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LOCKING PHENOMENA IN FINITE ELEMENT ANALYSIS OF DEEP BEAM AND REMOVAL METHOD

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Abstract. Since earlier 30 years ago, Finite Element Methods (FEM) has become an indispensable tool of engineers for analysis mechanical behaviour of structures. Generally displacement models are used in most usual problems. Under certain material conditions, such elements may provide inaccurate results and exhibit slow convergence. Locking phenomena are the cause of these problems and this situation could limit the computation of many kinds of structures. This paper presents a deep beam element based upon mixed formulation, which may illuminate these locking phenomena.

1. INTRODUCTION

Finite Element Methods is based upon the virtual work principle ("displacement models") [1], it is easy for computation, however it may provide inaccurate results and exhibit slow convergence. This phenomenon is characterized by a severe underestimation of the displacements, e.g. the structural response is too stiff. Since the late 1970s the term **locking** is employed.

The locking phenomena have some types, volumetric locking occurs in modeling nearly incompressible and incompressible materials, shear locking in bending dominated problems. Thereto, there are some other types of locking, such as membrane locking, thickness locking, in the framework of the paper, these types of locking are not discussed.

By the effect of locking phenomena, FEM may provide inaccurate results and exhibit slow convergence. Therefore, so many methods are used for locking removal. Selective Reduced Integration method (SRI) has also been proved effective in overcoming the locking, e.g., by Naylor [13], Hughes, Malkus and Hughes [11], and others. It may produce rank deficiency [12], leading to the so-called zero-energy modes and unstable solutions by many situations. The Enhanced Assumed Strain method has been developed by many researchers, e.g., Simo and Rifai [14], Simo and Armero [15], Andelfinger and Ramn [7], and others. However, the computation is too complicated.

The mixed formulation for the nearly incompressible and incompressible materials was first introduced by Herrmann in 1965 [8], then various mixed formulations were applied to rubber-like elasticity, plasticity and incompressible flow problem. The **new mixed formulation** in this work which will be discussed in this paper be applied a mathematical formulation for giving more accurate numerical solutions.

2. WHAT IS LOCKING?

Unfortunately, a unique, rigorous definition of locking does not seem to exist [3]. From a most simple an general point of view one could state that:

Locking means the effect of a reduced rate of convergence in dependence of a "critical" parameter. In the limit of the parameter being infinite, rate of convergence may be zero.

For example, in the case of transverse shear locking of plate elements, this parameter is the slenderness of the plate, in volumetric locking it is the bulk modulus.

2.1 Volumetric locking

The volumetric locking occurs normally in traditional displacement Finite Element Analysis (FEA) of 3D, axisymetric and plane problem. In fact, the effect of volumetric locking is not at all when ν=0.0. Therefore, the plane stress problems are also influenced by a slight of volumetric locking.

When Poisson's ratio $\nu \rightarrow 0.5$, the bulk modulus:

$$\kappa = \frac{E}{(3-6\nu)}, \tag{2.1}$$

tends to ∞. The ill-conditioning will occur in traditional displacement FEA of 3D, axisymetric and plane strain problems. This phenomenon is called Volumetric Locking.

An infinite bulk modulus means that any deformation preserves the volume of infinitesimal portions of the body, in other words, the material behaves incompressible.

In solid mechanics this effect can occur e.g. for rubber materials, but also for metals in the range of plastic deformations.

The corresponding constraint when ν=0.5 in elastic and plastic deformation

$$\varepsilon_v = u_{,x} + v_{,y} + w_{,z} = 0. \tag{2.2}$$

If the approximation spaces (i.e. displacement field) used for the formulation of a finite element are not well balanced, the corresponding constraint can not be vanished.

2.2 Shear locking

The shear locking can occur in shear deformation beam, plate and shell elements which is subjected to a pure bending situation.

From the classical theory of elasticity, the shear strain must vanish:

$$\gamma_{xy} = u_{,y} + v_{,x} = 0, \tag{2.3}$$

where $u_{,y} = \partial u/\partial y$, $v_{,x} = \partial v/\partial x$

Consider a 4-node element, with the field function defined as:

$$u = a_0 + a_1x + a_2y + a_3xy, \tag{2.4}$$

$$v = b_0 + b_1x + b_2y + b_3xy, \tag{2.5}$$

we can check the shear strain γ_{xy}:

$$\gamma_{xy} = u_{,y} + v_{,x} = a_2 + b_1 + a_3x + b_3y, \tag{2.6}$$

is generally non-zero, since the presence of inconsistent term a_3 and b_3. This does not satisfy with the coressponding constrain γ_{xy}=0.

Hence, it causes the so-called shear locking for the traditional displacement FEA.

2.3 The effect of locking

The functional of the principle of minimum potential energy principle is written as

$$\Pi_P = \frac{1}{2}\int_\Omega \varepsilon^T D \varepsilon_d \Omega - W, \tag{2.7}$$

where ε is the strain vector, D is the elasticity matrix and W is the term for body forces and boundary conditions. The integration is taken over the area of the whole domain [1].

In the case of plane strain, the Π_P can be written as

$$\Pi_P = G\int_\Omega \left\{ \left(\varepsilon_x^2 + \varepsilon_y^2\right) + \frac{1}{2}\gamma_{xy}^2 + \frac{\nu}{1-2\nu}\varepsilon_v^2 \right\} d\Omega - W, \tag{2.8}$$

with
$$\varepsilon_v = \varepsilon_x + \varepsilon_y. \tag{2.9}$$

The volumetric locking occurs when $\nu \to 0.5$ or $\nu = 0.5$.

Obviously, when Π_P is applied the numerical analysis, the denominator $(1-2\nu)$ contained in it will make troubles as $\nu \to 0.5$ or $\nu = 0.5$.

In the case of plane stress, the Π_P can be written as

$$\Pi_P = \int_\Omega \left\{ \frac{E}{2(1-\nu^2)} \left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y\right) + \frac{G}{2}\gamma_{xy}^2 \right\} d\Omega - W. \tag{2.10}$$

E and G are the Young's and shear module and

$$\Pi_P = \Pi_E + \Pi_G, \tag{2.11}$$

where Π_E is the strain energy contribution from the normal strain and Π_G comes from the shear strains.

A **deep beam** will illustrate the accuracy of the element for plane stress structure [5]. From the classical pure bending theory of elasticity, the shear strain must equal zero [6] and the shear energy Π_G must vanish. However, the shear strain is generally non-zero in traditional displacement FEA, that means the shear energy Π_G can not vanish. This is the effect of shear locking in traditional displacement FEA.

3. MIXED FORMULATION

3.1 Introduction

A new mixed formulation is proposed in this paper, it is based on the virtual work principle to associate with the locking consraints. In addition, the identity 1≡ β+(1-β) is also used for the variation. In all the previous formulations in this part we use an **irreducible formulation** [2], using the displacement **u** as the primary variable.

3.2 Mixed variational principle for plane stress elasticity (MVP)

For plain stress problems, they are two-dimentional, the thickness is assumed to be unity for simplicity.

As the starting point, consider the well-known the virtual work principle [4] which were written as

$$\int_\Omega \delta\varepsilon^T D\varepsilon_d\Omega - \int_\Omega \bar{F}\,\delta u_d\Omega - \int_{S_p} \bar{P}\,\delta u_d S = 0, \tag{3.1}$$

where Ω is the domain of the body considered, subjected to the body force $\vec{F}$.

The total boundary of Ω is $S = S_p + S_u$; on S_p is prescribed the surface traction $\vec{P}$, while on S_u the prescribed displacement $\vec{u}$.

D is an elasticity matrix containing the appropriate material properties.

$$D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}, \tag{3.2}$$

E is the Young's modulus and ν Poisson's ratio.

u and ε are the displacement vector and the strain vector.

$$u = [uv]^T, \tag{3.3}$$

$$\varepsilon = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]^T = \left[\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]^T, \tag{3.4}$$

and ε_v is the volume strain:

$$\varepsilon_v = \varepsilon_x + \varepsilon_y \quad \text{in} \quad V. \tag{3.5}$$

The corresponding trace of stress tensor σ is denoted by p:

$$p = \sigma_x + \sigma_y \quad \text{in} \quad V. \tag{3.6}$$

ε_v and p are related by the volumetric constitutive law

$$\varepsilon_v - \frac{1-\nu}{E}p = 0 \quad \text{or} \quad p - \frac{E}{1-\nu}\varepsilon_v = 0 \quad \text{in} \quad V. \tag{3.7}$$

γ_{xy} and τ_{xy} are related by the shear constitutive law

$$\gamma_{xy} - \frac{1}{G}\tau_{xy} = 0 \quad \text{or} \quad \tau_{xy} - G\gamma_{xy} = 0. \tag{3.8}$$

G is the shear modulus

$$G = \frac{E}{2(1+\nu)}. \tag{3.9}$$

Matrix D is splited into three parts

$$D = G\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + G\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2\nu G}{1-\nu}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.10}$$

$$D = GD_{d1} + GD_{d2} + \frac{2\nu G}{1-\nu} D_v. \tag{3.11}$$

The equilibrium equation (3.1) is rewritten using (3.10)

$$G\int_\Omega \delta\varepsilon^T D_{d1}\varepsilon d\Omega + G\int_\Omega \delta\varepsilon^T D_{d2}\varepsilon d\Omega + \frac{2\nu G}{1-\nu}\int_\Omega \delta\varepsilon^T D_v \varepsilon d\Omega$$
$$- \int_\Omega \overline{F}\delta u d\Omega - \int_{S_p} \overline{P}\delta u dS = 0, \tag{3.12}$$

$$G\int_\Omega \delta\varepsilon^T D_{d1}\varepsilon d\Omega + G\int_\Omega \delta\varepsilon^T I_1 \gamma_{xy} d\Omega + \frac{2\nu G}{1-\nu}\int_\Omega \delta\varepsilon^T I_0 \varepsilon_v d\Omega$$
$$- \int_\Omega \overline{F}\delta u d\Omega - \int_{S_p} \overline{P}\delta u dS = 0, \tag{3.13}$$

where

$$I_0 = [1 \quad 1 \quad 0]^T, \tag{3.14}$$
$$I_1 = [0 \quad 0 \quad 1]^T. \tag{3.15}$$

The identity $1 \equiv \beta + (1-\beta)$, where β is a free parameter function of x_i over Ω, called the slitting factor, should be used to split the troubling coefficient $1/(1-\nu)$ into two parts [9]:

$$\frac{1}{1-\nu} \equiv \beta\frac{1}{1-\nu} + (1-\beta)\frac{1}{1-\nu}. \tag{3.16}$$

Since the splitting factor is arbitrary, let

$$\beta \quad = \quad 1-\nu \tag{3.17}$$

(3.16) become

$$\frac{1}{1-\nu} \equiv 1 + \nu\frac{1}{1-\nu} \tag{3.18}$$

The equilibrium equation (3.13) is rewritten using (3.18)

$$G\int_\Omega \delta\varepsilon^T D_{d1}\varepsilon_d d\Omega + G\int_\Omega \delta\varepsilon^T I_1\gamma_{xy} d\Omega + 2\nu G\int_\Omega \delta\varepsilon^T I_0\varepsilon_v d\Omega$$
$$+ \frac{2\nu^2 G}{1-\nu}\int_\Omega \delta\varepsilon^T I_0\varepsilon_v d\Omega - \int_\Omega \overline{F}\delta u d\Omega - \int_{S_p} \overline{P}\delta u dS = 0, \tag{3.19}$$

$$\int_\Omega \delta\varepsilon^T D_\beta \varepsilon_d d\Omega + G\int_\Omega \delta\varepsilon^T I_1\gamma_{xy} d\Omega + \frac{2\nu^2 G}{1-\nu}\int_\Omega \delta\varepsilon^T I_0\varepsilon_v d\Omega$$
$$- \int_\Omega \overline{F}\delta u_d \Omega - \int_{S_p} \overline{P}\delta u dS = 0, \tag{3.20}$$

where

$$D_\beta = GD_{d1} + 2\nu GD_v = 2G \begin{bmatrix} 1+\nu & \nu & 0 \\ \nu & 1+\nu & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.21}$$

Now the equilibrium equation (3.20) is rewritten using (3.7) and (3.8), treating p and τ_{xy} as an independent variables, as

$$\int_\Omega \delta\varepsilon^T D_\beta \varepsilon_d \Omega + \int_\Omega \delta\varepsilon^T I_1 \tau_{xy} d\Omega + \frac{\nu^2}{1+\nu}\int_\Omega \delta\varepsilon^T I_0 p_d \Omega$$
$$- \int_\Omega \overline{F}\delta u_d \Omega - \int_{S_p} \overline{P}\delta u_d S = 0 \tag{3.22}$$

and in addition we shall impose a weak form of (3.7) and (3.8), i.e.,

$$\int_\Omega \delta p \left[\frac{\nu^2}{1+\nu} I_0 \varepsilon - \frac{\nu^2(1-\nu)}{E(1+\nu)} p\right] d\Omega = 0, \tag{3.23}$$

$$\int_\Omega \delta\tau_{xy} \left[I_1 \varepsilon - \frac{1}{G}\tau_{xy}\right] d\Omega = 0. \tag{3.24}$$

In this paper u is expanded in independent bilinear functions in four-node quadrilateral elements show in Fig. 1:

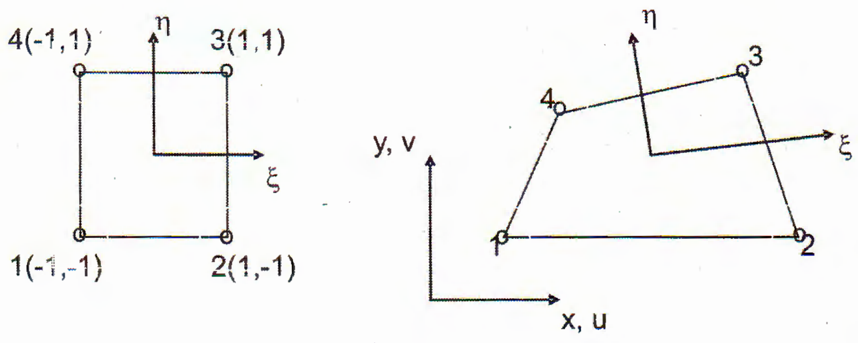

Fig. 1. Physical and reference planes for 4-node quadrilateral.

$$u = \sum_{i=1}^{NEL} N_i^e a_i^e, \tag{3.25}$$

$$\varepsilon = \sum_{i=1}^{NEL} B_i a_i^e, \tag{3.26}$$

$$a^e = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4]^T, \tag{3.27}$$

$$N^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e \end{bmatrix}. \tag{3.28}$$

The element shape functions are given by

$$N_k^e = \frac{1}{4}(1+\xi_k\xi)(1+\eta_k\eta) \quad k = 1,2,3,4, \tag{3.29}$$

$$B = \overline{\nabla} N, \tag{3.30}$$

where

$$\overline{\nabla} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}, \tag{3.31}$$

where NEL is the total number of elements, $\xi_k = \pm 1$ and $\eta_k = \pm 1$ are the coordinates of node k in the reference plane (ξ, η).

The equations (3.22), (3.23) and (3.24) gives the mixed approximation in the form

$$\begin{bmatrix} A & C & H \\ C^T & -V & 0 \\ H^T & 0 & -L \end{bmatrix} \begin{Bmatrix} a^e \\ p \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}, \tag{3.32}$$

where

$$\boldsymbol{A} = \int_\Omega B^T D_\beta B d\Omega, \tag{3.33}$$

$$\boldsymbol{C} = \int_\Omega \frac{\nu^2}{1+\nu} B^T I_o d\Omega, \tag{3.34}$$

$$\boldsymbol{V} = \int_\Omega \frac{\nu^2(1-\nu)}{E(1+\nu)}{}_d\Omega, \tag{3.35}$$

$$\boldsymbol{H} = \int_\Omega B^T I_1 d\Omega, \tag{3.36}$$

$$\boldsymbol{L} = \int_\Omega \frac{1}{G} d\Omega, \tag{3.37}$$

$$\boldsymbol{f}_1 = \int_\Omega N^T \overline{F} d\Omega + \int_\Omega N^T \overline{P} d\Omega, \tag{3.38}$$

$$\boldsymbol{f}_2 = 0, \tag{3.39}$$

$$\boldsymbol{f}_3 = 0. \tag{3.40}$$

All the integrals above are carried out in full integrating rule by 2×2 Gausian quadrature in the reference plane. Solving equation (3.32) gives the nodal displacement vector $\mathbf{a}^e$, τ_{xy} and the trace vector $\mathbf{p}$ (double hydrostatic pressure in plane stress elasticity).

4. NUMERICAL EXAMPLES

4.1 Patch test

The Patch Test has been originally proposed in the mid of sixties as a simple means to prove the convergence of an element. Beside the theoretical analysis there is also the possibility of the numerical verification. The below example is the numerical verification of the Patch Test for plane stress element that was presented by Macneal R H in 1985 [10].

- The geometry

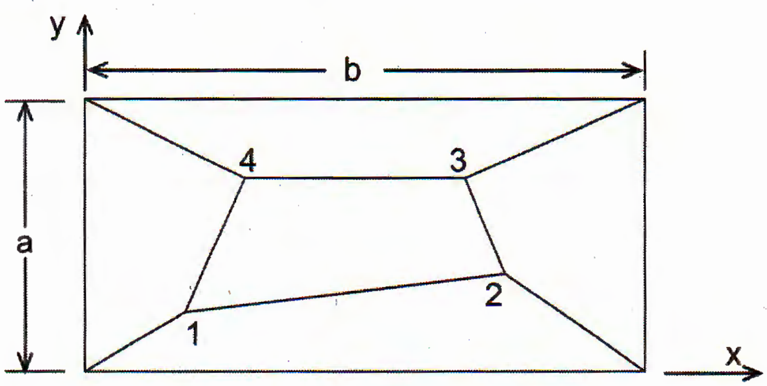

Fig. 2. The geometry of plane stress problem

Size $a = 0.12$; $b = 0.24$; thickness = 0.001.

- Element meshing

Table 1. The coordinates of nodes

Nodes	Coordinates	
	X	Y
1	0.04	0.02
2	0.18	0.03
3	0.16	0.08
4	0.18	0.08

- Material

$$E = 1.0 \times 10^6; \quad \nu = 0.25.$$

- Prescribed displacements of 4 nodes at 4 corners of the plate

$$u = 10^{-3}(x + \frac{y}{2}), \quad v = 10^{-3}(\frac{x}{2} + y).$$

- Theoretical solution

$$\varepsilon_x = \varepsilon_y = \varepsilon_{xy} = 10^{-3}; \quad \sigma_x = \sigma_y = 1333; \quad \tau_{xy} = 400 \quad \text{(Macneal R H, 1985) [10]}.$$

- Numerical results and discussion

Table 2. The results of MVP element

Element	$\varepsilon_x = \varepsilon_y = \varepsilon_{xy}$	$\sigma_x = \sigma_y$	τ_{xy}
1	10^{-3}	1333	400
2	10^{-3}	1333	400
3	10^{-3}	1333	400
4	10^{-3}	1333	400
5	10^{-3}	1333	400
Reference	10^{-3}	1333	400

Constant stress state is satisfactory. MVP element really pass the patch test.

4.2 Two-dimensional plan stress problem

A cantilever beam with the dimensions shown in Fig. 3 will illustrate the accuracy of the element for plane stress structures. Results for two different loading conditions and for some the different meshes are shown in Table 3. They are compared with exact solution and with the traditional displacement FEM.

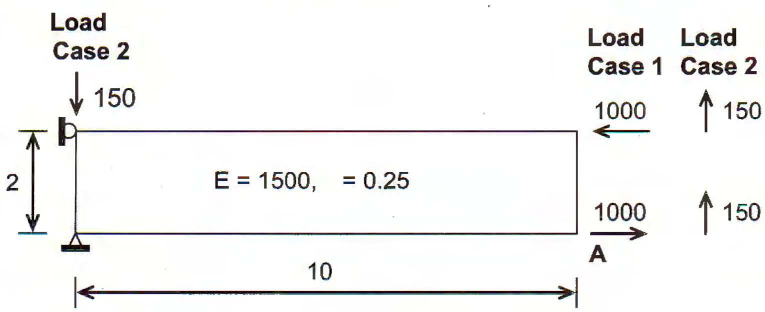

Fig. 3. Cantilever beam – plane stress

Table 3. Results of for problem in Fig. 3

Number of element	Displacement u_2 at A			
	Load case 1		Load case 2	
	u_{dispm}	u_{MVP}	u_{dispm}	u_{MVP}
1 element	9.03	100	9.27	77.5
5 elements	68.2	100	70.0	101.5
10 elements	85.7	100	88.0	102.25
15 elements	90.0	100	92.4	102.4
20 elements	91.6	100	94.0	102.44
100 elements	93.66	100	96.15	102.5
Beam theory	**100**		**103**	

$(\cdot)_{\text{dispm}}$ stands for displacement FEM and $(\cdot)_{\text{MVP}}$ stands for MVP element.

Obviously, from the diagram the shear locking effects the rate of convergence with traditional displacement FEM. The results of MVP element avoid the shear locking, this results are obtained by Matlab-Code.

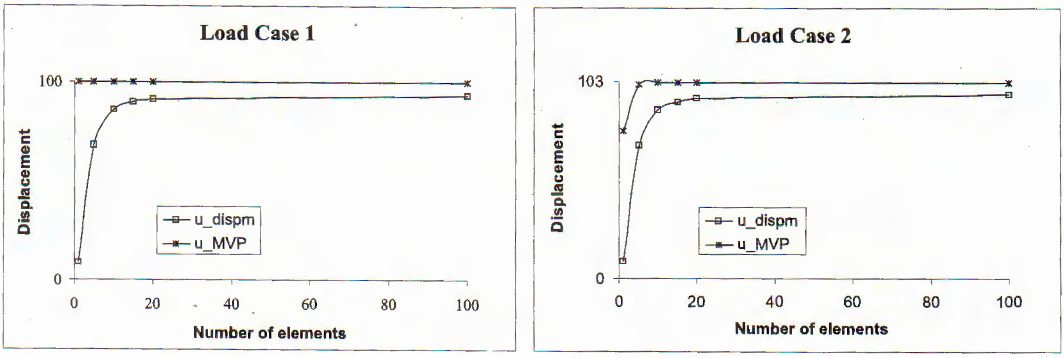

Fig. 4. The diagram of displacement with load case 1 and load case 2

In fact, the effect of volumetric locking is not at all when $\nu = 0.0$. Therefore, the plane stress problems are not only effected by shear locking, but also effected by a slight of volumetric locking. So both contraints of volumetric locking and shear locking are associated with the mixed variational principle. The three fields ($\mathbf{u}$-$\mathbf{p}$-τ_{xy}) mixed formulation is obtained.

5. CONCLUSIONS

In fact, the traditional displacement FEM has been ensured convergence to the correct result. Therefore, the simple way of locking removal is to employ a rather fine mesh, the correct result will be obtained. However, the amount of computation is too large.

The traditional displacement FEA is based on the the virtual work principle. Generally, this method has not any constraint, in order to remove locking, the constraint of volumetric locking or the constraint of shear locking must add to the problem. The mixed formulation is mainly the method to solve the problem of the virtual work principle and the constraints.

The mixed models in this paper are based on the virtual work principle to associate with the corresponding contraints and using the identity $1 \equiv \beta + (1 - \beta)$ for the variation. Above tests resulted in the following conclusions:

· This method gives more accurate numerical solutions than those based on the traditional displacement FEM under the same number of elements.

It avoids both locking phenomena: volumetric locking and shear locking.

However, due to the limited time, the mixed models are only applied to the 2-D elasticity problems. The mixed models apply to 3-D, plate, shell and plasticity deformation problems... will be found in the other researches.

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HIỆN TƯỢNG "NGHẼN" TRONG PHÂN TÍCH PHẦN TỬ HỮU HẠN CỦA "DẦM NGẮN" VÀ CÁCH LOẠI TRỪ

Từ hơn 30 năm trước đây, Phương pháp Phần tử Hữu hạn đã trở thành một công cụ không thể thiếu được của người kỹ sư trong việc phân tích ứng xử cơ học của kết cấu. Thông thường mô hình chuyển vị được sử dụng trong hầu hết các bài toán. Dưới những điều kiện nào đó, những phần tử này có thể cho kết quả không chính xác và sự hội tụ chậm. Những hiện tượng "nghẽn" là nguyên nhân của vấn đề, và trạng thái này có thể hạn chế sự tính toán của nhiều loại kết cấu. Bài báo này trình bày một phần tử "dầm ngắn" đặt trên cơ sở của mô hình hỗn hợp, nhằm làm sáng tỏ những hiện tượng "nghẽn" này .