Vietnam Journal of Mechanics, VAST, Vol. 27, No. 1 (2005), pp. 23 – 31

# ON A FORM OF LYAPUNOV EXPONENTS
## (II: VERIFICATION AND ILLUSTRATION)

NGUYEN VAN DINH
*Institute of Mechanics*

**Abstract.** The form of Lyapunov exponents proposed in the part I is verified and illustrated by examining some differential equations with well-known exact solutions.

## 1. INTRODUCTION

Having shown that Lyapunov exponents - the average rate of exponential expansion or contraction - defined in [1] as:

$$\lambda(y_0) = \lim_{t\to\infty} \ln\big(\|y(t)\|/\|y(0)\|\big) \tag{1.1}$$

can be expressed through the unit vector $u(t) = y(t)/\|y(t)\|$, that is

$$\lambda(u_0) = \lim_{t\to\infty} \frac{1}{t}\int_0^t u'(\tau)A(\tau)u(\tau)d\tau. \tag{1.2}$$

In this second part, for verification and illustration, we present some examples in which, exact solutions of the differential equations under examination are well-known.

## 2. LYAPUNOV EXPONENTS ASSOCIATED WITH EQUILIBRIUM STATE IN AUTONOMOUS SYSTEM

In this case:

$$X(t) = c \text{ (constant) or } \quad X_j(t) = c_j \quad (j = 1, 2, \dots, n), \tag{2.1}$$

$A(t) = [A_{jk}]$ is a constant matrix with elements

$$A_{jk} = \partial F_j(c_1, c_2, \dots, c_n)/\partial x_k \quad (j, k = 1, 2, \dots, n). \tag{2.2}$$

As known [2], the structure of the disturbance $y(t)$ (solution of the equation of variation) depends on the roots of the characteristic equation

$$|A - \chi I| = 0, \tag{2.3}$$

and Lyapunov exponents associated with the equilibrium state (2.1) are the real characteristic roots and the real parts of conjugate complex characteristic roots.

Do not verifying and illustrating (1.2) by a general presentation we examine only some concrete examples.

**Example 1.** Consider the equation of variation

$$\dot{y}_1 = y_1, \quad \dot{y}_2 = y_1 + y_2 - y_3, \quad \dot{y}_3 = y_3. \tag{2.4}$$

The matrix $A$ and the characteristic equation are:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad |A - \chi I| = \begin{bmatrix} 1-\chi & 0 & 0 \\ 1 & 1-\chi & -1 \\ 0 & 0 & 1-\chi \end{bmatrix} = (1-\chi)^3 = 0. \tag{2.5}$$

$\chi = 1$ is the unique characteristic root of multiplicity 3. Since rank $[A - \chi I] = 1$ we have $3 - 1 = 2$ groups of solutions respectively corresponding to two eigenvectors (solutions of the equation $[A - \chi I]a = 0$)

$$a_1 = (\alpha, 0, \alpha) \quad \text{and} \quad a_2 = (0, \beta, 0), \tag{2.6}$$

where $\alpha$, $\beta$ are two arbitrary constants.

The first group contains only one solution expressed through $a_1$, that is:

$$y_1 = \alpha e^t, \quad y_2 = 0, \quad y_3 = \alpha e^t. \tag{2.7}$$

The second group contains two solutions expressed through $a_2$, that is:

$$y_1 = y_3 = 0, \quad y_2 = \beta e^t \tag{2.8}$$

and

$$y_1 = \gamma e^t, \quad y_2 = \beta t e^t, \quad y_3 = (\gamma - \beta) e^t \tag{2.9}$$

where $\gamma$ is a new constant.

We have

$$\begin{aligned} \lambda(u) = u'Au &= u_1^2 + u_2 u_1 + u_2^2 - u_1 u_3 + u_3^2 \\ &= (u_1^2 + u_2^2 + u_3^2) + u_2(u_1 - u_3) = 1 + u_2(u_1 - u_3). \end{aligned} \tag{2.10}$$

The differential equations governing the variation of the unit vector $u(t) = y(t)/\|y(t)\|$ are

$$\begin{aligned} \dot{u}_1 &= u_1 - \lambda(u) u_1 = -(u_1 - u_3) u_2 u_1 \\ \dot{u}_2 &= u_1 + u_2 - u_3 - \lambda(u) u_2 = u_1 - (u_1 - u_3) u_2^2 \\ \dot{u}_3 &= u_3 - \lambda(u) u_3 = -(u_1 - u_3) u_2 u_3. \end{aligned} \tag{2.11}$$

The solution (2.7) gives rise to a constant solution $u$ of (2.1) that is

$$u_1 = \frac{\alpha e^t}{\|y\|} = \frac{\alpha e^t}{e^t \sqrt{\alpha^2 + \alpha^2}} = \frac{1}{\sqrt{2}}, \quad u_2 = 0, \quad u_3 = \frac{1}{\sqrt{2}}. \tag{2.12}$$

The solution $u$ corresponding to (2.8) is also a constant vector

$$u_1 = u_2 = 0, \quad u_3 = 1. \tag{2.13}$$

However, the solution (2.9) gives rise to a varying solution $u$:

$$u_1 = \frac{\gamma}{\Delta}, \quad u_2 = \frac{\beta t}{\Delta}, \quad u_3 = \frac{(\gamma - \beta)}{\Delta} \quad \text{where } \Delta = \sqrt{\gamma^2 + \beta^2 t^2 + (\gamma - \beta)^2}. \tag{2.14}$$

Using (1.2), from (2.12) or (2.13), (2.14), we can verify that $\lambda = 1$. For instance, with (2.14).

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\int_0^t \lambda(u)d\tau = \lim_{t\to\infty}\frac{1}{t}\int_0^t \left\{1 + \frac{\beta\tau}{\Delta}\left[\frac{\gamma}{\Delta} - \frac{(\gamma-\beta)}{\Delta}\right]\right\}d\tau = \lim_{t\to\infty}\frac{1}{t}\int_0^t d\tau = 1. \quad (2.15)$$

**Example 2.** Consider the equation of variation

$$\dot{y}_1 = y_1 + y_2, \quad \dot{y}_2 = y_2 - y_3, \quad \dot{y}_3 = y_2 + y_3. \quad (2.16)$$

We have

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \ |A - \chi I| = \begin{bmatrix} 1-\chi & 1 & 0 \\ 0 & 1-\chi & -1 \\ 0 & 1 & 1-\chi \end{bmatrix} = (1-\chi)\big((1-\chi)^2+1\big) = 0. \quad (2.17)$$

There are one single real characteristic roots $\chi = 1$ and a couple of single conjugate complex characteristic roots $1 \pm i$.

Since

$$u'Au = u_1^2 + u_1u_2 + u_2^2 - u_2u_3 + u_2u_3 + u_3^2 = 1 + u_1u_2 \quad (2.18)$$

the differential equations governing $u$ are

$$\dot{u}_1 = u_2 - u_1^2u_2, \quad \dot{u}_2 = u_3 - u_1u_2^2, \quad \dot{u}_3 = u_2 - u_1u_2u_3. \quad (2.19)$$

Corresponding to $\chi = 1$, the eigenvector is $a_1(\alpha, 0, 0)$ and the solution $y$ is

$$y_1 = \alpha e^t, \quad y_2 = y_3 = 0 \quad (2.20)$$

The last equation give rise to a constant solution $u_1 = 1$, $u_2 = u_3 = 0$ from which, using (1.2) we can easily verify that $\lambda = 1$.

Corresponding to conjugate complex characteristic roots $1 \pm i$, we have:

- Two conjugate complex eigenvectors

$$a_2(\beta, i\beta, \beta) \quad \text{and} \quad \overline{a}_2(\beta, -i\beta, \beta). \quad (2.21)$$

- Two corresponding real solutions

$$\begin{aligned} \widetilde{y}_1 &= \beta e^t \cos t, \quad \widetilde{y}_2 = \beta e^t \sin t, \quad \widetilde{y}_3 = \widetilde{y}_1, \\ \widetilde{\widetilde{y}}_1 &= \beta e^t \sin t, \quad \widetilde{\widetilde{y}}_2 = \beta e^t \cos t, \quad \widetilde{\widetilde{y}}_3 = \widetilde{\widetilde{y}}_1. \end{aligned} \quad (2.22)$$

- Two varying solutions $u$:

$$\widetilde{u}_1 = \widetilde{u}_3 = \frac{\cos t}{\Delta}, \quad \widetilde{u}_2 = \frac{\sin t}{\Delta} \quad \text{and} \quad \widetilde{\widetilde{u}}_1 = \widetilde{\widetilde{u}}_3 = \frac{\sin t}{\Delta}, \quad \widetilde{\widetilde{u}}_2 = \frac{\cos t}{\Delta}, \quad (2.23)$$

where $\Delta = \sqrt{1+\cos^2 t}$.

From (2.23) we can verify (1.2):

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \int_0^t (1+\widetilde{u}_1\widetilde{u}_2)d\tau = \lim_{t\to\infty} \int_0^t \left(1+\frac{\sin\tau\cos\tau}{1+\cos^2\tau}\right)d\tau$$

$$= 1 - \lim_{t\to\infty} \frac{1}{2t} \int_0^t \frac{d(1+\cos^2 t)}{1+\cos^2 t} = 1 - \lim_{t\to\infty} \frac{1}{2t}\ln(1+\cos^2 t) = 1. \qquad (2.24)$$

## 3. LYAPUNOV EXPONENTS ASSOCIATED WITH PERIODIC MOTIONS

In this case $X(t)$ is a periodic vector function with periodic $T$, $A(t)$ is a periodic matrix and the structure of $y(t)$ depends on the logarithm of the roots $\rho$ of the characteristic equation

$$|P - \rho I| = 0, \qquad (3.1)$$

where $P$ is a constant matrix connecting a fundamental matrix solution $Y(t)$ of the equation of variation and its "derived" $Y(t+T)$.

**Example 3.** Consider the equations of variation

$$\begin{aligned}\dot{y}_1 &= (1+\cos t)y_1 + y_2,\\ \dot{y}_2 &= y_1 + (1+\cos t)y_2,\end{aligned} \qquad A(t) = \begin{bmatrix} 1+\cos t & 1 \\ 1 & 1+\cos t\end{bmatrix}. \qquad (3.2)$$

It is easy to verify the following two linearly independent solutions

$$y_1^{(1)} = e^{\sin t}, \quad y_2^{(1)} = -e^{\sin t} \quad \text{with 0 as Lyapunov exponent}, \qquad (3.3)$$

$$y_1^{(2)} = e^{2t}e^{\sin t}, \quad y_2^{(2)} = e^{2t}e^{\sin t} \quad \text{with 2 as Lyapunov exponent}. \qquad (3.4)$$

We have

$$\lambda)u) = u'Au = (1+\cos t)u_1^2 + 2u_1u_2 + (1+\cos t)u_2^2 = (1+\cos t) + 2u_1u_2. \qquad (3.5)$$

Hence, the differential equations governing $u$ are:

$$\begin{aligned}\dot{u}_1 &= (1+\cos t)u_1 + u_2 - \lambda(u)u_1 = u_2(1-2u_1^2),\\ \dot{u}_2 &= u_1 + (1+\cos t)u_2 - \lambda(u)u_2. = u_1(1-2u_2^2).\end{aligned} \qquad (3.6)$$

The solution (3.3) gives rise to a constant solution $u$:

$$u_1 = \frac{e^{\sin t}}{\sqrt{(e^{\sin t})^2 + (-e^{\sin t})^2}} = \frac{\sqrt{2}}{2}, \quad u_2 = -\frac{\sqrt{2}}{2}. \qquad (3.7)$$

Using (3.7) we calculate the corresponding Lyapunov exponent

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda(u) d\tau = \lim_{t\to\infty} \frac{1}{t} \int_0^t (1 + \cos\tau + 2u_1 u_2) d\tau = \lim_{t\to\infty} \frac{1}{t} \int_0^t \cos\tau d\tau = 0. \quad (3.8)$$

The solution $u$ corresponding to the solution (3.4) is:

$$u_1 = u_2 = \frac{\sqrt{2}}{2}$$

and its Lyapunov exponent is equal to 2.

**Example 4.** Consider the equations of variation

$$\begin{aligned} \dot{y}_1 &= (1+\cos t)y_1 - (1+\sin t)y_2, \\ \dot{y}_2 &= (1+\sin t)y_1 + (1+\cos t)y_2, \end{aligned} \qquad A(t) = \begin{bmatrix} 1+\cos t & -(1+\sin t) \\ 1+\sin t & 1+\cos t \end{bmatrix}. \quad (3.9)$$

We have

$$\lambda(u) = (1+\cos t)(u_1^2 + u_2^2) = 1 + \cos t. \quad (3.10)$$

Since $\lambda(u)$ does not depend on $u$, we can immediately calculate the unique Lyapunov exponent

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda(u) d\tau = \lim_{t\to\infty} \frac{1}{t} \int_0^t (1 + \cos\tau) d\tau = 1. \quad (3.11)$$

The differential equations governing $u$ are simple enough

$$\begin{aligned} \dot{u}_1 &= (1+\cos t)u_1 - (1+\sin t)u_2 - \lambda(u)u_1 = -(1+\sin t)u_2 \\ \dot{u}_2 &= (1+\sin t)u_1 + (1+\cos t)u_2 - \lambda(u)u_2 = (1+\cos t)u_1 \end{aligned} \quad (3.12)$$

and admit two solutions:

$$u_1^{(1)} = \cos(\cos t - t), \quad u_2^{(1)} = -\sin(\cos t - t), \quad (3.13)$$

$$u_1^{(2)} = \sin(\cos t - t), \quad u_2^{(2)} = \cos(\cos t - t). \quad (3.14)$$

Let us illustrate the "inverse" relation i.e. we use the solutions (3.13), (3.14) of the equations (3.12) to solve the equations of variation (3.9).

Corresponding to (3.3), the first family of solution $y(t)$ can be found in the form

$$y_1^{(1)} = \rho_1(t)u_1^{(1)} = \rho_1(t)\cos(\cos t - t), \quad y_2^{(1)} = \rho_1(t)u_2^{(1)} = \rho_1(t)\sin(\cos t - t), \quad (3.15)$$

where $\rho_1(t)$ is a function to be determined.

Substituting (3.15) into the first equations of (3.9) yields

$$\dot{\rho}_1 = (1+\cos t)\rho_1. \quad (3.16)$$

Hence

$$\rho_1(t) = C_1 e^{t+\sin t}, \quad C_1 \text{ is arbitrary constant,} \tag{3.17}$$

$$y_1^{(1)} = C_1 e^{t+\sin t}\cos(\cos t - t), \quad y_2^{(1)} = -C_1 e^{t+\sin t}\sin(\cos t - t). \tag{3.18}$$

The second family of solution $y(t)$ corresponding to (3.14) is:

$$y_1^{(2)} = C_2 e^{t+\sin t}\sin(\cos t - t), \quad y_2^{(2)} = C_2 e^{t+\sin t}\cos(\cos t - t). \tag{3.19}$$

$C_2$ is an arbitrary constant.

**Example 5.** Consider the equations of variation

$$\begin{aligned}\dot{y}_1 &= (1+\cos t)y_1 + (1+\sin t)y_2,\\ \dot{y}_2 &= (1+\sin t)y_1 + (1+\cos t)y_2,\end{aligned} \qquad A(t) = \begin{bmatrix} 1+\cos t & 1+\sin t \\ 1+\sin t & 1+\cos t\end{bmatrix}. \tag{3.20}$$

We have:

$$\lambda(u) = 1 + \cos t + 2(1+\sin t)u_1 u_2, \tag{3.21}$$

$$\begin{aligned}\dot{u}_1 &= (1+\sin t)u_2(1-2u_1^2),\\ \dot{u}_2 &= (1+\sin t)u_1(1-2u_2^2),\end{aligned} \tag{3.22}$$

which admits two constant solutions

$$u_1 = \frac{\sqrt{2}}{2}, \quad u_2 = -\frac{\sqrt{2}}{2} \tag{3.23}$$

and

$$u_1 = u_2 = \frac{\sqrt{2}}{2}. \tag{3.24}$$

The Lyapunov exponents corresponding to (3.23) and (3.24) are, respectively

$$\lambda(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \Big[1+\cos\tau - 2(1+\sin\tau)\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\Big]d\tau = 0, \tag{3.25}$$

$$\lambda(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \Big[1+\cos\tau + 2(1+\sin\tau)\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\Big]d\tau = 2. \tag{3.26}$$

Using the solution $u$ (3.23) and its Lyapunov exponent (3.25) we can found the first family of solutions $y$ in the form

$$y_1^{(1)} = \rho_1(t), \quad y_2^{(1)} = -\rho_1(t). \tag{3.27}$$

Substituting (3.27) into the first equation of (3.20) yields

$$\dot{\rho}_1 = (\cos t - \sin t)\rho_1 \tag{3.28}$$

from which, it follows:

$$\rho_1 = C_1 e^{\sin t + \cos t}, \quad y_1^{(1)} = -y_2^{(1)} = C_1 e^{\sin t + \cos t}. \tag{3.29}$$

The second family of solutions $y$ corresponding to (3.24), (3.26) is

$$y_1^{(2)} = y_2^{(2)} = C_2 e^{2t} e^{\sin t - \cos t}. \tag{3.30}$$

## 4. EXAMPLES IN THE CASE GENERAL

For the general case, we examine two examples. Note that, if it is necessary, the lim is replaced by $\overline{\lim}$ [2].

**Example 6.** Consider the following equations of variation given by Lyapunov [2]

$$\begin{aligned} \dot{y}_1 &= y_1 \cos\ln(t+1) + y_2 \sin\ln(t+1), \\ \dot{y}_2 &= y_1 \sin\ln(t+1) + y_2 \cos\ln(t+1). \end{aligned} \tag{4.1}$$

We have

$$\begin{aligned} \lambda(u) &= (u_1^2 + u_2^2)\cos\ln(t+1) + 2u_1u_2 \sin\ln(t+1) \\ &= \cos\ln(t+1) + 2u_1u_2\sin\ln(t+1), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \dot{u}_1 &= u_1\cos\ln(t+1) + u_2\sin\ln(t+1) - \lambda(u)\cdot u_1 = u_2\sin\ln(t+1)(1-2u_1^2), \\ \dot{u}_2 &= u_1\sin\ln(t+1) + u_2\cos\ln(t+1) - \lambda(u)\cdot u_2 = u_1\sin\ln(t+1)(1-2u_2^2), \end{aligned} \tag{4.3}$$

which admits two constant solutions:

$$u_1 = -u_2 = \frac{\sqrt{2}}{2} \tag{4.4}$$

and

$$u_1 = u_2 = \frac{\sqrt{2}}{2}. \tag{4.5}$$

Using (1.2), we can calculate the Lyapunov exponent corresponding to (4.4)

$$\lambda = \overline{\lim_{t\to\infty}} \frac{1}{t}\Big\{(t+1)\cos\ln(t+1) - 1\Big\} = 1. \tag{4.6}$$

The same Lyapunov exponent is obtained for (4.5):

$$\lambda = \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t \big[\cos\ln(\tau+1) + \sin\ln(\tau+1)\big]d\tau = \overline{\lim_{t\to\infty}} \frac{1}{t}(t+1)\sin\ln(t+1) = 1. \tag{4.7}$$

From (4.4), (4.6), the first family of solutions of (4.1) can be found in the form

$$y_1^{(1)} = \rho_1 e^t, \quad \rho_2^{(1)} = -\rho_1 e^t. \tag{4.8}$$

Substituting (4.8) into the first equation of (4.1) yields

$$\dot{\rho}_1 = \rho_1(\cos\ln(t+1) - \sin\ln(t+1) - 1), \tag{4.9}$$

from which, it follows:

$$\rho_1 = C_1 e^{-t} \cdot e^{(t+1)\cos\ln(t+1)}, \tag{4.10}$$

$$y_1^{(1)} = C_1 e^{(t+1)\cos\ln(t+1)}, \quad y_2^{(1)} = -C_1 e^{(t+1)\cos\ln(t+1)}. \tag{4.11}$$

The second family of solutions of (4.1) corresponding to (4.5), (4.7) is

$$y_1^{(2)} = C_2 e^{(t+1)\sin\ln(t+1)} = y_2^{(2)}. \tag{4.12}$$

**Example 7.** Consider the equations of variation

$$\begin{aligned} \dot{y}_1 &= y_1\cos\ln(t+1) - y_2\sin\ln(t+1) \\ \dot{y}_2 &= y_1\sin\ln(t+1) + y_2\cos\ln(t+1), \end{aligned} \quad A = \begin{bmatrix} \cos\ln(t+1) & -\sin\ln(t+1) \\ \sin\ln(t+1) & \cos\ln(t+1) \end{bmatrix}. \tag{4.13}$$

We have

$$\lambda(u) = u_1^2\cos\ln(t+1) + u_2^2\cos\ln(t+1) = \cos\ln(t+1). \tag{4.14}$$

Since $\lambda(u)$ does not depend on $u$, we can calculate directly the Lyapunov exponent

$$\begin{aligned} \lambda &= \overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t \cos\ln(t+1)dt = \lim_{t\to\infty}\frac{1}{t}\Big\{\frac{t+1}{2}\big(\cos\ln(t+1) + \sin\ln(t+1)\big) - \frac{1}{2}\Big\} \\ &= \overline{\lim_{t\to\infty}}\frac{1}{2}\big(\cos\ln(t+1) + \sin\ln(t+1)\big) = \overline{\lim_{t\to\infty}}\frac{\sqrt{2}}{2}\sin\Big(\ln(t+1) + \frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}. \end{aligned} \tag{4.15}$$

The differential equations governing $u$ are

$$\begin{aligned} \dot{u}_1 &= u_1\cos\ln(t+1) - u_2\sin\ln(t+1) - \lambda(u)u_1 = -u_2\sin\ln(t+1), \\ \dot{u}_2 &= u_1\sin\ln(t+1) + u_2\cos\ln(t+1) - \lambda(u)u_2 = u_1\sin\ln(t+1). \end{aligned} \tag{4.16}$$

The first solution of (4.16) is found in the form

$$u_1 = \cos\theta, \quad u_2 = \sin\theta. \tag{4.17}$$

Substituting (4.17) into the first equation of (4.16) yields:

$$\dot{\theta} = \sin\ln(t+1), \tag{4.18}$$

from which, it follows

$$\theta = \frac{1}{2}(t+1)\big(\sin\ln(t+1) - \cos\ln(t+1)\big). \tag{4.19}$$

Using (4.15), (4.17), and (4.19), the first of solutions of (4.13) is found in the form

$$y_1^{(1)} = \rho_1 \cos\theta e^{\frac{\sqrt{2}}{2}t}, \quad y_2^{(1)} = \rho_1 \sin\theta e^{\frac{\sqrt{2}}{2}t}. \tag{4.20}$$

On account of (4.18), substituting (4.20) into the first equation of (4.13) gives

$$\dot{\rho}_1 = \rho_1\Big(\cos\ln(t+1) - \frac{\sqrt{2}}{2}\Big). \tag{4.21}$$

From (4.21) it follows

$$\rho_1 = C_1 e^{-\frac{\sqrt{2}}{2}} \cdot e^{\frac{t+1}{2}(\cos\ln(t+1)+\sin\ln(t+1))}. \tag{4.22}$$

Hence

$$\begin{aligned} y_1^{(1)} &= C_1 \cos\frac{t+1}{2}\big(\sin\ln(t+1) - \cos\ln(t+1)\big)e^{\frac{t+1}{2}(\cos\ln(t+1)+\sin\ln(t+1))}, \\ y_2^{(1)} &= C_1 \sin\frac{t+1}{2}\big(\sin\ln(t+1) - \cos\ln(t+1)\big)e^{\frac{t+1}{2}(\cos\ln(t+1)+\sin\ln(t+1))}. \end{aligned} \tag{4.23}$$

The second solution of (4.16) is found in the form

$$u_1 = \cos\theta, \quad u_2 = -\sin\theta \tag{4.24}$$

and the corresponding family of solutions $y$ is

$$y_1^{(2)} = y_2^{(2)} = C_2 \cos\frac{t+1}{2}\big(\sin\ln(t+1) - \cos\ln(t+1)\big)e^{\frac{t+1}{2}(\cos\ln(t+1)+\sin\ln(t+1))}. \tag{4.25}$$

## 5. CONCLUSION

In this part II, some examples are presented to confirm that (1.2) is really another form of the Lyapunov exponent (1.1). The use of (1.2) in practice will be treated.

This publication is completed with the financial support from The Council for Natural Science of Vietnam.

## REFERENCES

1. Ali H. Nayfeh, Balakumar Balachamdran. *Applied nonlinear dynamics*, John Willey, 1995.
2. Malkin I. G., *Theory of stability of motion*, Nauka Mowcow 1996.

*Received December 5, 2003*
*Revised June 10, 2004*

## VỀ MỘT DẠNG CỦA SỐ MŨ LYAPUNOV (II: KIỂM TRA VÀ MINH HỌA)

Ở phần II, một số thí dụ được trình bày cho thấy dạng (1.2) đúng là dạng khác của số mũ Liapunov (1.1).