procedure described in section 2 for finding a periodic solution of the obtained equation.

When the angular velocity Ω of input link 2 is assumed to be constant in the steady state, the loop equations can be expressed in the form

$$\begin{aligned} l_2 \cos \Omega t + l_3 \cos \varphi_3 - l_4 \cos \varphi_4 - l_1 \cos \theta = 0, \\ l_2 \sin \Omega t + l_3 \sin \varphi_3 - l_4 \sin \varphi_4 - l_1 \sin \theta = 0, \end{aligned} \tag{22}$$

where l_1, l_2, l_4 is the length of ground link 1, input link 2 and output link 4 respectively, l_3 the undeformed length of coupler 3. Rotation angles $\varphi_3(t), \varphi_4(t)$ of the coupler and output link as well as their time derivatives $\dot{\varphi}_3$, $\ddot{\varphi}_3$, $\dot{\varphi}_4$, $\ddot{\varphi}_4$ can be determined from Eq. (22)

using a recursive algorithm as Newton-Raphson method. Neglecting higher order non-linear terms, we obtain the partial differential equation of the relative transverse vibration for the coupler in the following simplified form [32]

$$\begin{gathered}\frac{\partial^4 w}{\partial x^4}+\alpha\frac{\partial^5 w}{\partial x^4\partial t}-\frac{\rho}{E}\frac{\partial^4 w}{\partial x^2\partial t^2}-\left[f_4(t)+f_3(t)-f_2(t)\frac{x^2}{2}\right]\frac{\partial^2 w}{\partial x^2}-\left[f_3(t)-f_2(t)x\right]\frac{\partial w}{\partial x}\\ +\frac{\mu}{EI}\frac{\partial^2 w}{\partial t^2}+\frac{c_y}{EI}\frac{\partial w}{\partial t}-f_2(t)w=-f_0(t)-f_1(t)\end{gathered}\tag{23}$$

where functions $f_i(t)$ for $i=0,1,\ldots,5$ are defined by

$$f_0(t)=\frac{\mu}{EI}\left[g\cos\varphi_3-l_2\Omega^2\sin(\Omega t-\varphi_3)\right]+\frac{c_y l_2}{EI}\Omega\cos(\Omega t-\varphi_3),\tag{24}$$

$$f_1(t)=\frac{1}{EI}\left(c_y\dot\varphi_3+\mu\ddot\varphi_3\right),\tag{25}$$

$$f_2(t)=\frac{\mu}{EI}\dot\varphi_3^2,\tag{26}$$

$$f_3(t)=\frac{\mu}{EI}\left[g\sin\varphi_3-l_2\Omega^2\cos(\Omega t-\varphi_3)\right]+\frac{c_x l_2}{EI}\Omega\sin(\Omega t-\varphi_3),\tag{27}$$

$$\begin{aligned}f_4(t)=&\frac{J_C\ddot\varphi_4+m_4gs_4\cos(\varphi_4+\alpha_4)+M_e}{EI\,l_4\sin(\varphi_4-\varphi_3)}+\\&+\cot(\varphi_4-\varphi_3)\left[\frac{\rho}{E}\ddot\varphi_3+f_0(t)\frac{l_3}{2}+f_1(t)\frac{{l_3}^2}{3}\right]+f_2(t)\frac{{l_3}^2}{3}-f_3(t)l_3.\end{aligned}\tag{28}$$

The boundary conditions at $x=0$ and $x=l$ must be specified for the solution of Eq. (23). These boundary conditions are given by

$$w(0,t)=\frac{\partial^2}{\partial x^2}w(0,t)=0,\quad w(l,t)=\frac{\partial^2}{\partial x^2}w(l,t)=0.\tag{29}$$

Using the mode superposition principle, a solution of Eq. (23) corresponding to the boundary conditions (29) is assumed in the form

$$w(x,t)=\sum_{i=1}^{n_f}q_i(t)\sin\frac{i\pi x}{l_3},\tag{30}$$

where $q_i(t)$ are generalized coordinates to be determined. Eq. (23) takes the following form

$$\begin{aligned}&\sum_{i=1}^{n_f}\left\{\left[\frac{\rho}{E}\left(i\frac{\pi}{l}\right)^2+\frac{\mu}{EI}\right]\ddot q_i+\left[\alpha\left(i\frac{\pi}{l}\right)^4+\frac{c_y}{EI}\right]\dot q_i+\left[\left(i\frac{\pi}{l}\right)^4+\left(i\frac{\pi}{l}\right)^2 f_4(t)-f_2(t)\right]q_i\right\}\sin\left(i\frac{\pi}{l}x\right)\\&+\sum_{i=1}^{n_f}\left(i\frac{\pi}{l}\right)^2 f_3(t)q_i x\sin\left(i\frac{\pi}{l}x\right)-\sum_{i=1}^{n_f}\left(i\frac{\pi}{l}\right)^2\frac{1}{2}f_2(t)q_i x^2\sin\left(i\frac{\pi}{l}x\right)-\sum_{i=1}^{n_f}\left(i\frac{\pi}{l}\right)^2 f_3(t)q_i\cos\left(i\frac{\pi}{l}x\right)\\&+\sum_{i=1}^{n_f}\left(i\frac{\pi}{l}\right)^2 f_2(t)q_i x\cos\left(i\frac{\pi}{l}x\right)=-f_0(t)-f_1(t)x.\end{aligned}\tag{31}$$

Multiplying both sides of Eq. (31) by $\sin(i\pi x/l)$ and then integrating the newly obtained equation from 0 to l, we get the system of ordinary differential equations

$$\begin{aligned}&\left[\frac{\rho}{E}\left(i\frac{\pi}{l}\right)^2+\frac{\mu}{EI}\right]\ddot{q}_i+\left[\alpha\left(i\frac{\pi}{l}\right)^4+\frac{c_y}{EI}\right]\dot{q}_i+\left[\left(i\frac{\pi}{l}\right)^4+\left(i\frac{\pi}{l}\right)^2 f_4(t)+j^2\frac{\pi^2}{2l}f_3(t)+\right.\\&\left.-\left(\frac{1}{4}+j^2\frac{\pi^2}{6}\right)f_2(t)\right]q_i-\sum_{\substack{i=1,i\neq j\\ i+j=2k+1}}^{n_f}\alpha_{ij}\left[\frac{2}{l}f_3(t)-f_2(t)\right]q_i-\sum_{\substack{i=1,i\neq j\\ i+j=2k+1}}^{n_f}\alpha_{ij}f_2(t)q_i=h_j(t),\end{aligned} \tag{32}$$

where we use indexes $j=0,1,2,\ldots,n_f$ and $k=1,2,\ldots$, and functions h_j,α_{ij} are

$$h_j=\begin{cases}\dfrac{2l}{j\pi}f_1(t) & \text{for } j=2k\\ -\dfrac{1}{j\pi}\left[4f_0(t)+2l\,f_1(t)\right] & \text{for } j=2k+1\end{cases} \tag{33}$$

$$\alpha_{ij}=\left[\frac{1}{(i-j)^2}-\frac{1}{(i+j)^2}\right]\ \text{for } i\neq j. \tag{34}$$

Eq. (32) can be rewritten in the compact matrix form as

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{B}\dot{\mathbf{q}}(t)+\mathbf{C}(\Omega t)\mathbf{q}(t)=\mathbf{h}(\Omega t), \tag{35}$$

where $\mathbf{M}$, $\mathbf{B}$, $\mathbf{C}$ are $n_f\times n_f$ matrices having coefficients as

$$m_{ij}=\left[\frac{\rho}{E}\left(i\frac{\pi}{l}\right)^2+\frac{\mu}{EI}\right]\delta_{ij},\quad b_{ij}=\left[\frac{\rho}{E}\left(i\frac{\pi}{l}\right)^4+\frac{c_y}{EI}\right]\delta_{ij}, \tag{36}$$

$$c_{ij}=\begin{cases}\left(i\dfrac{\pi}{l}\right)^4+\left(i\dfrac{\pi}{l}\right)^2 f_4(t)+\dfrac{j^2\pi^2}{2l}f_3(t)-\left(\dfrac{1}{4}+\dfrac{j^2\pi^2}{6}\right)f_2(t) & \text{for } i=j\\ -\alpha_{ij}f_2(t) & \text{for } i\neq j,\ i+j=2k\\ -\alpha_{ij}\left[\dfrac{2}{l}f_3(t)-f_2(t)\right] & \text{for } i\neq j,\ i+j=2k+1\end{cases} \tag{37}$$

with function $\delta_{ij}=\begin{cases}1 & \text{for } i=j\\ 0 & \text{for } i\neq j.\end{cases}$

The initial value for the periodic solution of Eq. (35) are then determined using Eq. (21), in which parameters $\gamma=1/4$, $\beta=1/6$ and step-size $h=10^{-4}(\text{sec.})$ were used for the numerical calculation. The calculating parameters are given in Tab. 1. The obtained results of the maximal value of coordinates q_i in Eq. (35) for a range of rotating speeds of the crank listed in Tab. 2. It can be clearly seen that the transverse vibration response of the rod can be closely approximated by the first mode since the higher modes are insignificant. The result for transverse vibrations of the connecting rod corresponding to different angular velocities of the crank are shown in Fig. 2. In addition, Fig. 3 shows a spectrum calculated by FFT that includes harmonic components of the rotating frequency, such as Ω and 2Ω. The spectrum indicates that the connecting rod performs stationary periodic transverse vibrations only.

Table 1. Calculating parameters

Parameters	Value	Parameters	Value
l_1(m)	1.0	ρ (kg m^{-3})	7860
l_2(m)	0.05	I(m^4)	45×10^{-10}
l_3(m)	0.8	c_x (kg m^{-1}s^{-1})	0.001
l_4(m)	0.8	c_y(kg m^{-1}s^{-1})	0.001
s_4(m)	0.4	J_C(kg m^2)	3.35
m_3(kg)	3.74	α (s)	10^{-4}
m_4(kg)	15	α_4(rad.)	0
E(Nm^{-2})	2.1×10^{11}	θ(rad.)	0
F(m^2)	6×10^{-4}		

Table 2. Calculating results for three modes $(n_f = 3)$

Crank speed (rpm)	max $\lvert q_1\rvert$(mm)	max $\lvert q_2\rvert$(mm)	max $\lvert q_3\rvert$(mm)
600	0.2750	0.0090	0.0011
900	0.6309	0.0204	0.0025
1200	1.1583	0.0364	0.0046
1500	1.8704	0.0572	0.0072

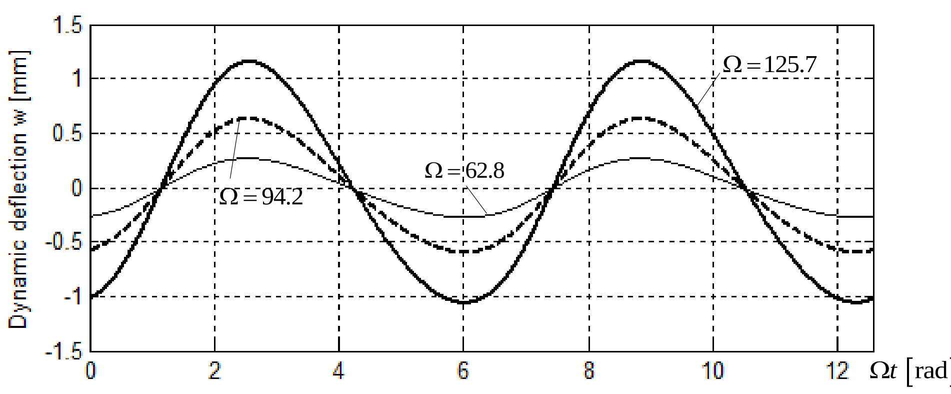

Fig. 2. Midpoint deflection of the flexible coupler versus crank rotating angle

To verify the correctness of the proposed procedure using Newmark equation, the procedure using the fourth-order Runge-Kutta formula presented in [29] is also applied to solve the same problem. The obtained results with both approaches are identical, but

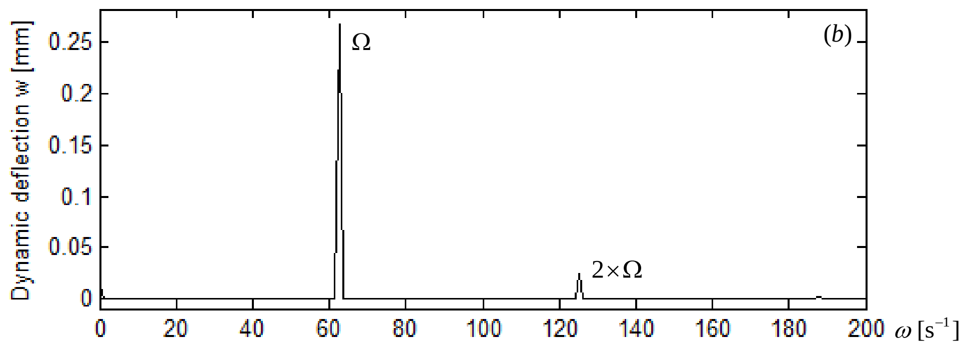

Fig. 3. Midpoint deflection of the flexible coupler at $\Omega = 62.8$ s^{-1}
(a) versus time and (b) versus angular frequency

the computation time with the proposed procedure is greatly reduced, especially in the cases of large number of time steps.

4. CONCLUDING REMARKS

In this study, the problem on the calculation of steady-state periodic vibrations of elastic transmission mechanisms is addressed. The differential equations of motion of the mechanism is established and linearized to obtain a system of linear differential equations having time-varying coefficients, known as a parametric vibration system.

A numerical algorithm based on Newmark integration method is proposed to determine initial conditions for the periodic solution of a system of linear differential equations with time-periodic coefficients. Using the obtained initial conditions, the periodic solution can be found by a common numerical integration method. For linear systems, this numerical procedure is simpler and easier to implement than one based on the fourth-order Runge-Kutta algorithm which was presented in [29,32].

Although this approach has been applied to only one example of a flexible four-bar mechanism, but the obtained results are wider applicability to more complicated transmission systems that perform the steady state motions. The problem of vibration control of elastic mechanisms, as addressed in [33,34], using the periodic solution of the linearized vibration equations will be the subject of our future investigation.

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