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HIGHER ORDER STOCHASTIC AVERAGING METHOD FOR MECHANICAL SYSTEMS HAVING TWO DEGREES OF FREEDOM

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ABSTRACT. Higher order stochastic averaging method is widely used for investigating single-degree-of-freedom nonlinear systems subjected to white and coloured random noises. In this paper the method is further developed for two-degree-of-freedom systems. An application to a system with cubic damping is considered and the second approximation solution to the Fokker-Planck (FP) equation is obtained.

1 Introduction

The stochastic averaging method was extended by Stratonovich (1963) and has a mathematically rigorous proof by Khasminskii (1963). At present, the stochastic averaging method (SAM) is widely used in different problems of stochastic mechanics, such as vibration, stability and reliability problems (see e.g. Ariaratnam and Tam, 1979; Bolotin, 1984; Roberts and Spanos, 1986; Zhu, 1988). It should be noted that principally only first order SAM has been applied in practice and usually to systems subject to white noise or wideband random processes. However, the effect of some nonlinear terms is lost during the first order averaging procedure. In order to overcome this insufficiency, different averaging procedures for obtaining approximate solutions have been developed (see e.g. Mitropolskii et al, 1992; Red-Horse and Spanos, 1992; Zhu and Lin, 1994; Zhu et al, 1997). Recently a higher order averaging procedure using FP equation is developed in (Anh, 1993) and then applied to the systems having one degree of freedom under white noise and coloured noise excitations (Anh and Tinh, 1995; Tinh, 1999). In the present paper this procedure is further developed to lightly nonlinear systems subject to white noise excitations. An application to the system with cubic damping is considered.

2 Higher approximate solutions to FP equation

Consider the equations of motion of a mechanical system with two degrees of freedom

$$
\begin{aligned}
&\ddot{x}_1 + c_{11}x_1 + c_{12}x_2 = \varepsilon f_{11}(x,\dot{x}) + \varepsilon^2 f_{12}(x,\dot{x}) + \sqrt{\varepsilon}\sigma_1\dot{\xi}(t),\\
&\ddot{x}_2 + c_{21}x_1 + c_{22}x_2 = \varepsilon f_{21}(x,\dot{x}) + \varepsilon^2 f_{22}(x,\dot{x}) + \sqrt{\varepsilon}\sigma_2\dot{\xi}(t),
\end{aligned} \tag{2.1}
$$

where c_{ij} $(i,j=1,2)$, σ_1, σ_2 are constants $\sigma_1>0$, $\sigma_2>0$, ε is a small positive parameter, f_{ij}, $(i,j=1,2)$ are functions of $x=(x_1,x_2)$ and $\dot{x}=(\dot{x}_1,\dot{x}_2)$. The random excitation $\dot{\xi}(t)$ is a Gaussian white noise process with unit intensity.

Suppose that the characteristic equation of the system (2.1)

$$D(\lambda) = \begin{vmatrix} c_{11} - \lambda & c_{12} \\ c_{21} & c_{22} - \lambda \end{vmatrix} = 0,$$

has two distinct positive solutions ω_1^2, ω_2^2, $(0 < \omega_1^2 < \omega_2^2)$.

Now we introduce the principal coordinates u_1, u_2 from the primary coordinates x_1, x_2 by the relations

$$\begin{aligned} x_1 &= u_1 + u_2, \\ x_2 &= d_1 u_1 + d_2 u_2, \end{aligned} \tag{2.2}$$

where

$$d_1 = \frac{\omega_1^2 - c_{11}}{c_{12}} = \frac{c_{21}}{\omega_1^2 - c_{22}}, \quad d_2 = \frac{\omega_2^2 - c_{11}}{c_{12}} = \frac{c_{21}}{\omega_2^2 - c_{22}}. \tag{2.3}$$

Substituting (2.2) into (2.1) we have

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 &= \varepsilon F_{11} + \varepsilon^2 F_{12} + \sqrt{\varepsilon} G_1 \dot{\xi}(t), \\ \ddot{u}_2 + \omega_2^2 u_2 &= \varepsilon F_{21} + \varepsilon^2 F_{22} + \sqrt{\varepsilon} G_2 \dot{\xi}(t), \end{aligned} \tag{2.4}$$

where

$$F_{1j} = \frac{f_{2j} - d_2 f_{1j}}{d_1 - d_2}, \; F_{2j} = \frac{d_1 f_{1j} - f_{2j}}{d_1 - d_2}, \; G_1 = \frac{\sigma_2 - d_2 \sigma_1}{d_1 - d_2}, \; G_2 = \frac{d_1 \sigma_1 - \sigma_2}{d_1 - d_2}, \quad (j = 1, 2). \tag{2.5}$$

According to the averaging method the state coordinates (u_1, u_2) are to be transformed into the variables $a = (a_1, a_2)$ and $\varphi = (\varphi_1, \varphi_2)$ by the change

$$\begin{aligned} u_j &= a_j \cos \varphi_j, \\ \dot{u}_j &= -\omega_j a_j \sin \varphi_j. \end{aligned} \qquad (j = 1, 2) \tag{2.6}$$

By using differentiation formula [8] the system of equations (2.4) is transformed into the following system of equations

$$\begin{aligned} \dot{a}_j &= \varepsilon A_{1j}(a, \varphi) + \varepsilon^2 A_{2j}(a, \varphi) - \sqrt{\varepsilon} G_j \frac{\sin \varphi_j}{\omega_j} \dot{\xi}(t), \\ \dot{\varphi}_j &= \omega_j + \varepsilon B_{1j}(a, \varphi) + \varepsilon^2 B_{2j}(a, \varphi) - \sqrt{\varepsilon} G_j \frac{\cos \varphi_j}{a_j \omega_j} \dot{\xi}(t), \quad (j = 1, 2), \end{aligned} \tag{2.7}$$

where it is denoted

$$\begin{aligned} A_{1j}(a, \varphi) &= -\frac{F_{j1}(a, \varphi)}{\omega_j} \sin \varphi_j + \frac{G_j^2 \cos^2 \varphi_j}{2 a_j \omega_j^2}, \\ B_{1j}(a, \varphi) &= -\frac{F_{j1}(a, \varphi)}{a_j \omega_j} \cos \varphi_j - \frac{G_j^2 \cos \varphi_j \sin \varphi_j}{a_j^2 \omega_j^2}, \quad (j = 1, 2), \\ A_{2j}(a, \varphi) &= -\frac{F_{j2}(a, \varphi)}{\omega_j} \sin \varphi_j, \quad B_{2j}(a, \varphi) = -\frac{F_{j2}(a, \varphi)}{a_j \omega_j} \cos \varphi_j, \end{aligned} \tag{2.8}$$

The Fokker-Planck (FP) equation for the stationary probability density function $W(a, \varphi)$ takes the form

$$\sum_{j=1}^{2} \omega_j \frac{\partial W}{\partial \varphi_j} = -\varepsilon[A_1, B_1]L[W] - \varepsilon^2[A_2, B_2]L[W], \tag{2.9}$$

where the operators $[A_j, B_j]L[.]$, $j = 1, 2$ are defined as follows

$$\begin{aligned}
[A_1, B_1]L[W] = &\sum_{j=1}^{2} \Big[\frac{\partial}{\partial a_j}(A_{1j}W) + \frac{\partial}{\partial \varphi_j}(B_{1j}W)\Big] \\
&- \sum_{j=1}^{2}\sum_{s=0}^{1} \Big\{\frac{\partial^2}{\partial a_j \partial a_s}\Big(\frac{G_j G_s \sin\varphi_j \sin\varphi_s}{2\omega_j\omega_s} W\Big) \\
&+ \frac{\partial^2}{\partial a_j \partial \varphi_s}\Big(\frac{G_j G_s \sin\varphi_j \cos\varphi_s}{2a_s\omega_j\omega_s} W\Big) + \frac{\partial^2}{\partial \varphi_j \partial \varphi_s}\Big(\frac{G_j G_s \cos\varphi_j \cos\varphi_s}{2a_j a_s\omega_j\omega_s} W\Big)\Big\},
\end{aligned} \tag{2.10}$$

$$[A_2, B_2]L[W] = \sum_{j=1}^{2} \Big[\frac{\partial}{\partial a_j}(A_{2j}W) + \frac{\partial}{\partial \varphi_j}(B_{2j}W)\Big].$$

We seek the solution of (2.9) in the form

$$W(a, \varphi) = W_0(a, \varphi) + \varepsilon W_1(a, \varphi) + \varepsilon^2 W_2(a, \varphi) + \cdots \tag{2.11}$$

Substituting (2.11) into (2.9) and comparing the coefficients of like powers of ε we obtain

$$\varepsilon^0: \quad \sum_{j=1}^{2} \omega_j \frac{\partial W_0}{\partial \varphi_j} = 0, \tag{2.12}$$

$$\varepsilon^1: \quad \sum_{j=1}^{2} \omega_j \frac{\partial W_1}{\partial \varphi_j} = -[A_1, B_1]L[W_0], \tag{2.13}$$

$$\varepsilon^2: \quad \sum_{j=1}^{2} \omega_j \frac{\partial W_2}{\partial \varphi_j} = -\big\{[A_2, B_2]L[W_0] + [A_1, B_1]L[W_1]\big\}, \tag{2.14}$$

....

From (2.12) we get

$$W_0 = W_0(a). \tag{2.15}$$

The arbitrary integration function $W_0(a)$ must be chosen from the condition for the function $W_1(a, \varphi)$ to be periodic to φ.

Thus, we get from (2.13)

$$\big\langle [A_1, B_1]L[W_0(a)] \big\rangle = 0, \tag{2.16}$$

where $\langle . \rangle$ is the averaging operator with respect to φ

$$\langle . \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} .(.) d\varphi_1 d\varphi_2 . \tag{2.17}$$

Substituting (2.10) into (2.16) yields

$$\sum_{j=1}^{2} \left\{ \frac{\partial}{\partial a_j} (\langle A_{1j} \rangle W_0(a)) - \frac{G_j^2}{4\omega_j^2} \frac{\partial^2 W_0(a)}{\partial a_j^2} \right\} = 0. \tag{2.18}$$

The second term $W_1(a, \varphi)$ in (2.11) is determined from (2.13), using Fourier expansion

$$[A_1, B_1] L[W_0(a)] = W_0(a) \sum_{k_1} \sum_{k_2} C_{k_1 k_2}(a) \exp[i(k_1\varphi_1 + k_2\varphi_2)], \tag{2.19}$$

where

$$C_{k_0 k_1}(a) = \frac{1}{(2\pi)^2 W_0(a)} \int_0^{2\pi} \int_0^{2\pi} [A_1, B_1] L[W_0(a)] \exp[-i(k_0\varphi_0 + k_1\varphi_1)] d\varphi_0 d\varphi_1. \tag{2.20}$$

Sustituting (2.19) into (2.13) yields

$$W_1(a, \varphi) = W_0(a) \Big[W_{10}(a) + \sum_{k_1} \sum_{k_2} \frac{C_{k_1 k_2}(a)}{k_1\omega_1 + k_2\omega_2} \exp[i(k_1\varphi_1 + k_2\varphi_2)] \Big], \tag{2.21}$$

where

$$k_1\omega_1 + k_2\omega_2 \neq 0. \tag{2.22}$$

The arbitrary integration function $W_{10}(a)$ must be chosen from the condition for the function $W_2(a, \varphi)$ to be periodic to φ. Similarly, we can find the third term $W_2(a, \varphi)$ in (2.11).

3 Application

In order to illustrate the procedure proposed we consider the system with cubic damping whose equation of motion takes the form

$$\begin{aligned} \ddot{x}_1 + c_{11}x_1 - c_{22}x_2 &= -2\varepsilon(h_{11}\dot{x}_1 - h_{12}\dot{x}_2) - \varepsilon^2 \beta \dot{x}_1^3 + \sqrt{\varepsilon}\sigma_1 \dot{\xi}(t), \\ \ddot{x}_2 - c_{22}x_1 + c_{22}x_2 &= -2\varepsilon h_{12}(\dot{x}_2 - \dot{x}_1) + \sqrt{\varepsilon}\sigma_2 \dot{\xi}(t). \end{aligned} \tag{3.1}$$

The physical model of this system is represented in Fig. 1. Where k_{11}, k_{12}, h_{11}, h_{12}, β, σ_1, σ_2 are positive constants, $m_1 = m_2 = 1$ and

$$\begin{aligned} c_{11} &= k_{11} + k_{12}, \quad c_{22} = k_{12}, \quad h_{11} = h_1 + h_{12}, \\ R_1 &= -\varepsilon^2 \beta \dot{x}_1^2 + \sqrt{\varepsilon}\sigma_1 \dot{\xi}(t), \quad R_2 = \sqrt{\varepsilon}\sigma_2 \dot{\xi}(t). \end{aligned} \tag{3.2}$$

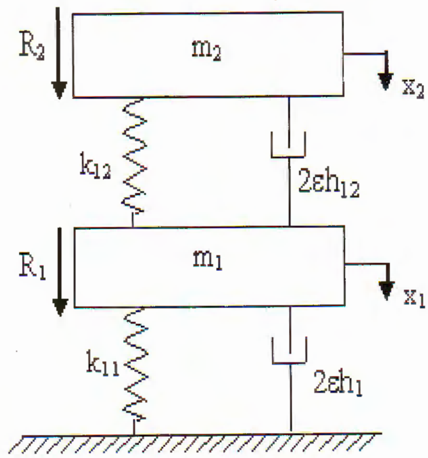

Fig. 1

In this case we have

$$
\begin{aligned}
&f_{11} = -2h_{11}\dot{x}_1 + 2h_{12}\dot{x}_2, \quad f_{12} = -\beta\dot{x}_1^3,\\
&f_{21} = 2h_{21}\dot{x}_1 - 2h_{12}\dot{x}_2, \quad f_{22} = 0,\\
&\omega_{1,2}^2 = \frac{c_{11} + c_{22} \mp \sqrt{(c_{11} - c_{22})^2 + 4c_{22}^2}}{2},\\
&d_{1,2} = \frac{c_{11} - c_{22}}{2c_{22}} \pm \frac{\sqrt{\triangle}}{2c_{22}},\\
&d_1 - d_2 = \frac{\sqrt{\triangle}}{c_{22}}, \quad \triangle = (c_{11} - c_{22})^2 + 4c_{22}^2.
\end{aligned}
\tag{3.3}
$$

Substituting (3.3) into (2.5) yields

$$
\begin{aligned}
&F_{11} = p_{11}\dot{x}_1 + p_{12}\dot{x}_2, \quad F_{12} = p_{13}\dot{x}_1^3,\\
&F_{21} = p_{21}\dot{x}_1 + p_{22}\dot{x}_2, \quad F_{22} = p_{23}\dot{x}_1^3,
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
&p_{11} = \frac{2(h_{12} + d_2 h_{11})}{d_1 - d_2}, \quad p_{12} = \frac{-2h_{12}(1 + d_2)}{d_1 - d_2}, \quad p_{13} = \frac{d_2\beta}{d_1 - d_2},\\
&p_{21} = \frac{2(d_1 h_{11} + h_{12})}{d_1 - d_2}, \quad p_{22} = \frac{2h_{12}(1 + d_2)}{d_1 - d_2}, \quad p_{23} = -\frac{d_1\beta}{d_1 - d_2},
\end{aligned}
\tag{3.5}
$$

From (2.8), using (2.2), (2.6) and (3.4), after calculations we obtain

$$
\langle A_{11} \rangle = \frac{p_{11} + p_{12}d_1}{2}a_1 + \frac{G_1^2}{4\omega_1^2 a_1}, \quad \langle A_{12} \rangle = \frac{p_{21} + p_{22}d_2}{2}a_2 + \frac{G_2^2}{4\omega_2^2 a_2}, \tag{3.6}
$$

$$
\langle A_{21} \rangle = \frac{3p_{13}}{8}(\omega_1^2 a_1^3 + 2\omega_2^2 a_1 a_2^2), \quad \langle A_{22} \rangle = \frac{3p_{23}}{8}(2\omega_1^2 a_1^2 a_2 + \omega_2^2 a_2^3). \tag{3.7}
$$

Substituting (3.6) into (2.18) we have

$$W_0(a) = Ca_1a_2\exp\Big[\frac{\omega_1^2}{G_1^2}(p_{11}+p_{12}d_1)a_1^2 + \frac{\omega_2^2}{G_2^2}(p_{21}+p_{22}d_2)a_2^2\Big]. \tag{3.8}$$

The parameters in the expression of $W_0(a)$ must satisfy the inequalities

$$\gamma_1 = \frac{\omega_1^2}{G_1^2}(p_{11}+p_{12}d_1) < 0, \quad \gamma_2 = \frac{\omega_2^2}{G_2^2}(p_{21}+p_{22}d_2) < 0. \tag{3.9}$$

Substituting (3.8) into (2.13), using (2.6), (2.8) and (3.5), after some calculations we obtain

$$\begin{aligned} W_{11}(a,\varphi) = &\frac{p_{11}+p_{12}d_1}{2\omega_1}\sin 2\varphi_1 + \frac{p_{21}+p_{22}d_2}{2\omega_2}\sin 2\varphi_2 \\ &+ \frac{1}{\omega_1^2-\omega_2^2}(\omega_2 s_{12}a_1a_2 - \frac{G_1G_2}{\omega_1}\frac{1}{a_1a_2})\sin\varphi_1\cos\varphi_2 \\ &+ \frac{1}{\omega_1^2-\omega_2^2}(\frac{G_1G_2}{\omega_2}\frac{1}{a_1a_2} - \omega_1 s_{12}a_1a_2)\cos\varphi_1\sin\varphi_2, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} s_{12} = &\frac{2\omega_1\omega_2}{G_1^2G_2^2}\big[G_2^2(p_{11}+d_1p_{12})(p_{11}+d_2p_{12}) + G_1^2(p_{21}+d_1p_{22})(p_{21}+d_2p_{22}) \\ &- 2\omega_1\omega_2(p_{11}+d_1p_{12})(p_{21}+d_2p_{22})\big]. \end{aligned} \tag{3.11}$$

Substituting (3.8) and (3.11) into (2.14) we have the equation for the arbitrary function $W_{10}(a)$ in the form

$$\sum_{j=1}^{2}\{\frac{\partial}{\partial a_j}[\langle A_{1j}\rangle W_0(a)W_{10}(a)] - \frac{G_j^2}{4\omega_j^2}\frac{\partial^2}{\partial a_j^2}[W_0(a)W_{10}(a)]\} = -\sum_{j=1}^{2}\frac{\partial}{\partial a_j}[< A_{2j} > W_0(a)]. \tag{3.12}$$

From (3.12), using (3.6) and (3.7) we have

$$W_{10}(a) = \alpha_1 a_1^2 + \alpha_2 a_2^2 + \alpha_{12}a_1^2a_2^2 + \alpha_{11}a_1^4 + \alpha_{22}a_2^4, \tag{3.13}$$

where

$$\begin{aligned} &\alpha_1 = -\frac{3\omega_1^2 d_1\beta}{2(d_1-d_2)\gamma_1}, \quad \alpha_2 = -\frac{3\omega_2^2 d_2\beta}{2(d_1-d_2)\gamma_2}, \\ &\alpha_{11} = \frac{3\omega_1^4 d_2(d_1-d_2)\beta}{8(\sigma_2-d_2\sigma_1)^2}, \quad \alpha_{22} = -\frac{3\omega_2^4 d_1(d_1-d_2)\beta}{8(\sigma_2-d_1\sigma_1)^2}, \\ &\alpha_{12} = \frac{3\omega_1^2\omega_2^2\beta(d_2G_2^2\gamma_1 - d_1G_1^2\gamma_2)}{2G_1^2G_2^2(\gamma_1+\gamma_2)}. \end{aligned} \tag{3.14}$$

Thus, the second order approximate solution of the FP equation (2.9) for the system (3.1) takes the form

$$W(a,\varphi) = W_0(a)\{1+\varepsilon[W_{10}(a)+W_{11}(a,\varphi)]\}. \tag{3.15}$$

where $W_0(a)$, $W_{11}(a)$ and $W_{10}(a)$ are defined in (3.8), (3.10) and (3.13), respectively. It is seen from (3.13) and (3.1) that the effect of the nonlinear term $\varepsilon^2\beta\dot{x}^3$ is shown in (3.13) and (3.14).

The approximate mean squares $E[x_1^2]$ and $E[x_2^2]$ are to be found

$$E[x_i^2] = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\infty}\int_0^{\infty} x_i^2 W(a,\varphi)da_1da_2d\varphi_1d\varphi_2, \quad (i=1,2). \tag{3.16}$$

Substituting (3.15) and x_i in (2.2), (2.6) into (3.16), after calculations we have

$$\begin{aligned}
E[x_1^2] &= -\frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2} + \varepsilon\frac{2}{\gamma_1^3\gamma_2^3}[\alpha_1\gamma_1\gamma_2^3 + \alpha_2\gamma_1^3\gamma_2 - 3\alpha_{12}\gamma_1\gamma_2(\gamma_1+\gamma_2)\\
&+ 2\alpha_{11}(\gamma_1^3+\gamma_1^2\gamma_2 - 2\gamma_1\gamma_2^2 - 6\gamma_2^3) + 2\alpha_{22}(\gamma_2^3+\gamma_2^2\gamma_1 - 2\gamma_2\gamma_1^2 - 6\gamma_1^3)],\\
E[x_2^2] &= -\frac{d_1^2\gamma_2+d_2^2\gamma_1}{\gamma_1\gamma_2} + \varepsilon\frac{2}{\gamma_1^3\gamma_2^3}[\alpha_1 d_1^2\gamma_1\gamma_2^3 + \alpha_2 d_2^2\gamma_1^3\gamma_2 - 3\alpha_{12}\gamma_1\gamma_2(d_1^2\gamma_1 + d_2^2\gamma_2)\\
&+ 2\alpha_{11}(d_2^2\gamma_1^3 + d_1^2\gamma_1^2\gamma_2 - 2d_2^2\gamma_1\gamma_2^2 - 6d_1^3\gamma_2^3)\\
&+ 2\alpha_{22}(d_1^2\gamma_2^3 + d_2^2\gamma_2^2\gamma_1 - 2d_1^2\gamma_2\gamma_1^2 - 6d_2^2\gamma_1^3)].
\end{aligned} \tag{3.17}$$

In the case $\beta = 0$ (linear system) we have

$$E[x_1^2] = -\frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}, \quad E[x_2^2] = -\frac{d_1^2\gamma_1+d_2^2\gamma_2}{\gamma_1\gamma_2}. \tag{3.18}$$

4 Conclusion

For many years the stochastic averaging method has been a very useful tool for investigating non-linear vibration systems subject to white noise and coloured noise excitations. In this paper, the higher order stochastic averaging method is applied to non-linear systems with two degrees of freedom subject to white noise excitations. The application to the system with cubic damping is considered and shows the effect of the non-linear term to the mean square response of the system.

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PHƯƠNG PHÁP TRUNG BÌNH NGẪU NHIÊN BẬC CAO
ĐỐI VỚI HỆ HAI BẬC TỰ DO

Phương pháp trung bình ngẫu nhiên bậc cao đã được áp dụng rộng rãi đối với các hệ dao động phi tuyến một bậc tự do chịu kích động ngẫu nhiên dạng ồn trắng và ồn màu. Trong bài báo này, phương pháp tiếp tục được trình bày đối với các hệ phi tuyến yếu hai bậc tự do chịu kích động ngẫu nhiên dạng ồn trắng. Sau đó phương pháp được áp dụng để xác định nghiệm xấp xỉ bậc hai của phương trình Fokker-Planck đối với hệ có cản phi tuyến bậc ba.