

A MODIFIED ASYMPTOTIC METHOD FOR A VAN-DER-POL OSCILLATOR WITH LARGE CUBIC RESTORING NONLINEARITY

NGUYEN VAN DINH
Institute of Mechanics

ABSTRACT. A modified asymptotic method is proposed and applied to evaluate the frequency of self-excited oscillation in a Van-Der-Pol oscillator with large cubic restoring nonlinearity. The result obtained can be used for large enough nonlinearity.

1 Introduction

In [3], to evaluate the frequency of steady self-excited oscillation in a Van-Der-Pol oscillator with large cubic restoring nonlinearity, a modified Poincaré method has been presented. There, assuming that the strongly nonlinear oscillator of interest is near certain linear one with unknown (to be evaluated) frequency, a formal small parameter is introduced and the governing differential equation is written in the form corresponding to weakly nonlinear system.

In this article, based on the same assumption and also on the variant of the asymptotic procedure presented in [1, 2], a modified asymptotic method is proposed to examine the problem considered. The results obtained are identical with those given in [3]. Compared with the modified Poincaré method, the modified asymptotic method is a little more complicated. This results from the fact that in the asymptotic method, we have to establish not only the algebraic equations determining steady state but the differential equations governing the amplitude-phase variation of the “general” solution. However, as compensation, the stability study is rather simple.

2 Systems under consideration. Frequency from the usual asymptotic method

Consider a Van-Der-Pol oscillator described by the differential equation

$$\ddot{x} + x = -\gamma x^3 + h(1 - x^2)\dot{x}, \quad (1.1)$$

where the significations of all the notations have been explained in [3]; $h > 0$ is assumed to be small but $\gamma > 0$ may be large enough.

The problem posed is to evaluate the frequency of steady self-excited oscillations satisfying the initial condition

$$\dot{x}(0) = 0. \quad (1.2)$$

For the sake of comparison, the usual asymptotic method (with a modified procedure) as presented in [2] is first used to discuss the case of small γ .

Being small, γ as h can be labeled by a small formal parameter ε . So, the system under consideration belongs to a weakly nonlinear type and its governing differential equation has the familiar form:

$$\ddot{x} + x = \varepsilon \{ -\gamma x^3 + h(1 - x^2)\dot{x} \}. \quad (1.3)$$

Following asymptotic expansions are used:

$$x = a \cos \psi + \varepsilon u_1(a, \psi) + \varepsilon^2 u_2(a, \psi) + \dots \quad (1.4)$$

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots \quad (1.5)$$

$$\dot{\psi} = 1 + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots \quad (1.6)$$

where a is the amplitude of order ε^0 of the first harmonic; ψ is the full phase angle with initial value $\psi(0) = 0$ (since the system considered is autonomous); A_i, B_i ($i = 1, 2, \dots$) are functions of a and u_i ($i = 1, 2, \dots$) are functions of (a, ψ) , 2π -periodic with respect to ψ . Note that u_i may contain the first harmonics $\sin \psi$ and $\cos \psi$.

Steady self-excited oscillation with a_0 as amplitude of order ε^0 is determined such that in each step of approximation (not at the end) following conditions are satisfied:

$$1 - \quad A_i(a_0) = 0 \quad \text{for every } i \quad (1.7)$$

$$2 - \quad \frac{\partial u_i(a_0, 0)}{\partial \psi} = 0 \quad \text{for every } i \quad (1.8)$$

It is not difficult to recognize that (1.7), (1.8) represent the stationarity condition and the initial condition respectively.

In the first approximation, we have:

$$\begin{aligned} & -2A_1 \sin \psi - 2aB_1 \cos \psi + \frac{\partial^2 u_1}{\partial \psi^2} + u_1 = -\gamma a^3 \cos^3 \psi - ha(1 - a^2 \cos^2 \psi) \sin \psi \\ & = -ha \left(1 - \frac{a^2}{4} \right) \sin \psi - \frac{3\gamma}{4} a^3 \cos \psi - \frac{\gamma a^3}{4} \cos 3\psi + \frac{ha^3}{4} \sin 3\psi. \end{aligned} \quad (1.9)$$

Hence:

$$A_1(a) = \frac{ha}{2} \left(1 - \frac{a^2}{4} \right), \quad B_1 = \frac{3\gamma}{8} a^2, \quad (1.10)$$

$$\begin{aligned} \frac{\partial^2 u_1}{\partial \psi^2} + u_1 &= -\frac{\gamma a^3}{4} \cos 3\psi + \frac{ha^3}{4} \sin 3\psi, \\ u_1 &= \frac{\gamma a^3}{32} \cos 3\psi - \frac{ha^3}{32} \sin 3\psi + a_1 \cos \psi + b_1 \sin \psi, \end{aligned} \quad (1.11)$$

where a_1, b_1 are two constants to be chosen.

By a_0 we denote the amplitude of order ε^0 of the self-excited oscillation; it is determined by the equation:

$$A_1(a) = 0 \quad \text{or} \quad a_0 = 2. \quad (1.12)$$

Using (1.12), the additional frequency $\varepsilon B_1(a)$ of order ε is:

$$B_1(a_0) = \frac{3\gamma a_0^2}{8} = \frac{3\gamma}{2}. \quad (1.13)$$

The initial condition (1.8) for $i = 1$ gives:

$$\frac{\partial u_1(a_0, 0)}{\partial \psi} = -\frac{3ha_0^3}{32} + b_1 = 0 \quad \text{or} \quad b_1 = b_{10} = \frac{3h}{4}. \quad (1.14)$$

In the second approximation, we have:

$$\begin{aligned} & -2A_2 \sin \psi - 2aB_2 \cos \psi + \frac{\partial^2 u_2}{\partial \psi^2} + u_2 = \\ & = -A_1 \frac{\partial A_1}{\partial a} \cos \psi + 2A_1 B_1 \sin \psi + aA_1 \frac{\partial B_1}{\partial a} \sin \psi + aB_1^2 \cos \psi - 2A_1 \frac{\partial^2 u_1}{\partial \psi \partial a} \\ & - 2B_1 \frac{\partial^2 u_1}{\partial \psi^2} - 3\gamma a^2 u_1 \cos^2 \psi + ha^2 u_1 \sin 2\psi \\ & + h(1 - a^2 \cos^2 \psi)(A_1 \cos \psi - aB_1 \sin \psi + \frac{\partial u_1}{\partial \psi}). \end{aligned} \quad (1.15)$$

Hence:

$$A_2(a) = \frac{-1}{2} \left\{ \frac{3\gamma a^2}{2} A_1 - h \left(1 - \frac{3a^2}{4} \right) a_1 - \frac{3\gamma ha^3}{8} + \frac{\gamma ha^5}{8} \right\}, \quad (1.16)$$

$$B_2(a) = \frac{-1}{2a} \left\{ \left[h \left(1 - \frac{3a^2}{4} \right) - \frac{\partial A_1}{\partial a} \right] A_1 - \frac{3\gamma a^2}{2} a_1 + h \left(1 - \frac{a^2}{4} \right) b_1 + \frac{h^2 a^5}{128} + \frac{15\gamma^2 a^5}{128} \right\}. \quad (1.17)$$

The stationarity condition (1.7) for $i = 2$ leads to the equations:

$$A_2(a_0) = 0 \quad \text{or} \quad 2ha_1 - 3\gamma h + 4\gamma h = 0 \quad \text{i.e.} \quad a_1 = a_{10} = \frac{-\gamma}{2}. \quad (1.18)$$

Using (1.12), (1.14), (1.18), the additional frequency of order ε^2 is obtained:

$$B_2(a_0) = -\frac{27\gamma^2 + h^2}{16}. \quad (1.19)$$

Thus, the first two formulas for frequency are:

$$\omega_I = 1 + \varepsilon B_1(a_0) = 1 + \varepsilon \frac{3\gamma}{2}, \quad (1.20)$$

$$\omega_{II} = 1 + \varepsilon B_1(a_0) + \varepsilon^2 B_2(a_0) = 1 + \varepsilon \frac{3\gamma}{2} - \varepsilon^2 \frac{27\gamma^2 + h^2}{16}. \quad (1.21)$$

In the second approximation the stability condition is

$$\left(\varepsilon \frac{\partial A_1}{\partial a} + \varepsilon^2 \frac{\partial A_2}{\partial a} \right)_0 < 0. \quad (1.22)$$

3 Frequency from a modified asymptotic method

In this section, a modified asymptotic method is proposed to determine steady state for the case of large cubic nonlinearity γ . As in [3], assuming that in the domain (x, \dot{x}) considered, the system of interest should be near certain linear one with unknown frequency ω , we rewrite (1.1) as:

$$\ddot{x} + \omega^2 x = \mu\{(\omega^2 - 1)x - \gamma x^3 + h(1 - x^2)\dot{x}\}, \quad (2.1)$$

where μ is the new small formal parameter introduced to indicate the smallness (by assumption) of the right hand side (especially, the smallness of the sum $(\omega^2 - 1)x - \gamma x^3$).

The asymptotic expansions are now of the form:

$$x = a \cos \psi + \mu u_1(a, \psi) + \mu^2 u_2(a, \psi) + \dots, \quad (2.2)$$

$$\dot{x} = \mu A_1(a) + \mu^2 A_2(a) + \dots, \quad (2.3)$$

$$\omega = \dot{\psi} = \omega_0 + \mu B_1(a) + \mu^2 B_2(a) + \dots, \quad (2.4)$$

where ω_0 (not 1 as in the case of small γ) is the approximate frequency of order μ^0 .

It is not difficult to establish following expressions:

$$\begin{aligned} \omega^2 x &= \omega_0^2 a \cos \psi + \mu\{\omega_0^2 u_1 + 2\omega_0 a B_1 \cos \psi\} \\ &+ \mu^2\{\omega_0^2 u_2 + 2\omega_0 u_1 B_1 + (2\omega_0 B_2 + B_1^2)a \cos \psi\} + \dots, \\ \dot{x} &= -\omega_0 a \sin \psi + \mu\left\{A_1 \cos \psi - a B_1 \sin \psi + \omega_0 \frac{\partial u_1}{\partial \psi}\right\} \\ &+ \mu^2\left\{A_2 \cos \psi - a B_2 \sin \psi + A_1 \frac{\partial u_1}{\partial a} + B_1 \frac{\partial u_1}{\partial \psi} + \omega_0 \frac{\partial u_2}{\partial \psi}\right\} + \dots, \\ \ddot{x} &= -\omega_0^2 \cos \psi + \mu\left\{-2\omega_0 A_1 \sin \psi - 2a\omega_0 B_1 \cos \psi + \omega_0^2 \frac{\partial^2 u_1}{\partial \psi^2}\right\} \\ &+ \mu^2\left\{-2\omega_0 A_2 \sin \psi - 2a\omega_0 B_2 \cos \psi + A_1 \frac{\partial A_1}{\partial a} \cos \psi\right. \\ &- 2A_1 B_1 \sin \psi - a A_1 \frac{\partial B_1}{\partial a} \sin \psi - a B_1^2 \cos \psi \\ &\left.+ 2\omega_0 A_1 \frac{\partial^2 u_1}{\partial \psi^2} + 2\omega_0 B_2 \frac{\partial^2 u_1}{\partial \psi^2} + \omega_0^2 \frac{\partial^2 u_2}{\partial \psi^2}\right\}. \end{aligned} \quad (2.5)$$

Substituting (2.2), (2.4), (2.5) into (2.1), then equating the terms of like powers of μ yield:

In the first approximation:

$$\begin{aligned} &-2\omega_0 A_1 \sin \psi + \omega_0^2 \left(\frac{\partial^2 u_1}{\partial \psi^2} + u_1\right) \\ &= (\omega_0^2 - 1)a \cos \psi - \gamma a^3 \cos^3 \psi - h(1 - a^2 \cos^2 \psi)\omega_0 a \sin \psi \\ &= -h\omega_0 a \left(1 - \frac{a^2}{4}\right) \sin \psi + \left(\omega_0^2 - 1 - \frac{3\gamma}{4}a^2\right)a \cos \psi + \frac{h\omega_0 a^3}{4} \sin 3\psi - \frac{\gamma a^3}{4} \cos 3\psi. \end{aligned} \quad (2.6)$$

Equating the terms of like harmonics in both sides of (2.6) we obtain:

$$-2\omega_0 A_1 = -h\omega_0 a \left(1 - \frac{a^2}{4}\right), \quad \omega_0^2 = 1 + \frac{3\gamma}{4}a^2, \quad (2.7)$$

$$\omega_0^2 \left(\frac{\partial^2 u_1}{\partial \psi^2} + u_1\right) = -\frac{\gamma a^3}{4} \cos 3\psi + \frac{h\omega_0 a^3}{4} \sin 3\psi. \quad (2.8)$$

The amplitude a_0 of order μ^0 of the steady self-excited oscillation is determined by the stationarity condition.

$$A_1(a) = 0 \quad \text{or} \quad a_0 = 2. \quad (2.9)$$

Using (2.9), from the second relation of (2.7), we obtain:

$$\omega_0^2 = 1 + \frac{3\gamma}{4}a_0^2 = 1 + 3\gamma. \quad (2.10)$$

The expression of u_1 is:

$$u_1 = \frac{\gamma a^3}{32\omega_0^2} \cos 3\psi - \frac{h\omega_0 a^3}{32\omega_0^2} \sin 3\psi + a_1 \cos \psi + b_1 \sin \psi, \quad (2.11)$$

where a_1, b_1 are two constants to be chosen.

To satisfy the initial condition (1.8) for $i = 1$, the constant b_1 should be taken so that:

$$\frac{\partial u_1(a_0, 0)}{\partial \psi} = 0 \quad \text{or} \quad b_1 = b_{10} = \frac{3h\omega_0 a_0^3}{32\omega_0^2} = \frac{3h}{4\sqrt{1+3\gamma}}. \quad (2.12)$$

Note that in the first approximation, B_1 is absent and a_1 remains indeterminate.

In the second approximation, we have:

$$\begin{aligned} & -2\omega_0 A_2 \sin \psi + \omega_0^2 \left(\frac{\partial^2 u_2}{\partial \psi^2} + u_2 \right) = \\ & = -A_1 \frac{\partial A_1}{\partial a} \cos \psi + 2A_1 B_1 \sin \psi + a \frac{\partial B_1}{\partial a} A_1 \sin \psi - 2\omega_0 A_1 \frac{\partial^2 u_1}{\partial \psi \partial a} \\ & - 2\omega_0 B_1 \left(\frac{\partial^2 u_1}{\partial \psi^2} + u_1 \right) + (\omega_0^2 - 1)a_1 + 2\omega_0 a B_1 \cos \psi - 3\gamma a^2 u_1 \cos^2 \psi \\ & + h\omega_0 a^2 \sin 2\psi \cdot u_1 + (A_1 \cos \psi - a B_1 \sin \psi + \omega_0 \frac{\partial u_1}{\partial \psi}) h(1 - a^2 \cos^2 \psi). \end{aligned} \quad (2.13)$$

Hence:

$$-2\omega_0 A_2 = 2A_1 B_1 + a A_1 \frac{\partial B_1}{\partial a} - h \left(1 - \frac{a^2}{4} \right) a B_1 - h\omega_0 \left(1 - \frac{3a^2}{4} \right) a_1 + \frac{\gamma h \omega_0 a^5}{32\omega_0^2}, \quad (2.14)$$

$$\begin{aligned} 0 & = 2\omega_0 a B_1 - \frac{3\gamma a^2}{2} a_1 - A_1 \frac{\partial A_1}{\partial a} + h \left(1 - \frac{3a^2}{4} \right) A_1 \\ & + h\omega_0 b_1 \left(1 - \frac{a^2}{4} \right) - \frac{3\gamma^2 a^5}{128\omega_0^2} + \frac{h^2 \omega_0^2 a^5}{128\omega_0^2}. \end{aligned} \quad (2.15)$$

The stationarity condition (1.7) for $i = 2$ leads to the equation:

$$A_2(a_0) = 0 \quad \text{or} \quad h\omega_0 \left(1 - \frac{3a_0^2}{4} \right) a_1 + \frac{\gamma h \omega_0 a_0^5}{32\omega_0^2} = 0, \quad (2.16)$$

from which we obtain:

$$a_1 = a_{10} = \frac{-\gamma}{2(1+3\gamma)}. \quad (2.17)$$

The additional frequency $B_1(a_0)$ of order μ can be deduced from (2.15):

$$B_1(a_0) = B_{10} = -\frac{9\gamma^2 + h^2\omega_0^2}{16\omega_0^3}. \quad (2.18)$$

Thus, the first two formulas for frequency are:

$$\omega_{I^*} = \omega_0 = \sqrt{1 + 3\gamma}, \quad (2.19)$$

$$\omega_{II^*} = \omega_0 + \varepsilon B_{10} = \sqrt{1 + 3\gamma} \left\{ 1 - \frac{9\gamma^2 + h^2(1 + 3\gamma)}{16(1 + 3\gamma)^2} \right\}. \quad (2.20)$$

In practice the stability condition in the first approximation is used and it is identical with the condition (1.2) given in section 1.

4 Comparison and conclusion

The formulas (1.20), (1.21) of ω_I , ω_{II} in section 1 and (2.19), (2.20) of ω_{I^*} , ω_{II^*} in section 2 are identical with the formulas (2.17), (2.18) and (3.16), (3.17) obtained in [3] by the standard and the modified Poincaré method, respectively.

The degree of accuracy of these formulas has been shown in [3] and this allows us to affirm that the modified asymptotic procedure presented may be useful.

It is not difficult for instance to affirm that the modified Poincaré method and the modified asymptotic method determine the same steady self-excited oscillations. Indeed, in [3] by using the dimensionless time $\tau = \omega t$ and by expanding

$$x = x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \dots$$

following differential equations are obtained:

$$\omega_0^2(x_0'' + x_0) = 0, \quad (3.1)$$

$$\omega_0^2(x_1'' + x_1) = (\omega_0^2 - 1)x_0 - \gamma x_0^3 + h(1 - x_0^2)\omega_0 x_0', \quad (3.2)$$

$$\begin{aligned} \omega_0^2(x_2'' + x_2) = & -2\omega_0\omega_1(x_1'' + x_1) + (\omega_0^2 - 1)x_1 + 2\omega_0\omega_1x_0 - 3\gamma x_0^2x_1 \\ & + h(1 - x_0^2)(\omega_0x_1' + \omega_1x_0') - 2hx_0x_1\omega_0x_0', \end{aligned} \quad (3.3)$$

where primes denote differentiation with respect to τ .

The general solution of (3.1) satisfying the initial condition $x'(0) = 0$ is $x_0 = a_0 \cos \tau$. For steady self-excited oscillation $A_1(a_0) = 0$, $A_2(a_0) = 0$, So the difference between the equations (2.6), (2.13) and (3.2), (3.3) consists only in the difference between the notations ($\psi \rightarrow \tau$, $u_1 \rightarrow x_1$, $u_2 \rightarrow x_2$, $B_1 \rightarrow \omega_1$).

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References

1. Mitropolskii Yu. A., Nguyen Van Dao, *Applied asymptotic method in nonlinear oscillations*, Kluwer Academic Publishers, 1997.
2. Nguyen Van Dinh, On a variant of the asymptotic procedure (for weakly nonlinear autonomous systems), *Vietnam Journal of Mechanics*, NCST of Vietnam, **25** (2003).

3. Nguyen Van Dinh and Tran Duong Tri, Frequency of a Van-Der-Pol oscillator with large cubic restoring nonlinearity, *Vietnam Journal of Mechanics*, NCST of Vietnam, 25 (2003).

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BIẾN THỂ CỦA PHƯƠNG PHÁP TIỆM CẬN CHO CHẤN TỬ VAN-DER-POL
CÓ PHI TUYẾN HỒI PHỤC MẠNH BẬC BA

Xét chấn tử Vandéc-pôn có phi tuyến phục hồi mạnh bậc ba với tự kích yếu. Một biến thể của phương pháp tiệm cận được đề xuất và áp dụng. Kết quả thu được trùng với kết quả cho bởi một biến thể của phương pháp Poincaré với độ chính xác được nâng cao rõ rệt và có thể chấp nhận kể cả khi độ phi tuyến khá mạnh.