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THE POINCARÉ METHOD FOR AN OSCILLATOR WITH QUADRATIC NONLINEARITY

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In [2], to evaluate free oscillation period of an undamped oscillator with large cubic restoring nonlinearity, a modified Poincaré method has been proposed. There, in the neighbourhood of the free oscillation of interest, the strongly nonlinear system under consideration is assumed to be near certain linear one with unknown (to be evaluated) frequency.

In the present paper, we deal with the case of quadratic non-linearity. An additional modification is introduced consisting in the elimination of constant derivation and second harmonic terms. The results obtained show that the domain of application of the Poincaré method can be enlarged.

1 Systems under consideration and exact free oscillation period

Consider an oscillator described by the differential equation

$$\ddot{x} + x + \beta x^2 = 0, \tag{1.1}$$

where $\beta > 0$ is coefficient of quadratic nonlinearity; other notations retain their significations explained in [2]. In the phase plane $Ox\dot{x}$, the origin $O(0,0)$ and the point $I\Big(\dfrac{-1}{\beta}, 0\Big)$ represent two equilibrium states: the first one is a stable center, the second one is a saddle. The domain where oscillations occur contains the center O and is bounded by an homoclinic orbit IJI starting from and ending at the same point I, intersecting the axis Ox at the point $J\Big(\dfrac{1}{2\beta}, 0\Big)$. If β is very small, the mentioned oscillatory domain is very large and the standard Poincaré method is applicable. Contrarily, if β is very large, the oscillatory domain enclosed by the homoclinic orbit IJI is very small and the problem of oscillation becomes insignificant. So, below, we are interested only in the case in which the quadratic nonlinearity coefficient β is of "medium magnitude" - not too small and also not too large.

The system under consideration is autonomous then, initial conditions for free oscillations can be chosen as:

$$x(0) = a_0 > 0, \qquad \dot{x}(0) = 0. \tag{1.2}$$

Obviously, on a_0, the following requirement is imposed

$$a_0 < \frac{1}{2\beta} = \frac{0.5}{\beta} \quad (\text{for } \beta = 1,\ a_0 < 0.5). \tag{1.3}$$

The potential energy has as expression

$$V(x) = \frac{x^2}{2} + \beta\frac{x^3}{3}. \tag{1.4}$$

By x_2, x_1, a_0 $\left(x_2 < -\frac{1}{\beta} < x_1 < 0 < a_0 < \frac{1}{2\beta}\right)$ we denote three roots of the equation

$$V(a_0) - V(x) = 0. \tag{1.5}$$

The exact period of the free oscillation satisfying the initial condition (1.2) is given by the integral [3]

$$T_{ex} = \sqrt{2}\int_{x_1}^{a_0} \frac{dx}{\sqrt{V(x_0) - V(x)}}. \tag{1.6}$$

2 Period from the standard Poincaré method

For the sake of comparison, the standard Poincaré method is first used to examine the case of small β.

Being small, β can be labeled by a formal small parameter ε; so, the differential equation (1.1) can be written as:

$$\ddot{x} + x = -\varepsilon\beta x^2. \tag{2.1}$$

Let $\omega = 2\pi/T$ be the unknown frequency of the free oscillation satisfying the initial conditions (1.2). By introducing the dimensionless time $\tau = \omega t$, the equation (2.1) becomes:

$$\omega^2 x'' + x = -\varepsilon\beta x^2, \tag{2.2}$$

where primes denote differentiation with respect to τ.

Then both unknowns x and ω are expanded in powers of ε, that is

$$x = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots, \tag{2.3}$$

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots \tag{2.4}$$

Corresponding to (1.2), the initial conditions of $x_0(\tau)$, $x_i(\tau)$ are:

$$x_0(\tau = 0) = a_0, \quad x_0'(\tau = 0) = 0, \tag{2.5}$$

$$x_i(\tau = 0) = 0, \quad x_i'(\tau = 0) = 0, \quad (i = 1, 2, \ldots). \tag{2.6}$$

Substituting (2.3), (2.4) into (2.2), equating coefficients of like power of ε yield:

$$x_0'' + x_0 = 0, \tag{2.7}$$
$$x_1'' + x_1 = -2\omega_1 x_0'' - \beta x_0^2, \tag{2.8}$$
$$x_2'' + x_2 = -2\omega_1 x_1'' - (\omega_1^2 + 2\omega_2)x_0'' - 2\beta x_0 x_1, \tag{2.9}$$
$$\dots \quad \dots \quad \dots$$

With regard to initial conditions (2.5), the solution x_0 is

$$x_0 = a_0 \cos\tau. \tag{2.10}$$

Using (2.10) we rewrite (2.8) as:

$$x_1'' + x_1 = 2\omega_1 a_0 \cos\tau - \frac{1}{2}\beta a_0^2(1 + \cos 2\tau). \tag{2.11}$$

To eliminate the secular term $2\omega_1 a_0 \cos\tau$ in (2.11), the additional frequency ω_1 should be taken:

$$\omega_1 = 0. \tag{2.12}$$

Then the solution x_1 satisfying the initial conditions (2.6) is:

$$x_1 = -\frac{1}{2}\beta a_0^2(1 - \cos\tau) + \frac{1}{6}pa_0^2(\cos 2\tau - \cos\tau). \tag{2.13}$$

With regard to (2.10), (2.12), (2.13), the differential equation (2.9) can be written as:

$$\begin{aligned} x_2'' + x_2 &= 2\omega_2 a_0 \cos\tau - 2\beta^2 a_0^3 \cos\tau\Big\{ -\frac{1}{2} + \frac{1}{3}\cos\tau + \frac{1}{6}\cos 2\tau\Big\} \\ &= -\frac{1}{3}\beta a_0^3 + (2\omega_2 \frac{5}{6}\beta a_0^2)a_0 \cos\tau - \frac{1}{3}a_0^3 \cos 2\tau - \frac{1}{6}\beta^2 a_0^3 \cos 3\tau. \end{aligned} \tag{2.14}$$

Eliminating secular term yields:

$$\omega_2 = -\frac{5}{12}\beta^2 a_0^2. \tag{2.15}$$

Thus, in the second approximation, the formulas for frequency and period are, respectively:

$$\omega = 1 - \frac{5}{12}\beta^2 a_0^2, \quad T_0 = \frac{2\pi}{\left(1 - \frac{5}{12}\beta^2 a_0^2\right)}. \tag{2.16}$$

Note that the condition $\omega > 0$ requies

$$a_0 < \frac{1}{\beta}\sqrt{\frac{12}{5}} \approx \frac{1.5492}{\beta}. \tag{2.17}$$

which differs enough from (1.3). This means that the range of validity of the standard Poincaré method is limited and can only be extended by suitable modifications.

3 Period from a modified Poincaré method

In this section, with additional modification, the modified Poincaré method proposed in [2] is applied to treat the case in which β is not too small.

The differential equation (1.1) is rearranged as:

$$\ddot{x} + \omega^2 x = (\omega^2 - 1)x - \beta x^2, \tag{3.1}$$

or, by using the dimensionless time $\tau = \omega t$

$$\omega^2(x'' + x) = (\omega^2 - 1)x - \beta x^2\,, \tag{3.2}$$

where ω is the unknown (to be evaluaed) frequency of the free oscillation satisfying the initial conditions (1.2).

Note that for $x = a\cos\tau$, the quadratic nonlinearity βx^2 "produces" the derivation $\frac{1}{2}\beta a^2$ and also the second harmonic $\frac{1}{2}\beta a^2\cos 2\tau$; these two components may be large enough and could not be "neutralized" by the first harmonic $(\omega^2 - 1)a\cos\tau$. Therefore, the right hand side of (3.2) should not be small and the differential equation (3.2) could not be considered as the one belonging to weakly nonlinear type.

Let us decompose x into two parts, that is

$$x = y + z \tag{3.3}$$

and rewrite (3.2) in the form:

$$\omega^2(y'' + y) = \mu\Big\{ -\omega^2(z'' + z) + (\omega^2 - 1)(y + z) - \beta(y + z)^2\Big\}, \tag{3.4}$$

where z should be chosen in such a way that the right hand side of (3.4) may be labeled by a new small parameter μ.

The three unknowns ω, y, z are then expanded in power of μ, that is:

$$\omega^2 = \omega_0^2 + \mu\omega_1 + \mu^2\omega_2 + \ldots, \tag{3.5}$$

$$y = y_0 + \mu y_1 + \mu^2 y_2 + \ldots, \tag{3.6}$$

$$z = z_0 + \mu z_1 + \mu^2 z_2 + \ldots \tag{3.7}$$

The initial conditions (1.2) can be written in the form:

$$y_0(0) = a, \qquad y_0'(0) = 0, \tag{3.8}$$

$$y_i(0) = 0, \qquad y_i'(0) = 0, \tag{3.9}$$

$$z_0'(0) = 0, \qquad z_i'(0) = 0, \tag{3.10}$$

$$a_0 = a + \mu z_1(0) + \mu^2 z_2(0) + \ldots \tag{3.11}$$

Substituting (3.5), (3.6), (3.7) into (3.4), equating the terms of like power of μ yield:

$$\omega_0^2(y_0'' + y_0) = 0, \tag{3.12}$$

$$\omega_0^2(y_1'' + y_1) = -\omega_0^2 z_0'' - z_0 + (\omega_0^2 - 1)y_0 - \beta(y_0 + z_0)^2, \tag{3.13}$$

$$\begin{aligned}\omega_0^2(y_2'' + y_2) = {} & -\omega_1(y_1'' + y_1) - \omega_0^2 z_1'' - \omega_1 z_0'' - z_1 \\ & + (\omega_0^2 - 1)y_1 + \omega_1 y_0 - 2\beta(y_0 + z_0)(y_1 + z_1),\end{aligned} \tag{3.14}$$

$$\ldots \quad \ldots \quad \ldots$$

The solution y_0 satisfying the initial condition (3.8) is

$$y_0 = a\cos\tau. \tag{3.15}$$

Using (3.15) and choosing

$$z_0 = b_0 + c_0\cos 2\tau, \tag{3.16}$$

the differential equation (3.13) becomes:

$$\begin{aligned}
\omega_0^2(y_1'' + y_1) &= (\omega_0^2 - 1)a\cos\tau - b_0 + (4\omega_0^2 - 1)c_0\cos 2\tau - \beta a^2\cos^2\tau - 2\beta a b_0\cos\tau \\
&\quad - 2\beta a c_0\cos\tau\cos 2\tau - \beta b_0^2 - 2\beta b_0 c_0\cos 2\tau - \beta c_0^2\cos^2 2\tau \\
&= (\omega_0^2 - 1)a\cos\tau - b_0 + (4\omega_0^2 - 1)c_0\cos 2\tau - \frac{1}{2}\beta a^2(1 + \cos 2\tau) \\
&\quad - 2\beta a b_0\cos\tau - \beta a c_0(\cos 3\tau + \cos\tau) - \beta b_0^2 \\
&\quad - 2\beta b_0 c_0\cos 2\tau - \frac{1}{2}\beta c_0^2(1 + \cos 4\tau).
\end{aligned} \tag{3.17}$$

Eliminating secular terms and also constant and second harmonic terms yields:

$$(\omega_0^2 - 1) - 2\beta b_0 - \beta c_0 = 0, \tag{3.18}$$

$$b_0 + \frac{1}{2}\beta a^2 + \beta b_0^2 + \frac{1}{2}\beta c_0^2 = 0, \tag{3.19}$$

$$(4\omega_0^2 - 1)c_0 - \frac{1}{2}\beta a^2 - 2\beta b_0 c_0 = 0. \tag{3.20}$$

From (3.18), it follows:

$$c_0 = \frac{1}{\beta}(\omega_0^2 - 1 - 2\beta b_0). \tag{3.21}$$

Substituting (3.21) into (3.19), (3.20) gives:

$$6\beta^2 b_0^2 + (6 - 4\omega_0^2)\beta b_0 + (\omega_0^2 - 1)^2 + \beta^2 a^2 = 0, \tag{3.22}$$

$$8\beta^2 b_0^2 + (8 - 20\omega_0^2)\beta b_0 + 8\omega_0^4 - 10\omega_0^2 + 2 - \beta^2 a^2 = 0, \tag{3.23}$$

from which we can deduce

$$\beta_0 = \frac{1}{44\omega_0^2}(20\omega_0^4 - 22\omega_0^2 + (2 - 7\beta^2 a^2)). \tag{3.24}$$

With regard to (3.24), the equation (3.23) can be transformed into the form:

$$136\omega_0^8 - (140 - 248\beta^2 a^2)\omega_0^4 + (2 - 7\beta^2 a^2)^2 = 0. \tag{3.25}$$

The formulae for ω_0^4 is given by the positive solution satisfying the condition $\omega_0\big|_{\beta=0} = 1$, that is

$$\omega_0^4 = \frac{1}{68}\left\{(35 - 62\beta^2 a^2) + \sqrt{(35 - 62\beta^2 a^2)^2 - 34(2 - 7\beta^2 a^2)^2}\right\}. \tag{3.26}$$

The existence of this root requires:

$$35 - 62\beta^2 a^2 > 0, \quad (35 - 62\beta^2 a^2)^2 - 34(2 - 7\beta^2 a^2)^2 \geq 0, \tag{3.27}$$

which leads to the inequality:

$$\beta a \leq 0.6982 \quad \text{or} \quad a < \frac{0.6982}{\beta}. \tag{3.28}$$

Although (3.28) is still unexact, it is better than (2.17): for $\beta = 1$, the value $a \approx 0.65$ corresponds to the limit value $a_0 = 0.5$.

The differential equation governing y_1 is rather simple

$$\omega_0^2(y_1'' + y_1) = -\beta a c_0 \cos 3\tau - \frac{1}{2}\beta c_0^2 \cos 4\tau, \tag{3.29}$$

and the solution y_1 satisfying the initial conditions (3.9) is:

$$y_1 = \frac{\beta a c_0}{8\omega_0^2}(\cos 3\tau - \cos\tau) + \frac{\beta c_0^2}{30\omega_0^2}(\cos 4\tau - \cos\tau). \tag{3.30}$$

Going on the presented procedure, by choosing $z_1 = b_1 + c_1 \cos 2\tau$, three linear algebraic equations for determining ω_1, b_1, c_1 are obtained

$$\begin{aligned}
&(1 + 2\beta b_0)b_1 + (\beta c_0)c_1 = \frac{\beta^2 c_0 a^2}{8\omega_0^2} + \frac{\beta^2 c_0^2 a}{30\omega_0^2} \\
&-(2\beta c_0)b_1 + (4\omega_0^2 - 1 - 2\beta b_0)c_1 + (8\omega_0 c_0)\omega_1 = \frac{\beta^2 c_0^3}{30\omega_0^2} - \frac{\beta^2 c_0^2 a}{30\omega_0^2}, \\
&(2\beta a)b_1 + (\beta a)c_1 - (2\omega_0 a)\omega_1 = \frac{\beta^2 b_0 c_0 a}{4\omega_0^2} + \frac{\beta^2 b_0 c_0^2}{15\omega_0^2} + \frac{\beta^2 c_0^3}{30\omega_0^2} - (\omega_0^2 - 1)\Big(\frac{\beta c_0 a}{8\omega_0^2} + \frac{\beta c_0^2}{30\omega_0^2}\Big).
\end{aligned} \tag{3.31}$$

Thus, in the second approximation, the period of the free oscillation satisfying the initial condition (1.2) is given by the formulae

$$a_0 = a + b_0 + c_0 + b_1 + c_1, \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0 + \omega_1}.$$

Remark. Instead of the form (3.8)-(3.11), the initial condition (1.2) can be written as:

$$\begin{aligned}
&y_0(0) = a_0, \quad y_0'(0) = 0, \quad y_i(0) = y_i'(0) = 0, \\
&z_0(0) = z_i(0) = z_0'(0) = z_i'(0) = 0, \quad (i = 1, 2, \ldots).
\end{aligned}$$

In this case, the form of z_0 is:

$$z_0 = b_0(1 - \cos\tau) + c_0(\cos 2\tau - \cos\tau).$$

However, the estimation (3.28) cannot be obtained.

4 Numerical results

For $\beta = 1$ (the limit value of the initial position is 0.5) numerical results obtained are shown in the following Table

a	$a_0 = a + b_0 + c_0 + b_1 + c_1$	T_{ex}	T_0	T
0.00	0.0000	6.2832	6.2832	6.2832
0.02	0.0399	6.2842	6.2842	6.2842
0.04	0.0395	6.2873	6.2873	6.2874
0.06	0.0588	6.2926	6.2922	6.2926
0.08	0.0799	6.3000	6.2991	6.3001
0.10	0.0967	6.3096	6.3077	6.3096
0.20	0.1865	6.3925	6.3756	6.3926
0.30	0.2690	6.5438	6.4786	6.5441
0.40	0.3434	6.7905	6.6080	6.7913
0.50	0.4048	7.1942	6.7524	7.1954
0.60	0.4631	7.9780	6.8997	7.9181
0.61	0.4684	8.1201	6.9154	8.0201
0.62	0.4740	8.3006	6.9322	8.1279
0.63	0.4802	8.5563	6.5912	8.2391
0.64	0.4877	9.0101	6.9745	8.3469
0.65	0.4892	10.9180	7.0080	8.4267

For relative error of order 4%, the period T from the modified Poincaré method is acceptable if $a_0 \leq 0.48$ (about 96% of the oscillatory domain) while the standard one is acceptable only if $a_0 \leq 0.4$ (about 80%). The domain of application of the standard Poincaré method is not too narrow; this results from the fact that the quadratic nonlinearity is not very strong and the oscillatory domain is limited

5 Conclusion

An oscillator with not too small quadratic nonlinearity is considered. The governing differential equation is written as for weakly nonlinear oscillator after eliminating the deviation and second harmonic terms. The results obtained can be used for a great part of the oscillatory domain.

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PHƯƠNG PHÁP POINCARÉ CHO CHẤN TỬ VỚI PHI TUYẾN BẬC HAI

Xét chấn tử không có cản, không bị kích động, có phi tuyến hồi phục bậc hai không quá nhỏ. Phương trình vi phân dao động được viết như với chấn tử phi tuyến yếu sau khi loại bỏ độ lệch và ác-mô-nic thứ hai, kết quả tính chu kỳ dao động tự do có thể chấp nhận với phần lớn miền dao động.