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ANALYSIS OF SOME NONLINEAR DETERMINISTIC OSCILLATORS USING EXTENDED AVERAGED EQUATION APPROACH

Ninh Quang Hai
Hanoi Architectural University

Abstract. The paper presents an application of extended averaged equation approach in investigating some nonlinear oscillation problems. The main idea of the method is briefly described and numerical simulations are carried out for some nonlinear oscillators. The results in analyzing oscillation systems with strong nonlinearity show advantages of the method.

1. INTRODUCTION

The method of moment equation is well known for analysis of random nonlinear oscillation phenomena and gives also good approximate solutions for systems with strong nonlinearity [15-16]. One way of extension the method to deterministic oscillation systems was given in [17]. In this paper, an extended averaged equation for deterministic one degree-of-freedom systems is presented and then some nonlinear oscillations are investigated in detail. The numerical results give good approximate solutions for the systems with weak, and strong nonlinearity.

2. EXTENSION OF MOMENT EQUATION METHOD TO DETERMINISTIC NONLINEAR VIBRATIONS

In order to describe briefly the main idea of the extended averaging approach which was presented in [17], one considers a oscillation of one-degree-of-freedom system governed by a nonlinear differential equation

$$\ddot{z} + f(z, \dot{z}) = 0, \tag{2.1}$$

where dots denote time differentiation, $f(z, \dot{z})$ is a nonlinear function of z, $\dot{z}$. At the same time, consider the corresponding linear equation

$$\ddot{x} + k^2 x = 0. \tag{2.2}$$

For an arbitrary differentiable function $\Psi(t,\, x,\, \dot{x},\, \dot{z})$ using equations (2.1) and (2.2), one gets.

$$\frac{d\Psi}{dt} = \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial z}\dot{z} + \frac{\partial \Psi}{\partial \dot{z}}(-f(z, \dot{z})) + \frac{\partial \Psi}{\partial x}\dot{x} + \frac{\partial \Psi}{\partial \dot{x}}(-k^2 x). \tag{2.3}$$

Denote the averaging operator (Borgoliubov & Mitropolskii) [1-3] as

$$< . >= \lim_{T\to\infty} \frac{1}{T}\int\limits_0^T (.)dt. \tag{2.4}$$

Suppose that

$$\left\langle \frac{d\Psi}{dt} \right\rangle = 0, \quad \left\langle \frac{\partial\Psi}{\partial t} \right\rangle = 0. \tag{2.5}$$

Thus, one gets

$$\left\langle \frac{d\Psi}{dt} \right\rangle = \left\langle \frac{\partial\Psi}{\partial z}\dot{z} \right\rangle + \left\langle \frac{\partial\Psi}{\partial \dot{z}}\left(-f(z,\dot{z})\right) \right\rangle + \left\langle \frac{\partial\Psi}{\partial x}\dot{x} \right\rangle + \left\langle \frac{\partial\Psi}{\partial \dot{x}}\left(-k^2x\right) \right\rangle = 0. \tag{2.6}$$

Here, we consider functions in the polynomial form

$$\Psi = r(t)z^m x^n, \quad \Psi = r(t)z^m \dot{z}^n, \quad \Psi = r(t)z^m \dot{x}^n,$$
$$\Psi = r(t)z^m \dot{x}^n, \quad \Psi = r(t)x^m \dot{z}^n, \quad \Psi = r(t)\dot{z}^m \dot{x}^n,$$

where m, $n = 0, 1, 2, \ldots$ and $r(t)$ is a function of t. It should be noted that the equation (2.6) could be referred to as an extended averaged equation, which is similar to the moment equations in the theory of random vibrations, where the averaging operator is taken in the probabilistic sense [4], [15-16]. The advantage of the equation (2.6) is that this equation can be applied to weakly and strongly nonlinear systems since the condition of small nonlinearity of the system is not used for establishing the equation (2.6). Furthermore, the equation (2.6) contains both the response $z(t)$ of the original system and the response $x(t)$ of its corresponding linear system. Thus, it can express links between the responses.

In order to close a set of averaged equations one needs some additional relations between the variables. For instance, in the classical equivalent linearization and averaging methods one puts $z(t) = x(t)$ [1-3]. Then, in these techniques, $x(t)$ represents only a linear system while $z(t)$ is from a nonlinear one. One way of overcoming this deficiency to express the response z in a polynomial form

$$z = x + \alpha_1 x^2 + \alpha_2 x^3 + \cdots + \alpha_i x^i + \ldots. \tag{2.7}$$

Thus, the problem is reduced to the problem of determining $x(t)$ (or of determining k^2) and the parameters α_1 (i =1, 2, ...). In other words, the problem of solving differential equation is reduced to the problem of solving a system of algebraic equations. For application, some nonlinear oscillation systems are investigated.

3. NUMERICAL SIMULATION ON SOME NONLINEAR OSCILLATORS

3.1. Oscillator with high nonlinearity degree

Consider the oscillator described by a nonlinear differential equation as follows

$$\ddot{z} + z + \varepsilon z^3 + \gamma z^5 = 0, \tag{3.1a}$$
$$z(0) = z_0, \tag{3.1b}$$
$$\dot{z}(0) = 0, \tag{3.1c}$$

together with its corresponding linear equation.

$$\ddot{x} + k^2 x = 0. \tag{3.2}$$

Now, in the equation (3.1a), one has

$$f(z) = z + \varepsilon z^3 + \gamma z^5. \tag{3.3}$$

Consider the case where Ψ does not depend on t. Taking "the lowest" polynomial functions $\Psi(z,\ \dot{z},\ x,\ \dot{x})$ from (2.6), one gets, for example: for

$$
\begin{aligned}
&<\dot{z}^2> - <zf(z,\dot{z})>=0 && \text{for} \quad \Psi = z\dot{z}, && (3.4)\\
&<xf(z,\dot{z})> - <\dot{z}\dot{x}>=0 && \text{for} \quad \Psi = x\dot{z}, && (3.5)\\
&<\dot{x}f(z,\dot{z})> + k^2 <x\dot{z}>=0 && \text{for} \quad \Psi = \dot{x}\dot{z}, && (3.6)\\
&<\dot{z}f(z,\dot{z})>=0 && \text{for} \quad \Psi = \dot{z}^2/2, && (3.7)\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

The equation (3.4), (3.7) are conventional averaged for original variables z, $\dot{z}$ while the equation (3.5), (3.6) contains z, $\dot{z}$, x, $\dot{x}$.

Here, we establish the response of the nonlinear equation (3.1) in the form

$$z(t) = x(t) + \alpha x^3(t), \tag{3.8}$$

where $x(t)$ is the solution of the equation (3.2), namely,

$$x = a\cos\varphi, \qquad \varphi = kt. \tag{3.9}$$

For a T-period solution $z(t)$, one gets

$$<.>= \frac{1}{T}\int_0^T (.)dt = \frac{1}{2\pi}\int_0^{2\pi} (.)dt. \tag{3.10}$$

Using (3.1b)-(3.1c), (3.8)-(3.10), after some calculations, from the equations (3.4), (3.5) one obtains the following equations:

$$
\begin{aligned}
&81920\pi^2a^2 - 4096a^2T^2 - 448\varepsilon a^4T^2 - 55\gamma a^6T^2 - 98304\pi^2 z_0 a - 8192 z_0 aT^2 -\\
&- 1280\varepsilon z_0 a^3T^2 - 210\gamma z_0 a^5T^2 + 147456\pi^2 z_0^2 - 20480 z_0^2T^2 - 2688\varepsilon z_0^2a^2T^2 -\\
&525\gamma z_0^2a^4T^2 - 5376\varepsilon z_0^3 aT^2 - 1100\gamma z_0^3a^5T^2 - 14784\varepsilon z_0^4T^2 - 2145\gamma z_0^4a^2T^2 -\\
&- 4290\gamma z_0^5 aT^2 - 12155\gamma z_0^6T^2 = 0 && (3.11)\\
&16384\pi^2 - 4096T^2 - 384\varepsilon a^2T^2 - 45\gamma a^4T^2 - 40\gamma z_0 a^3T^2 -\\
&- 2688\varepsilon z_0^2T^2 - 330\gamma z_0^2a^2T^2 - 2145\gamma z_0^4T^2 = 0, && (3.12)
\end{aligned}
$$

with two unknowns: the amplitude a and the period T (or the frequency k) of $x(t)$.

As a result, the solution $z(t)$ of the original nonlinear system (3.1) can be obtained from (3.8). The period T_p obtained by the proposed method is compared with the exact period T_E in the Table 1 for $z_0 = 1$ and different values of ε and γ.

Table 1. The period of free oscillation of the system 3.1

$\varepsilon = \gamma$	T_E	T_p	*error*
0.1	5.9023	5.8912	- 0.19%
1	4.1320	4.1173	0.36%
10	1.6916	1.6772	- 0.85%
100	0.5545	0.5491	-0.97%
1000	0.1759	0.1741	- 1%

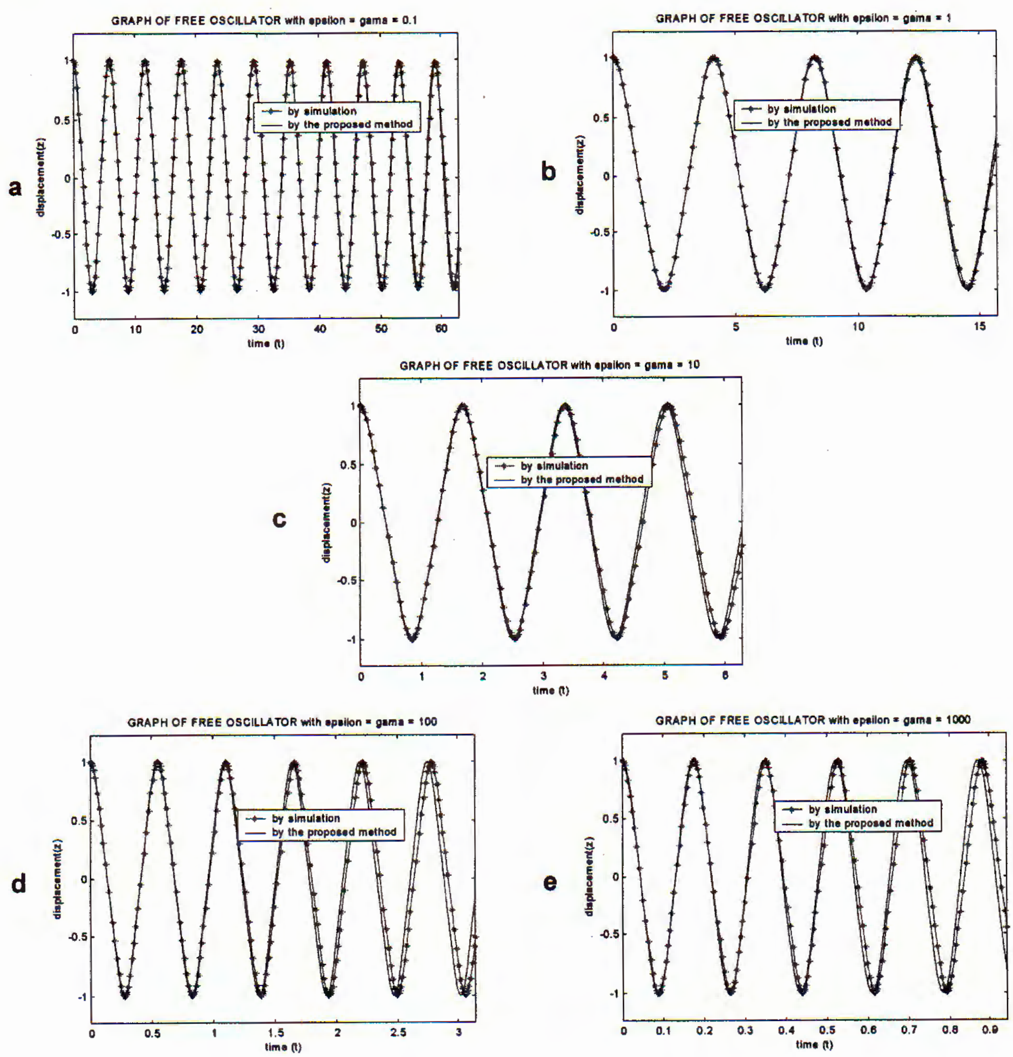

Fig. 1. Graphs of the free oscillator with higher nonlinearity

a. with $\varepsilon = \gamma = 0.1$; b. with $\varepsilon = \gamma = 1$

c. with $\varepsilon = \gamma = 10$; d. with $\varepsilon = \gamma = 100$; e. with $\varepsilon = \gamma = 1000$.

The graphs obtained by the proposed method and by numerical simulation are presented in the Figures 1(a - e).

It can be seen from the Table 1 and from the Figs. 1.(a - e) that the proposed method can give results with very high accuracy for the weakly nonlinear systems as well as for the strongly nonlinear ones. Furthermore, it should be noted that for $\gamma = 0$, we, again, obtain the Duffing oscillator which was investigated in [17].

3.2. Oscillator with absolute term in nonlinearity

Consider the oscillator governed by a following nonlinear differential equation

$$\ddot{z} + z + \varepsilon z|z| = 0, \tag{3.13a}$$

$$z(0) = z_0, \tag{3.13b}$$

$$\dot{z}(0) = z_0, \tag{3.13c}$$

together with its corresponding linear equation

$$\ddot{x} + k^2 x = 0. \tag{3.14}$$

Now, in the equation (3.13a), one has

$$f(z) = z + \varepsilon z|z|. \tag{3.15}$$

Using (3.13a)-(3.13b), (3.8)-(3.10), after some calculations, from the equations (3.4), (3.5) one obtains the following equations:

$$\begin{aligned} &-\frac{1}{4}\pi a + \frac{9}{8}\pi k^2 - \frac{3}{4}\pi a k^2 + \frac{5}{8}\pi a^2 k^2 - \frac{1}{8}\pi a^2 - \frac{8}{63}\varepsilon a^3 sign(a) - \\ &-\frac{512}{315}\varepsilon \mathrm{sign}(a) - \frac{64}{105}\varepsilon a \mathrm{sign}(a) - \frac{32}{105}\varepsilon a^2 \mathrm{sign}(a) - \frac{5}{8}\pi = 0, \end{aligned} \tag{3.16}$$

$$\begin{aligned} &-\frac{3}{4}\pi a + \frac{3}{4}\pi a k^2 + \frac{1}{4}\pi a^2 k^2 - \frac{1}{4}\pi a^2 - \frac{8}{35}\varepsilon a^3 \mathrm{sign}(a) - \\ &-\frac{64}{105}\varepsilon a^2 \mathrm{sign}(a) - \frac{64}{35}\varepsilon a \mathrm{sign}(a) = 0, \end{aligned} \tag{3.17}$$

with two unknowns: the amplitude a and the frequency k (or the period T) of $x(t)$. As a result, the solution $z(t)$ of the original nonlinear system (3.13) can be obtained from (3.8). The period T_p obtained by the proposed method is compared with the exact period T_E in the Table 2 for $z_0 = 1$ and different values of ε.

The graphs obtained by the proposed method and by numerical simulation for different values of ε are presented in the Figs. 2.(a - e). It can be seen from the Table 2 and from the Figs. 2.(a - e) that the proposed method can give results with very high accuracy for the weakly nonlinear systems as well as for the strongly nonlinear ones.

Table 2. The period of free oscillation of the system 3.2

ε	T_E	T_P	$Error$
0.1	6.037	6.0327	- 0.0007%
1	4.6357	4.6282	- 0.0016%
10	2.0534	2.0521	-0.0006%
-50	0.9617	0.9602	- 0.0016 %
100	0.6843	0.6830	- 0.0019%
500	0.3075	0.3069	- 0.002 %

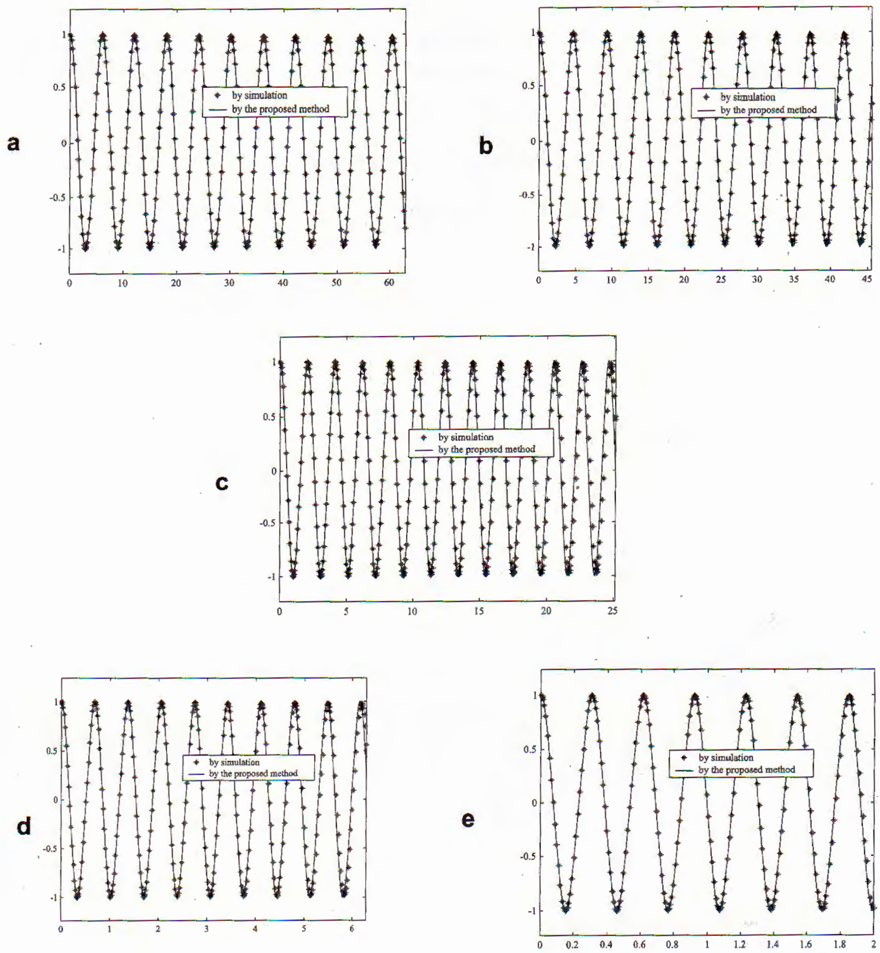

Fig. 2. Graphs of free oscillator with absolute term in nonlinearity
a. $\varepsilon = 0.1$; b. with $\varepsilon = 1$; c. with $= 10$; d. with $\varepsilon = 100$; e. with $\varepsilon = 500$.

3.3. Self-excited oscillator

Consider the VanderPol oscillator

$$\ddot{z} + z + \varepsilon(z^2 - 1)\dot{z} = 0, \tag{3.18}$$

together with its corresponding linear system

$$\ddot{x} + k^2 x = 0. \tag{3.19}$$

Now, in the equation (3.18), one has

$$f(z, \dot{z}) = z + \varepsilon(z^2 - 1)\dot{z}. \tag{3.20}$$

In order to close a set of the equations, as well as to express the non-linearity of the solution $z(t)$ of the nonlinear equation (3.23), we propose

$$z(t) = x(t) + \alpha\dot{x}(t) + \beta x^2(t)\dot{x}(t), \tag{3.21}$$

where $x(t)$ is the solution of the linear equation (3.19). Substituting (3.21) into equations (3.4) - (3.7) and using (3.9), (3.10), after some calculations, one obtains the following equations:

$$\begin{aligned}&64k^2 + 8\varepsilon a^4k^4\alpha^2\beta - 16\varepsilon a^2k^2\beta + 8\varepsilon a^4k^2\beta + 30.72\varepsilon a^8k^4\beta^3 + \\ &+ 16\varepsilon a^2k^4\alpha^3 + 3\varepsilon a^6k^4\alpha\beta^2 + 16\varepsilon a^2k^2\alpha - 64\varepsilon k^2\alpha - 64 = 0,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&- 5\varepsilon a^6k^2\beta^2 - 16a^2\beta - 16\varepsilon a^2 + 64k^2\alpha + 16a^2k^2\beta - \\ &- 16\varepsilon a^4k^2\alpha\beta - 16\varepsilon a^2k^2\alpha^2 - 64\alpha + 64\varepsilon = 0,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&8k^2 + 5a^4k^4\beta^2 - a^4k^2\beta^2 - 8k^2\alpha^2 + \\ &+ 8k^4\alpha^2 + 4a^2k^4\alpha\beta - 4a^2k^2\alpha\beta - 8 = 0,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&272a^6k^2\beta^2 + 128a^2 - 512 + 512a^4k^2\alpha\beta + 256a^2k^2\alpha^2 - \\ &- 320a^4k^2\beta^2 + 128a^2k^4\alpha^4 + 648^4k^4\alpha\beta^3 + 112a^6k^4\alpha\beta^2 - \\ &- 256a^2k^2\alpha\beta - 512k^2\alpha^2 + 13a^{10}k^4\beta^4 = 0,\end{aligned} \tag{3.25}$$

with the 4 unknowns: the amplitude a, the frequency k (or the period T) of $x(t)$, the solution polynomial coefficients α and β.

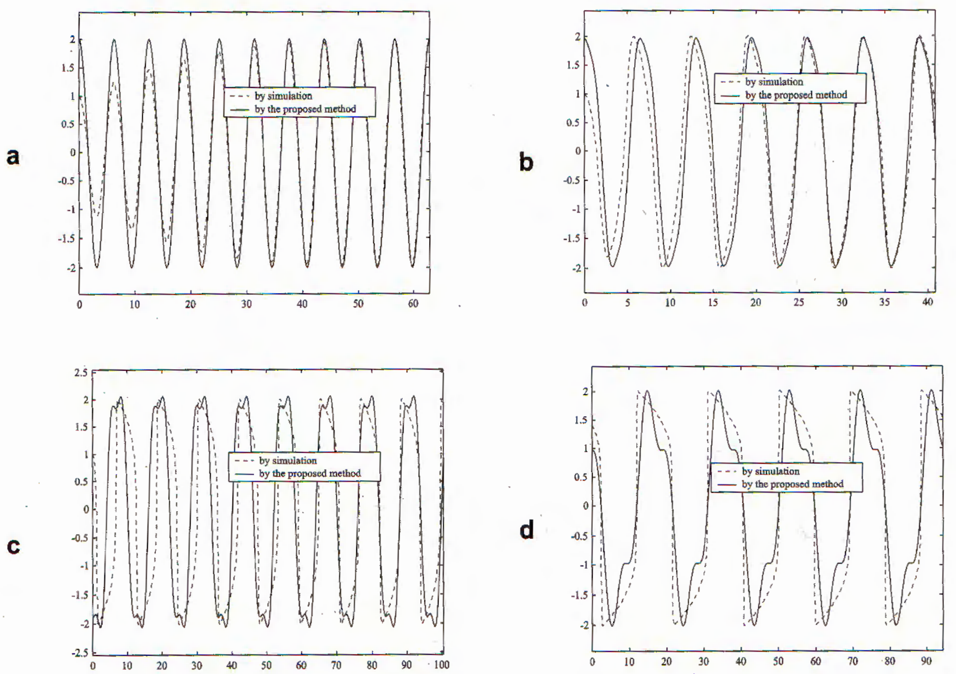

Fig. 3. Graphs of Vanderpol oscillator
a. with $\varepsilon = 0.1$; b. with $\varepsilon = 1$; c. with $\varepsilon = 5$; d. with $\varepsilon = 10$.

As a result, the solution $z(t)$ of the original nonlinear equation (3.18) can be obtained from (3.21). The oscillation amplitude Z_p and the oscillation period T_p obtained by the proposed method are compared with the simulation quantities Z_S and T_S in the Table 3 for different values of ε.

Table 3. The period of the Vanderpol oscillator

ε	z_S	$z_P(error)$	T_S	$T_P(error)$
0.1	1.9950	1.9997 (0.0023%)	6.2949	6.2871 (-0.0012%)
1	2.0080	1.9739 (-0.017%)	6.6628	6.4901 (-0.026%)
5	2.0212	2.0641 (0.02%))	11.4562	11.9872 (0.046%)
10	2.0141	2.0982 (0.04%)	18.8690	19.7828 (0.05%)

The graphs obtained by the proposed method and by numerical simulation are presented in the Figures 3(a-d).

It can be seen from the table 3 and from the graphs in Fig. 3(a - d) that the results (amplitudes, periods, ...) of the Vanderpol oscillator obtained by the proposed method are very close to the ones obtained by numerical simulations in the weakly nonlinear cases as well as in the strongly non linear ones.

4. CONCLUSION

The paper presents in detail the extended averaged equations involving the variables of the original nonlinear and of the corresponding linear systems as well as the representation of a periodicc solution of nonlinear systems by a polynomial of harmonic solution of its corresponding linear systems. Thus, a possible way to determine the solution polynomial coefficients and the linear system can be derived. The proposed method can be applied to both stochastic oscillations and deterministic oscillations. The extended averaged equation is established not using the condition of small nonlinearity. Thus, the method can be applied to weakly nonlinear systems and strongly nonlinear ones, as well. Numerical simulations are carried out in some nonlinear oscillators. The obtained results of the method give good approximate solutions for the systems with weak, middle and strong nonlinearity.

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ÁP DỤNG PHƯƠNG PHÁP PHƯƠNG TRÌNH TRUNG BÌNH SUY RỘNG TRONG VIỆC PHÂN TÍCH MỘT SỐ HỆ DAO ĐỘNG PHI TUYẾN

Bài báo trình bày việc sử dụng phương pháp phương trình mô men - phương pháp đóng của lĩnh vực dao động ngẫu nhiên trong việc nghiên cứu một số hệ dao động phi tuyến tiền định một bậc tự do. Thông qua hệ một bậc tự do, bài báo tóm lược ý tưởng chính của phương pháp "phương trình trung bình suy rộng" và cách biểu diễn nghiệm của hệ dao động phi tuyến dưới dạng đa thức của nghiệm của phương trình tuyến tính tương ứng. Các kết quả mô phỏng số của kỹ thuật này trên một số hệ dao động phi tuyến cho ta thấy ưu điểm của phương pháp đối với các hệ có độ phi tuyến yếu và mạnh.