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ON A VARIANT OF THE ASYMPTOTIC PROCEDURE
(II: WEAKLY NONLINEAR NON-AUTONOMOUS SYSTEMS)

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ABSTRACT. A variant of the asymptotic method is proposed to construct steady solution of weakly nonlinear non-autonomous oscillating systems. The amplitude and the dephase angle of order ε^0 are used as variables, the uniqueness of the asymptotic expansions is assured by stationarity conditions.

1 Introduction

In [3], to determine steady state in weakly nonlinear autonomous oscillating systems, a variant of the asymptotic procedure has been proposed, consisting of two modifications: - the approximate amplitude a of order ε^0 of the first harmonic is chosen as variable in asymptotic expansions and - the arbitrariness of the latter is removed by initial conditions and by stationarity conditions.

In the present paper, the case of non-autonomous system is considered. Besides the mentioned amplitude of order ε^0, the dephase angle θ of same order is used as the second variable and the additional stationarity conditions are used to assure the uniqueness of the asymptotic expansions. It is shows that steady state (stationary oscillation) can be successively determined in each step of approximation and the solution obtained is identical with that given by the Poincaré method.

2 Systems under consideration. The usual asymptotic procedure

Consider a weakly nonlinear non-autonomous oscillating system described by the differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, \omega t), \tag{1.1}$$

where ω is the exciting frequency; $f(x, \dot{x}, \omega t)$ is a function of $(x, \dot{x}, \omega t)$, 2π-periodic with respect to ωt; the significations of other notations have been explained in [3].

For simplicity, $f(x, \dot{x}, \omega t)$ is assumed to be a finite Fourier series in t with polynomial in $(x, \dot{x})$ coefficients.

To be able to make a comparison, the usual asymptotic procedure is briefly recalled [1]. First, following asymptotic expansions are used:

$$x = a\cos\psi + \varepsilon u_1(a, \theta, \psi) + \varepsilon^2 u_2(a, \theta, \psi) + \dots, \quad \psi = \omega t + \theta, \tag{1.2}$$

$$\dot{a} = \varepsilon A_1(a, \theta) + \varepsilon^2 A_2(a, \theta) + \dots, \tag{1.3}$$

$$\dot{\theta} = \varepsilon B_1(a, \theta) + \varepsilon^2 B_2(a, \theta) + \dots, \tag{1.4}$$

where a and θ are the full amplitude and the full dephase angle of the first harmonic, respectively; A_i, B_i $(i = 1, 2, \dots)$ are functions of (a, θ); u_i $(i = 1, 2, \dots)$ are functions of (a, θ, ψ), 2π-periodic with respect to ψ, do not containing the first harmonics $\sin\psi$, $\cos\psi$.

Substituting (1.2) into (1.1), using (1.3), (1.4), expanding the right hand side in Taylor series of ε, equating the terms of like powers of ε yield in the first approximation

$$
\begin{aligned}
-2\omega A_1 \sin\psi - 2\omega a B_1 \cos\psi + \omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) &= f^{(1)}(a, \theta, \psi) \\
&= f(a\cos\psi, -\omega a \sin\psi, \psi - \theta).
\end{aligned}
\tag{1.5}
$$

Expanding $f^{(1)}$ in Fourier series, that is:

$$
f^{(1)}(a, \theta, \psi) = f_0^{(1)}(a, \theta) + \sum_{n=1}^{N_1} \big[S_n^{(i)}(a, \theta) \sin n\psi + C_n^{(i)}(a, \theta) \cos n\psi\big], \tag{1.6}
$$

then equating the terms of like harmonics, we obtain:

$$
A_1(a, \theta) = \frac{-1}{2\omega} S_1^{(1)}(a, \theta), \quad B_1(a, \theta) = \frac{-1}{2\omega a} C_1^{(1)}(a, \theta), \tag{1.7}
$$

$$
\omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = f_0^{(1)}(a, \theta) + \sum_{n=2}^{N_1} \big[S_n^{(1)}(a, \theta) \sin n\psi + C_n^{(1)}(a, \theta) \cos n\psi\big]. \tag{1.8}
$$

The expression of u_1 is of the form:

$$
u_i(a, \theta, \psi) = \frac{1}{\omega^2}\Big\{f_0^{(i)}(a, \theta) - \sum_{n=2}^{N_i} \frac{1}{n^2 - 1}\big[S_n^{(i)}(a, \theta) \sin n\psi + C_n^{(i)}(a, \theta) \cos n\psi\big]\Big\}. \tag{1.9}
$$

The same procedure leads to A_i, B_i, u_i in the ith approximation $(i = 2, 3, \dots)$.

In the end i.e. in the nth approximation, the full amplitude a_* and the full dephase angle θ_* of stationary oscillation are determined by stationarity conditions which are two following equations:

$$
\begin{aligned}
A(a, \theta) &= \varepsilon A_1(a, \theta) + \dots + \varepsilon^n A_n(a, \theta) = 0, \\
B(a, \theta) &= \varepsilon B_1(a, \theta) + \dots + \varepsilon^n B_n(a, \theta) = 0.
\end{aligned}
\tag{1.10}
$$

The stability conditions take the form:

$$
\Big(\frac{\partial A}{\partial a} + \frac{\partial B}{\partial\theta}\Big)_* < 0, \quad \Big(\frac{\partial A}{\partial a}\frac{\partial B}{\partial\theta} - \frac{\partial A}{\partial\theta}\frac{\partial B}{\partial a}\Big)_* > 0. \tag{1.11}
$$

3 A variant of the asymptotic procedure

In this section, a variant of the asymptotic procedure is applied to discuss the problem of interest.

First, in the expansions (1.2), (1.3), (1.4), the variables a and θ are now understood as the amplitude and the dephase angle of order ε^0 (not the full amplitude and the full dephase angle) of the first harmonic. Consequently, the requirement on the absence of the first harmonics $\sin\psi$, $\cos\psi$ in u_i $(i = 1, 2, 3, \dots)$ must be rejected i.e. u_i may contain the sum $(a_i \cos\psi + b_i \sin\psi)$ where a_i, b_i are constants to be chosen.

The identity (1.5) retains its form; so, from (1.5) we obtain the same expressions (1.7) of $A_1(a,\theta)$, $B_1(a,\theta)$ and the same differential equation (1.8) governing u_1.

However, since $u_1(a,\theta,\psi)$ may contain the first harmonics $\sin\psi$ and $\cos\psi$, the expression (1.9) - is replaced by

$$u_1(a,\theta,\psi) = \frac{1}{\omega^2}\Big\{f_0^{(1)}(a,\theta) - \sum_{n=2}^{N_1}\frac{1}{n^2-1}\big[S_n^{(1)}(a,\theta)\sin n\psi + C_n^{(1)}(a,\theta)\cos n\psi\big]\Big\} + a_1\cos\psi + b_1\sin\psi \tag{2.1}$$

where a_1, b_1 are constants to be chosen.

The amplitude and the dephase angle (a_0,θ_0) of order ε^0 of stationary oscillation are immediately determined in the first approximation by the equations

$$A_1(a,\theta) = \frac{-1}{2\omega}S_1^{(1)}(a,\theta) = 0, \quad B_1(a,\theta) = \frac{-1}{2\omega a}C_1^{(1)}(a,\theta) = 0. \tag{2.2}$$

In the second approximation we have

$$\begin{aligned}
&-2\omega A_2\sin\psi - 2\omega a B_2\cos\psi + \omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) = f_2(a,\theta,\psi)\\
&= \Big(aB_1^2 - A_1\frac{\partial A}{\partial a} - B_1\frac{\partial A_1}{\partial\theta}\Big)\cos\psi + \Big(2A_1B_1 + aA_1\frac{\partial A_1}{\partial a} + aB_1\frac{\partial B_1}{\partial\theta}\Big)\sin\psi\\
&-2\omega A_1\frac{\partial^2 u_1}{\partial\psi\partial a} - 2\omega B_1\frac{\partial^2 u_1}{\partial\psi\partial\theta} - 2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2} + u_1 f_x + \Big(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\Big)f_{\dot{x}}\\
&= \big\{f_0^{(2)}(a,\theta) + a_1 f_0'(a,\theta) + b_1 f_0''(a,\theta)\big\}\\
&+ \sum_{n=1}^{N_2}\Big\{\big[S_n^{(2)}(a,\theta) + a_1 S_n'(a,\theta) + b_1 S_n''(a,\theta)\big]\sin n\psi\\
&+ \big[C_n^{(2)}(a,\theta) + a_1 C_n'(a,\theta) + b_1 C_n''(a,\theta)\big]\cos n\psi\Big\}.
\end{aligned} \tag{2.3}$$

Equating the terms of like harmonics, we get:

$$\begin{aligned}
A_2(a,\theta) &= \frac{-1}{2\omega}\big\{S_1^{(2)}(a,\theta) + a_1 S_1'(a,\theta) + b_1 S_1''(a,\theta)\big\},\\
B_2(a,\theta) &= \frac{-1}{2\omega a}\Big\{C_1^{(2)}(a,\theta) + a_1 C_1'(a,\theta) + b_1 C_1''(a,\theta)\Big\},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) &= \big\{f_0^{(2)}(a,\theta) + a_1 f_0'(a,\theta) + b_1 f_0''(a,\theta)\big\}\\
&+ \sum_{n=2}^{N_2}\Big\{\big[S_n^{(2)}(a,\theta) + a_1 S_n'(a,\theta) + b_1 S_n''(a,\theta)\big]\sin n\psi\\
&+ \big[C_n^{(2)}(a,\theta) + a_1 C_n'(a,\theta) + b_1 C_n''(a,\theta)\big]\cos n\psi\Big\}.
\end{aligned} \tag{2.5}$$

Note that A_2, B_2 as u_2 remain inderterminate because of the presence of a_1, b_1. A question arises: how to choose a_1 and b_1? It is natural to impose on A_2, B_2 the conditions

$$A_2(a_0, \theta_0) = 0, \quad B_2(a_0, \theta_0) = 0 \tag{2.6}$$

which are called stationarity conditions (not $\varepsilon A_1 + \varepsilon^2 A_2 = 0$, $\varepsilon B_1 + \varepsilon^2 B_2 = 0$ and not a, b but $a = a_0$, $b = b_0$). In detailed form:

$$\begin{aligned} a_1 S_1'(a_0, \theta_0) + b_1 S_1''(a_0, \theta_0) + S_1^{(2)}(a_0, \theta_0) = 0, \\ a_1 C_1'(a_0, \theta_0) + b_1 C_1''(a_0, \theta_0) + C_1^{(2)}(a_0, \theta_0) = 0. \end{aligned} \tag{2.7}$$

Assuming that the determinant

$$D = (S_1' C_1'' - S_1'' C_1')_0 \neq 0, \tag{2.8}$$

we obtain:

$$a_1 = a_{10} = D_1/D, \quad b_1 = b_{10} = D_2/D, \tag{2.9}$$

where

$$D_1 = (S_1^{(1)} C_1^{(2)} - S_1^{(2)} C_1'')_0, \quad D_2 = (S_1^{(2)} C_1' - S_1' C_1^{(2)})_0. \tag{2.10}$$

Now, with a_1, b_1 given by (2.8), the expressions of A_2, B_2 and that of u_1 are fully determined.

The expression of $u_2(a, \theta, \psi)$ is:

$$\begin{aligned} u_2(a, \theta, \psi) = {} & \frac{1}{\omega^2}\{f_0^{(2)}(a, \theta) + a_{10} f_0'(a, \theta) + b_{10} f_0''(a, \theta)\} + a_2 \cos\psi + b_2 \sin\psi \\ & - \frac{1}{\omega^2} \sum_{n=2}^{N_2} \frac{1}{n^2 - 1} \Big\{ \big[S_n^{(2)}(a, \theta) + a_{10} S_n'(a, \theta) + b_{10} S_n''(a, \theta)\big] \sin n\psi \\ & + \big[C_n^{(2)}(a, \theta) + a_{10} C_n'(a, \theta) + b_{10} C_n''(a, \theta)\big] \cos n\psi \Big\}, \end{aligned} \tag{2.11}$$

where a_2, b_2 are two constants to be chosen.

In the third approximation, we determine a_2, b_2 using the stationaity conditions

$$A_3(a_0, \theta_0) = 0, \qquad B_3(a_0, \theta_0) = 0 \tag{2.12}$$

(not $\varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 = 0$, $\varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3 = 0$ and not $a_1 \theta$ but a_0, θ_0).

Continuing this procedure, in the nth approximation, we determine a_{n-1}, b_{n-1} using the stationary conditions

$$A_n(a_0, \theta_0) = 0, \quad B_n(a_0, \theta_0) = 0. \tag{2.13}$$

It is necessary to insist again that stationarity conditions are not expressed by two equations as in the usual asymptotic procedure but by a set of equations

$$\begin{aligned} A_1(a_0, \theta_0) = 0, \quad & B_1(a_0, \theta_0) = 0, \\ A_2(a_0, \theta_0) = 0, \quad & B_2(a_0, \theta_0) = 0, \\ \dots \quad & \dots \\ A_n(a_0, \theta_0) = 0, \quad & B_n(a_0, \theta_0) = 0. \end{aligned}$$

Also note that the expressions of $A_i(a, \theta)$, $B_i(a, \theta)$ $(i = 2, 3, \dots, n)$ differ from the corresponding ones in the usual asymptotic procedure.

The stability conditions take the same formal form (1.11) given on Section 1, (a_*, θ_* must be replaced by a_0, θ_0).

4 Comparison with the Poincaré method

As in [3], let us compare the variant presented in Section 2 with the Poincaré method. Following the latter, the dimensionless time $\tau = \omega t$ is introduced and the differential equation (1.1) is written as:

$$\omega^2(x'' + x) = \varepsilon f(x, \omega x', \tau), \tag{3.1}$$

where primes denote derivation with respect to τ.

Expanding $x(\tau)$ in powers of ε, that is:

$$x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots, \tag{3.2}$$

then equating the terms of like powers of ε in both sides of (3.1) yield:

$$\omega^2(x_0'' + x_0) = 0, \tag{3.3}$$

$$\omega^2(x_1'' + x_1) = f(x_0, \omega x_0', \tau), \tag{3.4}$$

$$\omega^2(x_2'' + x - 2) = x_1 f_x(x_0, \omega x_0', \tau) + \omega x_1' f_{\dot{x}}(x_0, \omega x_0', \tau), \tag{3.5}$$

$$\dots \quad \dots \quad \dots \quad \dots$$

The general solution of (3.3) takes the form

$$x_0 = a_0 \cos(\tau + \theta_0), \tag{3.6}$$

where a_0, θ_0 are two constants to be determined.

With regard to (3.6), the differential equations (3.4), (3.5) become:

$$\omega^2(x_1'' + x_1) = f(a_0 \cos(\tau + \theta_0), -\omega a_0 \sin(\tau + \theta - 0), \tau), \tag{3.7}$$

$$\begin{aligned} \omega^2(x_2'' + x_2) = {} & x_1 f_x(a_0 \cos(\tau + \theta_0), -\omega a_0 \sin(\tau + \theta_0), \tau) \\ & + \omega_1 x_1' f_{\dot{x}}(a \cos(\tau + \theta_0), -\omega a_0 \sin(\tau + \theta_0), \tau), \end{aligned} \tag{3.8}$$

$$\dots \quad \dots \quad \dots$$

Except the absence of A_1, B_1, A_2, B_2, the difference between (3.7), (3.8) and (1.5), (2.2) is of formal character and consists only in the difference between the notations ($\tau + \theta_0 = \omega t + \theta_0 \to \psi$, $\tau \to \psi - \theta_0$, $x_1 \to u_1$, $x_2 \to u_2$, $(\ )' = \dfrac{\partial}{\partial \tau} \to \dfrac{\partial}{\partial \psi}$). However, for stationary oscillation $A_1 = B_1 = A_2 = B_2 = 0$. So, stationary oscillation obtained in Section 2 by the variant of the asymptotic procedure coincides with that determined by (3.7) by the Poincaré method.

5 Example

As an illustration, let us consider the system:

$$\ddot{x} + x = \varepsilon\{h(1 - x^2)\dot{x} + e \cos t\}, \quad h = 0.04, \ e = 0.03. \tag{4.1}$$

In the first approximation, we have:

$$
\begin{aligned}
&-2A_1\sin\psi - 2aB_1\cos\psi + \Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = f^{(1)}(a,\theta,\psi) = \\
&= \Big\{e\sin\theta - ha\Big(1-\frac{a^2}{4}\Big)\Big\}\sin\psi + e\cos\theta\cos\psi + \frac{1}{4}ha^3\sin 3\psi
\end{aligned}
\tag{4.2}
$$

from which we obtain:

$$
A_1(a,\theta) = -\frac{1}{2}\Big\{e\sin\theta - ha\Big(1-\frac{a^2}{4}\Big)\Big\},\quad B_1(a,\theta) = \frac{-e}{2a}\cos\theta,
\tag{4.3}
$$

$$
\frac{\partial^2 u_1}{\partial\psi^2} + u_1 = \frac{1}{4}ha^3\sin 3\psi,\quad u_1(a,\theta,\psi) = -\frac{ha^3}{32}\sin 3\psi + a_1\cos\psi + b_1\sin\psi.
\tag{4.4}
$$

By solving the equations

$$
A_1(a,\theta) = 0,\quad B_1(a,\theta) = 0,
\tag{4.5}
$$

three stationary oscillations are determined

$$
\theta_{01} = -\frac{\pi}{2},\quad a_{01} \approx 2.3028;
\tag{4.6}
$$

$$
\theta_{02} = \frac{\pi}{2},\quad a_{02} = 1;\quad \theta_{03} = \frac{\pi}{2},\quad a_{03} \approx 1.3028.
\tag{4.7}
$$

In the second approximation, we have to examine the identity (2.3); for the example considered, it has the form:

$$
\begin{aligned}
\dots\ &+ \Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) = \dots + u_1 f_x + \frac{\partial u_1}{\partial\psi} f_{\dot{x}} = \\
&= \Big(-\frac{ha^3}{32}\sin 3\psi + a_1\cos\psi + b_1\sin\psi\Big)(2ha^2\cos\psi\sin\psi) \\
&+ \Big(\frac{-3ha^3}{32}\cos 3\psi - a_1\sin\psi + b_1\cos\psi\Big)h(1-a^2\cos^2\psi) \\
&= \Big[\frac{h^2a^5}{128} + b_1\frac{h}{2}\Big(1-\frac{a^2}{2}\Big)\Big]\cos\psi + a_1\frac{h_2}{2}\Big(\frac{3a^2}{2} - 1\Big)\sin\psi + (\text{higher harmonics}),
\end{aligned}
\tag{4.8}
$$

where non-written terms do not contain A_1, B_1, A_2, B_2.

By substituting $a = a_0$, $\theta = \theta_0$ then by vanishing the coefficient of the first harmonics $\cos\psi$ and $\sin\psi$ in (4.8) we obtain:

$$
a_{10} = 0,\quad b_{10} = -ha_0^5/64\Big(1-\frac{a_0^2}{2}\Big).
\tag{4.9}
$$

The stability conditions are

$$
\begin{aligned}
&\varepsilon\Big(\frac{\partial A_1}{\partial a} + \frac{\partial B_1}{\partial\theta}\Big)_0 + \varepsilon^2 = \varepsilon\Big\{\frac{h}{2}\Big(1-\frac{3a_0^2}{4}\Big) + \frac{e}{2a_0}\sin\theta_0\Big\} + \dots < 0, \\
&\varepsilon^2\Big(\frac{\partial A_1}{\partial a}\frac{\partial B_1}{\partial\theta} - \frac{\partial A_1}{\partial\theta}\frac{\partial B_1}{\partial a} + \varepsilon^3\dots\Big)_0 = \varepsilon^2\Big(\frac{he}{4a_0}\Big(1-\frac{3a_0^2}{4}\Big)\sin\theta_0 + \frac{e^2}{4a_0^2}\cos^2\theta_0\Big) + \varepsilon^3\dots > 0.
\end{aligned}
\tag{4.10}
$$

It is easy to verify that the stationary oscillation (3.14) is stable and other two stationary oscillations (3.15) are unstable.

6 Conclusion

A variant of the asymptotic procedure is proposed for weakly non-linear non autonomous systems. It differs from the usual one by two modifications: 1) the use of the amplitude and the dephase angle order ε^0 as variables in asymptotic expansion, 2) the use of the stationarity condition in each step of approximation. The formulas obtained are identical with those given the well-known Poincaré method.

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MỘT BIẾN THỂ CỦA TRÌNH TỰ TIỆM CẬN
(II: TRƯỜNG HỢP HỆ PHI TUYẾN YẾU, KHÔNG ÔTÔNÔM)

Một biến thể của trình tự tiệm cận trình bầy ở [3] cho hệ ôtônôm phi tuyến yếu được mở rộng và áp dụng cho hệ không ôtônôm phi tuyến yếu. Biên độ và pha ở cấp ε^0 được dùng làm biến cho khai triển tiệm cận và tính dừng được sử dụng để đảm bảo tính duy nhất của khai triển. Dao động được xác định dần ở từng bước của trình tự tiệm cận.