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CALCULATING PERIODIC VIBRATIONS OF NONLINEAR AUTONOMOUS MULTI-DEGREE-OF--FREEDOM SYSTEMS BY THE INCREMENTAL HARMONIC BALANCE METHOD

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ABSTRACT. In this paper the incremental harmonic balance method is used to calculate periodic vibrations of nonlinear autonomous multip-degree-of-freedom systems. According to Floquet theory, the stability of a periodic solution is checked by evaluating the eigenvalues of the monodromy matrix. Using the programme MAPLE, the authors have studied the periodic vibrations of the system multi-degree van der Pol form.

1 Introduction

Periodic solutions play a very important role in study of nonlinear dynamic systems. Perturbation techniques have been employed in classical nonlinear analysis [1], [2]. However, these perturbation techniques are restricted to solve weakly nonlinear problems only. It is still unknown whether the nonlinearity is weak enough for which the techniques can be applied adequately. The numerical integration (NI) method can treat strongly nonlinear problems very well. However, for the cases of systems with low damping, the convergence rate of getting a steady-state solution will be extremely slow. For complicated problems, periodic solutions of nonlinear system can't be sought by NI method if suitable initial conditions are unknown.

It is also known that the harmonic balance (HB) method can be used in nonlinear dynamic analysis. When a high accuracy is desired, the calculation is difficult. In comparison, the incremental harmonic balance (IHB) method can give accurate results and without difficulty in increasing the accuracy of the results [4, 5, 6, 7, 8].

In this paper the incremental harmonic balance (IHB) method is applied to calculate periodic solution of multi-degree-of-freedom nonlinear autonomous systems. There are two main advantages of the proposed method. Firstly, it is not subjected to limitations of small exciting parameters and weak nonlinearity. Secondly, it is particularly simple and economical for computer implementation because only linear algebraic equations have to be formed and solved in each incremental step.

2 IHB formulation

Let us consider a nonlinear autonomous system, which is derived by the differential equations

$$g_k(\ddot{x}_i, \dot{x}_i, x, \lambda) = 0, \quad (k, i = 1, \cdots, n), \tag{2.1}$$

where λ is a parameter. It is useful to change the time scale by introducing the "dimensionless time" defined by the relations

$$\tau = \omega t, \quad x_i' = \omega \dot{x}_i, \quad x'' = \omega^2 \ddot{x}_i, \tag{2.2}$$

where ω is the unknown frequency, primes denote differentiation with respect to τ. Substituting the formula (2.2) into equations (2.1) we obtain

$$f_k(x_i'', x_i', x_i, \omega, \lambda) \equiv g_k(\omega^2 \ddot{x}_i, \omega \dot{x}_i, x_i, \lambda) = 0, \quad (k, i = 1, \cdots, n), \tag{2.3}$$

The procedure of the IHB method for seeking periodic solution is mainly divided into two steps. The first step is the Ritz-Galerkin procedure, in which an approximate solution of the problem is found by satisfying the governing nonlinear equation in the average sense. By calculating the approximate solution with Ritz-Galerkin method we assume that the unknown x_i^0 can be expanded into a truncated Fourier series

$$x_i^0 = \sum_{r=1}^{R_C} a_{in_r} \cos n_r \tau + \sum_{r=1}^{R_S} b_{im_r} \sin m_r \tau, \quad (i = 1, ..., n). \tag{2.4}$$

The values a_{in_r} $(i = 1, ..., n; r = 1, ..., R_C)$ and b_{im_r} $(i = 1, ..., n; r = 1, ..., R_S)$ are found from following conditions (Lau and Yuen, 1991).

$$\int_0^{2\pi} f_k(x_i'', x_i', x_i, \omega, \lambda) \cos n_r \tau d\tau = 0 \quad (k, i = 1, ..., n; r = 1, ..., R_C), \tag{2.5}$$

$$\int_0^{2\pi} f_k(x_i'', x_i', x_i, \omega, \lambda) \sin m_r \tau d\tau = 0 \quad (k, i = 1, ..., n; r = 1, ..., R_S). \tag{2.6}$$

The second step of the IHB method is a Newton-Raphson process to improve the solutions x_i^0 . The small increments (symbolized by Δ) are added to the current solution of (2.3), i. e: λ and x_i^0 (i = 1, , n), then we get the neighboring solutions.

$$x_i = x_i^0 + \Delta x_i, \quad (i = 1, \cdots, n), \quad \omega = \omega_0 + \Delta\omega, \quad \lambda = \lambda_0 + \Delta\lambda. \tag{2.7}$$

Expanding (2.3) by Taylor's series about the current solution yields

$$\mathbf{f}(\mathbf{x}_0'' + \Delta\mathbf{x}'', \mathbf{x}_0' + \Delta\mathbf{x}', \mathbf{x}_0 + \Delta\mathbf{x}, \tau, \lambda_0 + \Delta\lambda) = \mathbf{f}_0 + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}''}\right|_0 \Delta\mathbf{x}''$$
$$+ \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}'}\right|_0 \Delta\mathbf{x}' + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_0 \Delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \omega}\right|_0 \Delta\omega + \left.\frac{\partial \mathbf{f}}{\partial \lambda}\right|_0 \Delta\lambda + \text{higher order terms.}$$

Neglecting all nonlinear terms of small increments yields the incremental equations

$$\mathbf{A}_0\Delta\mathbf{x}'' + \mathbf{B}_0\Delta\mathbf{x}' + \mathbf{C}_0\Delta\mathbf{x} = -\mathbf{f}_0 - \mathbf{p}_0\Delta\omega - \mathbf{q}_0\Delta\lambda, \tag{2.8}$$

where

$$\begin{aligned}
\mathbf{f}_0 &= \mathbf{f}(\mathbf{x}_0'', \mathbf{x}_0', \mathbf{x}_0, \omega, \lambda_0) = [(f_1)_0 \cdots (f_n)_0]^T, \\
\mathbf{x}_0 &= \left[x_1^0 \cdots x_n^0\right]^T, \quad \Delta\mathbf{x} = [\Delta x_1 \cdots \Delta x_n], \\
\mathbf{A}_0 &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}''}\bigg|_0 = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1''}\right)_0 & \cdots & \left(\frac{\partial f_1}{\partial x_n''}\right)_0 \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_n}{\partial x_1''}\right)_0 & \cdots & \left(\frac{\partial f_n}{\partial x_n''}\right)_0 \end{bmatrix}, \quad \mathbf{B}_0 = \frac{\partial \mathbf{f}}{\partial \mathbf{x}'}\bigg|_0 = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1'}\right)_0 & \cdots & \left(\frac{\partial f_1}{\partial x_n'}\right)_0 \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_n}{\partial x_1'}\right)_0 & \cdots & \left(\frac{\partial f_n}{\partial x_n'}\right)_0 \end{bmatrix}, \\
\mathbf{C}_0 &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\bigg|_0 = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)_0 & \cdots & \left(\frac{\partial f_1}{\partial x_n}\right)_0 \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_n}{\partial x_1}\right)_0 & \cdots & \left(\frac{\partial f_n}{\partial x_n}\right)_0 \end{bmatrix}, \quad \begin{aligned} \mathbf{p}_0 &= \frac{\partial \mathbf{f}}{\partial \omega}\bigg|_0 = \left[\left(\frac{\partial f_1}{\partial \omega}\right)_0 \cdots \left(\frac{\partial f_n}{\partial \omega}\right)_0\right]^T, \\ \mathbf{q}_0 &= \frac{\partial \mathbf{f}}{\partial \lambda}\bigg|_0 = \left[\left(\frac{\partial f_1}{\partial \lambda}\right)_0 \cdots \left(\frac{\partial f_n}{\partial \lambda}\right)_0\right]^T \end{aligned}
\end{aligned} \tag{2.9}$$

Assume that increments Δx_i can be expanded into Fourier series

$$\Delta x_i = \sum_{r=1}^{R_C} \Delta a_{in_r} \cos n_r\tau + \sum_{r=1}^{R_S} \Delta b_{im_r} \sin m_r\tau, \quad (i = 1, \cdots, n). \tag{2.10}$$

By using symbols

$$\begin{aligned}
\mathbf{s} &= [\cos n_1\tau \quad \cdots \quad \cos n_{R_C}\tau \quad \sin m_1\lambda \quad \cdots \sin m_{R_S}]^T, \\
\mathbf{a}_i^0 &= \left[a_{in_1}^0 \quad \cdots \quad a_{in_{R_C}} \quad b_{im_1} \quad \cdots b_{im_{R_S}}\right]^T, \quad (i = 1, \cdots, n). \\
\Delta\mathbf{a}_i &= \left[\Delta a_{in_1}^0 \quad \cdots \quad \Delta a_{in_{R_C}} \quad \Delta b_{im_1} \quad \cdots \Delta b_{im_{R_S}}\right]^T, \quad (i = 1, \cdots, n). \\
\mathbf{Y} &= \begin{bmatrix} \mathbf{s}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{s}^T & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{s}^T \end{bmatrix}, \quad \mathbf{a}_0 = \left[\mathbf{a}_1^{0T} \quad \cdots \quad \mathbf{a}_n^{0T}\right]^T, \quad \Delta\mathbf{a} = [\Delta\mathbf{a}_1^T \quad \cdots \quad \Delta\mathbf{a}_n^T]^T,
\end{aligned} \tag{2.11}$$

we have

$$\mathbf{x}_0 = \mathbf{Y}\,\mathbf{a}_0, \quad \Delta\mathbf{x} = \mathbf{Y}\,\Delta\mathbf{a}. \tag{2.12}$$

With these equations, we obtain

$$\mathbf{x}' = \mathbf{Y}'\mathbf{a}_0, \quad \mathbf{x}_0'' = \mathbf{Y}''\mathbf{a}_0, \quad \Delta\mathbf{x}' = \mathbf{Y}'\Delta\mathbf{a}, \quad \Delta\mathbf{x}'' = \mathbf{Y}''\Delta\mathbf{a}. \tag{2.13}$$

By substituting (2.12) and (2.13) into (2.8) we get

$$(\mathbf{A}_0\mathbf{Y}'' + \mathbf{B}_0\mathbf{Y}' + \mathbf{C}_0\mathbf{Y})\Delta\mathbf{a} = -\mathbf{f}_0 - \mathbf{p}_0\Delta\omega - \mathbf{q}_0\Delta\lambda. \tag{2.14}$$

Therefore (2.14) is a system of linear algebraic equations, whose coefficients are periodic functions of τ . Now, two sides of (2.14) are multiplied by $\mathbf{Y}^T$ and then integrated for a period. At last we get

$$\mathbf{K}\Delta\mathbf{a} = \mathbf{r} + \mathbf{p}\Delta\omega + \mathbf{q}\Delta\lambda, \tag{2.15}$$

where

$$\mathbf{K} = \int_0^{2\pi} \mathbf{Y}^T(\mathbf{A}_0\mathbf{Y}'' + \mathbf{B}_0\mathbf{Y}' + \mathbf{C}_0\mathbf{Y})d\tau,$$
$$\mathbf{r} = -\int_0^{2\pi} \mathbf{Y}^T\mathbf{f}_0 d\tau, \quad \mathbf{p} = -\int_0^{2\pi} \mathbf{Y}^T\mathbf{p}_0 d\tau, \quad \mathbf{q} = -\int_0^{2\pi} \mathbf{Y}^T\mathbf{q}_0 d\tau. \tag{2.16}$$

Equation (2.15) is a system of $L = n(R_C + R_S)$ linear algebraic equations for $L + 2$ unknowns $\Delta a_{in_r}, \Delta a_{im_r}\, \Delta\omega, \Delta\lambda$. Therefore, two of these unknowns have to be fixed to make the number of equations equal to that of the unknowns. Because ω is unknown frequency of the looking periodic solution, which is corresponding with $\lambda = \lambda_0$, we set $\Delta\lambda = 0$ and $\Delta b_{im_{R_S}}$. Then the increment $\Delta\mathbf{x}$ is calculated according to (2.12) and the neighboring solution $\mathbf{x}$ is updated by

$$\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x}. \tag{2.17}$$

The loop is continued until $||\Delta\mathbf{a}|| < \varepsilon$, where ε is a small positive error chosen. We now consider the stability of periodic solutions of the autonomous system

$$\mathbf{f}(\mathbf{x}'', \mathbf{x}', \mathbf{x}, \lambda) = \mathbf{0}, \tag{2.18}$$

where $\mathbf{f}$ and $\mathbf{x}$ are n-dimensional vectors. The periodic solution of (2.18) be denoted by $\mathbf{x}_0(\tau)$ and have the minimal period $T = 2\pi$ We superimpose a disturbance $\mathbf{y}(\tau)$ is on $\mathbf{x}_0(\tau)$ and obtain

$$\mathbf{x}(\tau) = \mathbf{x}_0(\tau) + \mathbf{y}(\tau). \tag{2.19}$$

Substituting (2.19) into (2.18), expanding f in a Taylor series about $\mathbf{x}_0, \dot{\mathbf{x}}_0, \ddot{\mathbf{x}}_0$, we obtain

$$\mathbf{f}(\mathbf{x}_0'' + \Delta\mathbf{y}'', \mathbf{x}_0' + \Delta\mathbf{y}', \mathbf{x}_0 + \Delta\mathbf{y}, \tau, \lambda) = \mathbf{f}_0 + \frac{\partial\mathbf{f}}{\partial\mathbf{x}''}\bigg|_0 \Delta\mathbf{y}'' + \frac{\partial\mathbf{f}}{\partial\mathbf{x}'}\bigg|_0 \Delta\mathbf{y}' + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}\bigg|_0 \Delta\mathbf{y} + \dots. \tag{2.20}$$

Retaining only linear terms in the disturbance, the equation (2.20) can be written as

$$\mathbf{A}(\tau)\mathbf{y}'' + \mathbf{B}(\tau)\mathbf{y}' + \mathbf{C}(\tau)\mathbf{y} = 0, \tag{2.21}$$

where

$$\mathbf{A}(\tau) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}''}\bigg|_0, \quad \mathbf{B}(\tau) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}'}\bigg|_0, \quad \mathbf{C}(\tau) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\bigg|_0. \tag{2.22}$$

The equation (2.21) is linear in $\mathbf{y}$, but with periodically varying coefficients in general. One can rewrite equation (2.21) as

$$\mathbf{u}' = \mathbf{P}(\tau)\mathbf{u}, \tag{2.23}$$

where $\mathbf{u} = [\mathbf{y}, \mathbf{y}']^T$, and the matrix

$$\mathbf{P}(\tau) = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ -\mathbf{A}^{-1}\mathbf{C} & -\mathbf{A}^{-1}\mathbf{B} \end{bmatrix}$$

is periodic in τ with period 2π. Proceeding along the lines discussed earlier, we use Floquet theory to treat (2.23) and determine the monodromy matrix Φ associated with a periodic solution of equation (2.18).

The stability of a periodic solution $\mathbf{x}_0(\tau)$ of the equation (2.18) is checked by evaluating the eigenvalues of the monodromy matrix Φ [3, 10]. If all of these eigenvalues are within the unit circle, then the periodic solution $\mathbf{x}_0(\tau)$ is asymptotically stable. If at least one of these eigenvalues is outside the unit circle, the associated solution is unstable. If some of these eigenvalues lie on the unit circle and other eigenvalues are within the unit circle, a nonlinear analysis is necessary to determine the stability.

3 Numerical example

Study problems of biped robots derived to the systems of differential equation of van der Pol form

$$\ddot{x}_1 + \omega_1^2 x_1 - d_1\left[(1 - h_1^2 x_1^2)\dot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2)\right] = 0, \tag{3.1}$$

$$\ddot{x}_2 + \omega_2^2 x_2 - d_2\left[(1 - h_2^2 x_2^2)\dot{x}_2 + c_2(\dot{x}_2 - \dot{x}_2)\right] = 0, \tag{3.2}$$

where ω_1, ω_2 are the natural frequencies of the linear system corresponding to (3.1), (3.2), h_1, h_2, c_1, c_2, d_1, d_2 are parameters. To consider behavior of the system which depends to the parameters, we introduce parameter λ by setting

$$d_1 = \lambda, \quad d_2 = \alpha\lambda. \tag{3.3}$$

Substituting (3.3) into equations (3.1), (3.2) we have the equations

$$\ddot{x}_1 + \omega_1^2 x_1 - \lambda\left[(1 - h_1^2 x_1^2)\dot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2)\right] = 0, \tag{3.4}$$

$$\ddot{x}_2 + \omega_2^2 x_2 - \lambda\alpha\left[(1 - h_2^2 x_2^2)\dot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1)\right] = 0, \tag{3.5}$$

For the parameters $\omega_1 = 1$, $\omega_2 = 1.1$, $h_1 = 25$, $h_2 = 25$, $c_1 = 4$, $c_2 = 4$, $\alpha = 2$, now we look for periodic solutions of the systems (3,4), (3.5) that depend on varying of λ.
It is clear that the larger λ is, the stronger nonlinearity of the system is. When λ is small enough, the nonlinerity of the system is weak. Computer simulations show that the system

has periodic solutions for $\lambda > 0.018$ only.
We now apply IHB method to look for periodic solutions of the system in some typical cases. By introducing the "dimensionless time" $\tau = \omega t$, (3.4), (3.5) have the form

$$\omega^2 x_1'' + \omega_1^2 x_1 - \lambda\omega\left[(1 - h_1^2 x_1^2)x_1' + c_1(x_1' - x_2')\right] = 0, \tag{3.6}$$

$$\omega^2 x_2'' + \omega_2^2 x_2 - \lambda\alpha\omega\left[(1 - h_2^2 x_2^2)x_2' + c_2(x_2' - x_1')\right] = 0, \tag{3.7}$$

where $x_1' = \frac{dx_1}{d\tau}$, $x_2' = \frac{dx_2}{d\tau}$, $x_1'' = \frac{d^2x_1}{d\tau^2}$, $x_2'' = \frac{d^2x_2}{d\tau^2}$.

In this case $n = 2$ because of symmetry of the system we consider a solution as folows:

$$\mathbf{x}_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \mathbf{Y}\mathbf{a}_0, \tag{3.8}$$

where

$$\begin{aligned} \underset{20\times 2}{\mathbf{Y}} &= \begin{bmatrix} \mathbf{s}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{s}^T \end{bmatrix}, \quad \underset{20\times 1}{\mathbf{a}_0} = \begin{bmatrix} \mathbf{a}_1^0 \\ \mathbf{a}_2^0 \end{bmatrix}^T, \\ \mathbf{s} &= [\cos\tau \quad \cos 3\tau \quad \cdots \quad \cos 9\tau \quad \sin 3\tau \quad \cdots \quad \sin 9\tau]^T, \\ \mathbf{a}_1^0 &= [a_{1,1}^0 \quad a_{1,3}^0 \quad \cdots \quad a_{1,9}^0 \quad b_{1,1}^0 \quad b_{1,3}^0 \quad \cdots \quad b_{1,9}^0]^T, \\ \mathbf{a}_2^0 &= [a_{2,1}^0 \quad a_{2,3}^0 \quad \cdots \quad a_{2,9}^0 \quad b_{2,1}^0 \quad b_{2,3}^0 \quad \cdots \quad b_{2,9}^0]^T. \end{aligned} \tag{3.9}$$

Using conditions (2.5) and (2.6), the coefficients $a_{i,j}^0$, $b_{i,j}^0$ $(i = 1,\ 2;\ j = 1,\ 2,\ 3, \cdots, 9)$ and ω_0 are found. Then the small increments are added to the current solution:

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + \Delta\mathbf{x}, \quad \Delta\mathbf{x} = [\Delta\mathbf{x}_1 \quad \Delta\mathbf{x}_2]^T, \\ \omega &= \omega_0 + \Delta\omega, \quad \lambda = \lambda_0 + \Delta\lambda. \end{aligned} \tag{3.10}$$

The increments $\Delta\mathbf{x}$ can be sought in the forms:

$$\begin{aligned} \Delta\mathbf{x} &= \mathbf{Y}\Delta\mathbf{a}, \quad \Delta\mathbf{a} = [\Delta\mathbf{a}_1^T \quad \Delta\mathbf{a}_2^T]^T, \\ \Delta\mathbf{a}_1 &= [\Delta a_{11} \quad \Delta a_{31} \quad \cdots \quad \Delta a_{91} \quad \Delta b_{11} \quad \Delta b_{31} \quad \Delta b_{31} \quad \cdots \quad \Delta b_{91}]^T, \\ \Delta\mathbf{a}_2 &= [\Delta a_{12} \quad \Delta a_{32} \quad \cdots \quad \Delta a_{91} \quad \Delta b_{12} \quad \Delta b_{32} \quad \Delta b_{32} \quad \cdots \quad \Delta b_{92}]^T. \end{aligned}$$

Now the equations (2.15) for the increments is the linear algebraic system of 20 equations for 22 unknown. Chosing $\Delta a_{19} = 0$, $\Delta\lambda = 0$ we calculate $\Delta\mathbf{a}$, $\Delta\omega$ and the neighboring solution $\mathbf{x}$ is updated by (3.10). The loop is continued until $\|\Delta\mathbf{a}\| < \varepsilon$, $|\Delta\omega| < \varepsilon$, $\varepsilon = 10^{-10}$ and the periodic solution is found.

For $\lambda = 0.016$ the periodic solution is shown in Fig. 1. This solution is unstable. The computated result by NI method is shown in Fig. 2.
For $\lambda = 0.02$ the stable periodic solution with $\omega = 1.03188098312$ is found and shown in Fig. 3. The computated result by NI method is shown in Fig. 4.
For larger values of λ , the nonlinearity of the system is stronger, but can be found periodic solutions. Fig. 5 shows the periodic solution with $\omega = 0.560675242144$ for $\lambda = 1.0$, and the computated results by using NI method are presented in Fig. 6. We now apply numerical calculation of Floquet multiplier to study stability of the periodic solutions [3, 10] which are found by IHB method. The Floquet multipliers of monodromy matrix Φ associated with the periodic solutions are shown in Table 1.

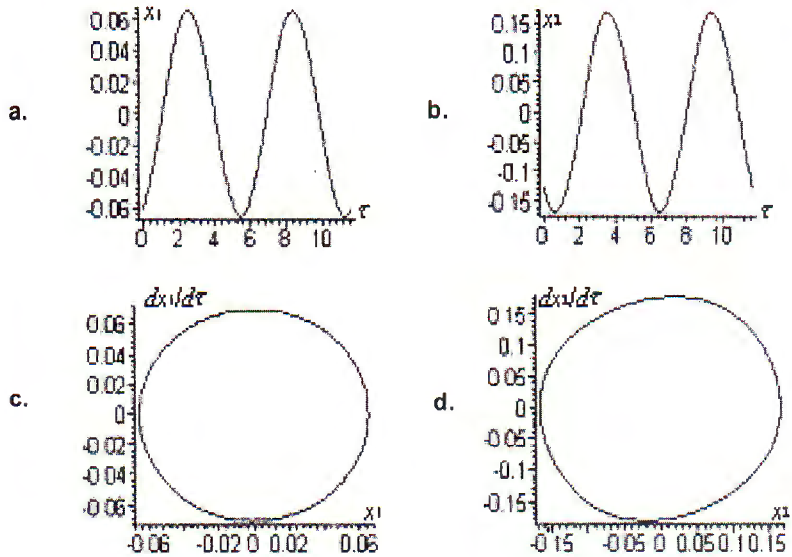

Fig 1. The periodic solution in the case of λ=0.016 by IHB method

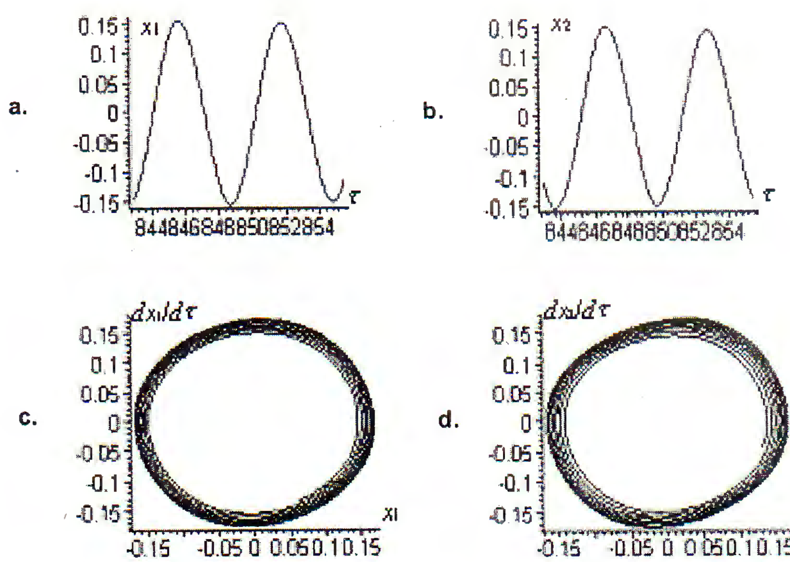

Fig 2. The periodic solution in the case of λ=0.016 by NI method

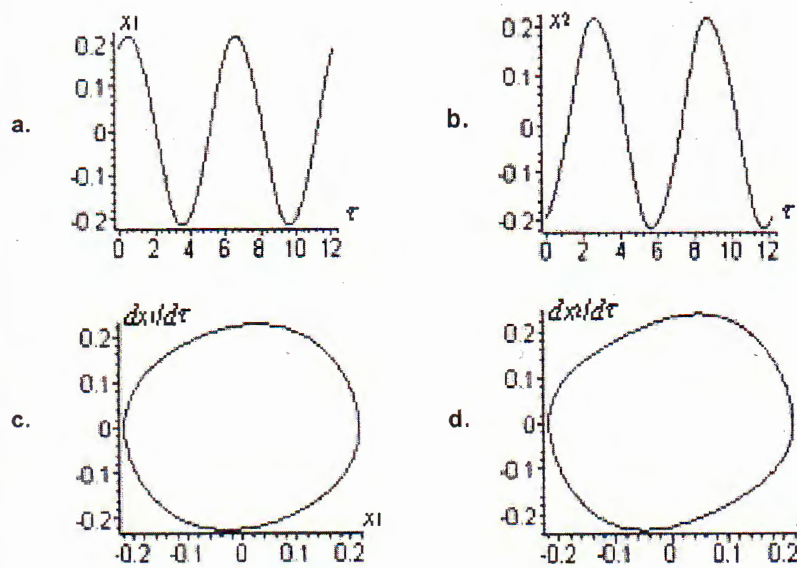

Fig 3. The periodic solution in the case of λ=0.02 by IHB method

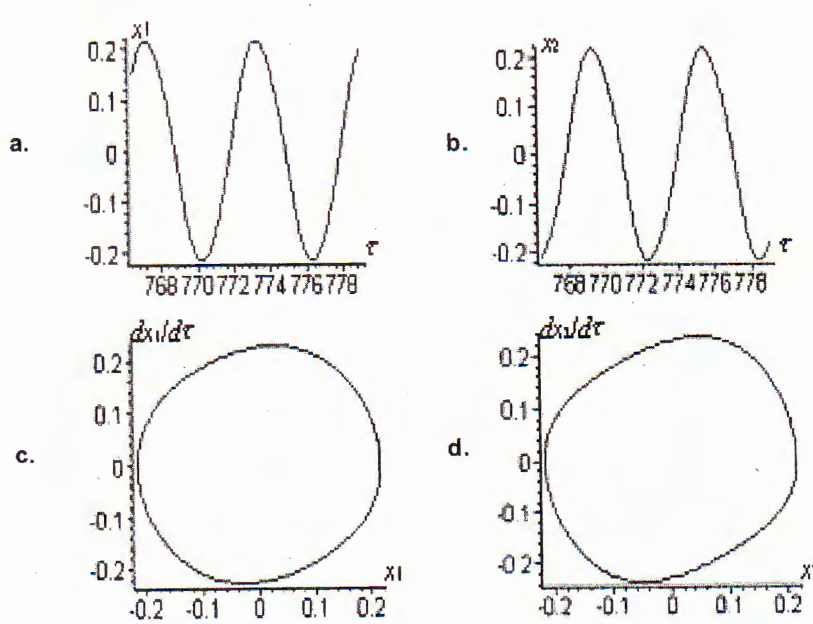

Fig 4. The periodic solution in the case of λ=0.02 by NI method

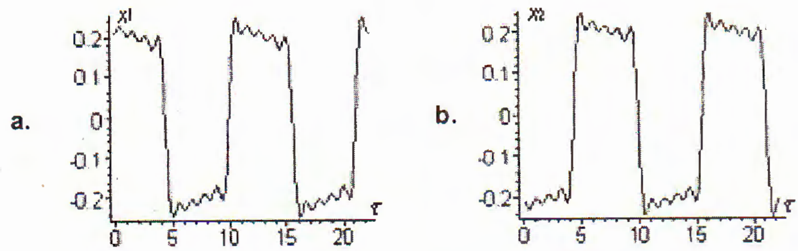

Fig 5. The periodic solution in the case of λ=1.0 by IHB method

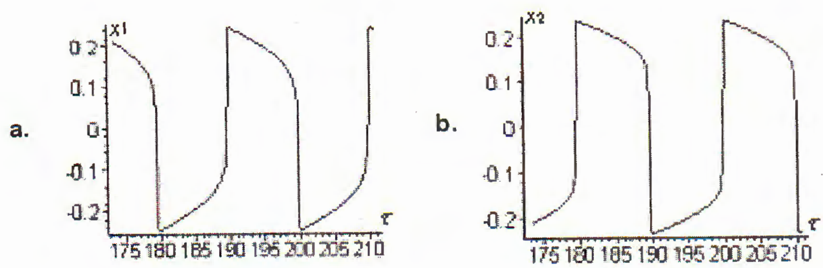

Fig 6. The periodic solution in the case of λ=1.0 by NI method

Table 1. The Floquet multipliers

λ	$\|\rho_1\|$	$\|\rho_2\|$	$\|\rho_3\|$	$\|\rho_4\|$
0.016	1.252781699	1.252781699	1.000000197	0.45230564
0.02	0.120558312	0.683333018	0.354077290	1.00000031
1.0	0.65507386	0.56961895	0.90627246×10^{-8}	0.9117628×10^{-9}

4 Conclusion

The incremental harmonic balance method presented by Lau (1983) and Pierre (1985) has been successfully applied to the investigation of periodic solutions of multi-degree-of-freedom nonlinear autonomous systems.

In the present paper the authors have studied in some detail periodic solutions of the motion equations in a kind of biped robots. Both stable and unstable periodic solutions are found by IHB method for some values of parameter λ.

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TÍNH TOÁN DAO ĐỘNG TUẦN HOÀN CỦA HỆ ÔTÔNÔM PHI TUYẾN NHIỀU BẬC TỰ DO BẰNG PHƯƠNG PHÁP CÂN BẰNG ĐIỀU HOÀ GIA LƯỢNG

Trong bài báo này trình bày việc tính toán dao động tuần hoàn của hệ ôtônôm phi tuyến nhiều bậc tự do bằng phương pháp cân bằng điều hoà gia lượng. Sử dụng định lý Floquet nghiên cứu sự ổn định của nghiệm tuần hoàn bằng cách xác định các trị riêng của ma trận đơn đạo. Sử dụng phần mềm MAPLE đã tính toán dao động tuần hoàn của hệ phương trình vi phân phi tuyến dạng van der Pol.