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ON THE ELASTOPLASTIC STABILITY PROBLEM OF THE THIN ROUND CYLINDRICAL SHELLS SUBJECTED TO COMPLEX LOADING PROCESSES WITH THE VARIOUS KINEMATIC BOUNDARY CONDITIONS

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ABSTRACT. In this paper, the elastoplastic stability of cylindrical shells simultaneously subjected to compression force along the generatrix and external pressure has been presented. Two types of considered kinematic boundary conditions are simply supported and clamped at the butt-ends. The expressions for determining the critical forces by using the Bubnov-Galerkin method [3] have been established. The sufficient condition of extremum for a long cylindrical shell also is considered. Some results of numerical calculation have been also given and discussed.

1 Stability problem

Let's consider a thin round cylindrical shell of length L, radius R and thickness h. We choose a orthogonal coordinate system $Ox_1x_2x_3$ so that the axis Ox_1 belonging to the middle surface and lying along the generatrix of the shell, $x_2 = R\theta_1$ with θ_1-the angle of circular arc and x_3 in the direction of the normal to the middle surface.

Assume that a material of shell is incompressible and shell is subjected to the compression force $p(t)$ along the generatrix and external pressure $q_1(t)$ which depend arbitrarily on a loading parameter t. One of the main aims of the stability problem is to find the moment t_* when the instability of the structure happens and respectively the critical loads $p^* = p(t_*)$, $q_1^* = q_1(t_*)$. Suppose that the unloading does not happen in the structure. We use the criterion of bifurcation of equilibrium state to investigate the proposed problem.

An investigation of the elastoplastic stability problem is always made two parts: pre-buckling process and post-buckling process.

1.1 Pre-buckling process

Suppose that at any moment t there exists a membrane plane stress state in the cylindrical shell

$$\sigma_{11} = -p(t) \equiv -p; \quad \sigma_{22} = -q_1(t)\frac{R}{h} \equiv -q(t) \equiv -q, \quad \sigma_{12} = \sigma_{13} = \sigma_{23} = \sigma_{33} = 0. \quad (1.1)$$

Thus

$$\sigma = \frac{\sigma_{11} + \sigma_{22}}{3} = -\frac{p+q}{3}, \quad \sigma_u = \sqrt{\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2} = \sqrt{p^2 - pq + q^2}.$$

Components of the strain velocity tensor determined according to the theory of elasto-plastic processes [1] are of the form

$$\dot{\varepsilon}_{11} = \frac{1}{N}\Big(-\dot{p} + \frac{1}{2}\dot{q}\Big) - Q(s,t)\Big(p - \frac{1}{2}q\Big),$$
$$\dot{\varepsilon}_{22} = \frac{1}{N}\Big(-\dot{q} + \frac{1}{2}\dot{p}\Big) - Q(s,t)\Big(q - \frac{1}{2}p\Big),$$
$$\dot{\varepsilon}_{33} = -(\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22}), \quad \dot{\varepsilon}_{12} = \dot{\varepsilon}_{13} = \dot{\varepsilon}_{23} = 0, \tag{1.2}$$

where

$$Q(s,t) = \Big(\frac{1}{\phi'} - \frac{1}{N}\Big)\frac{p\dot{p} + q\dot{q} - \frac{1}{2}p\dot{q} - \frac{1}{2}\dot{p}q}{p^2 - pq + q^2}, \quad \phi' = \phi'(s), \quad N = \frac{\sigma_u}{s}.$$

The arc-length of the strain trajectory is given respectively by the formula

$$\frac{ds}{dt} = \frac{2}{\sqrt{3}}\big(\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{11}\dot{\varepsilon}_{22} + \dot{\varepsilon}_{22}^2\big)^{1/2} \equiv F(s,t). \tag{1.3}$$

So, we can determine, from equations (1.1) ÷ (1.3) associating with boundary conditions and the equilibrium equations, stress and strain states at any point M in the cylindrical shell at any moment of the prebuckling process.

1.2 Post-buckling process

As shown in [1, 4], the system of stability equations of the cylindrical shell is written in the form

$$\beta_1\frac{\partial^4\varphi}{\partial x_1^4} + \beta_3\frac{\partial^4\varphi}{\partial x_1^2\partial x_2^2} + \beta_5\frac{\partial^4\varphi}{\partial x_2^4} + \frac{N}{R}\frac{\partial^2\delta w}{\partial x_1^2} = 0, \tag{1.4}$$

$$\alpha_1\frac{\partial^4\delta w}{\partial x_1^4} + \alpha_3\frac{\partial^4\delta w}{\partial x_1^2\partial x_2^2} + \alpha_5\frac{\partial^4\delta w}{\partial x_2^2} + \frac{9}{Nh^2}\Big(p\frac{\partial^2\delta w}{\partial x_1^2} + q\frac{\partial^2\delta w}{\partial x_2^2} - \frac{1}{R}\frac{\partial^2\varphi}{\partial x_1^2}\Big) = 0, \tag{1.5}$$

where the coefficients α_i, β_i $(i = 1,3,5)$ are calculated as follows

$$\beta_1 = 1 + \frac{1}{4}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2q-p)^2}{p^2 - pq + q^2},$$
$$\beta_3 = 2 + \frac{1}{2}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2p-q)(2q-p)}{p^2 - pq + q^2},$$
$$\beta_5 = 1 + \frac{1}{4}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2p-q)^2}{p^2 - pq + q^2},$$

$$\alpha_1 = 1 - \frac{3}{4}\Big(1 - \frac{\phi'}{N}\Big)\frac{p^2}{p^2 - pq + q^2},$$
$$\alpha_3 = 2 - \frac{3}{2}\Big(1 - \frac{\phi'}{N}\Big)\frac{pq}{p^2 - pq + q^2},$$
$$\alpha_5 = 1 - \frac{3}{4}\Big(1 - \frac{\phi'}{N}\Big)\frac{q^2}{p^2 - pq + q^2}. \tag{1.6}$$

In order to solve the stability problem of the cylindrical shell, we consider two types of kinematic boundary conditions following

* the shell is simply supported at the planes $x_1 = 0$ and $x_1 = L$

* the shell is clamped at the planes $x_1 = 0$ and $x_1 = L$.

Hereafter we will study the solution of these two stability problems

2 Solving the elastoplastic stability problem of simply supported cylindrical shell

We find the solution δw which satisfies the mentioned boundary conditions in the form

$$\delta w = \sum_{m=1}^{M}\sum_{n=1}^{M} A_{mn} \sin\frac{m\pi x_1}{L}\sin\frac{nx_2}{R}\,. \tag{2.1}$$

It is easy to see that the system of functions $\delta w_{mn} = \sin\frac{m\pi x_1}{L}\sin\frac{nx_2}{R}$ is linearly independence.

Substituting this expression into (1.4), we can obtain the particular solution φ as follows

$$\varphi = \sum_{m=1}^{M}\sum_{n=1}^{M} B_{mn} \sin\frac{m\pi x_1}{L}\sin\frac{nx_2}{R}, \tag{2.2}$$

where

$$B_{mn} = \frac{N}{R}\Big(\frac{m\pi}{L}\Big)^2 A_{mn}\Big[\beta_1\Big(\frac{m\pi}{L}\Big)^4 + \beta_3\Big(\frac{m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2 + \beta_5\Big(\frac{n}{R}\Big)^4\Big]^{-1}. \tag{2.3}$$

Now we pass to find the expression determining the critical forces by the Bubnov-Galerkin method. For doing that, one need to realize the following steps:

a) Substituting the expressions of δw and φ from (2.1), (2.2) into (1.5).

b) Multiplying both sides of that stability equation by

$$\delta w_{ij} = \sin\frac{i\pi x_1}{L}\sin\frac{jx_2}{R}\,.$$

c) Integrating the received just equation following x_1 and x_2.

Finally, we reach

$$\int_0^L\int_0^{2\pi R}\Big\{\alpha_1\frac{\partial^4\delta w}{\partial x_1^4} + \alpha_3\frac{\partial^4\delta w}{\partial x_1^2\partial x_2^2} + \alpha_5\frac{\partial^4\delta w}{\partial x_2^4} + \frac{9}{h^2N}\Big(p\frac{\partial^2\delta w}{\partial x_1^2} + q\frac{\partial^2\delta w}{\partial x_2^2}\Big) - \frac{9}{h^2NR}\frac{\partial^2\varphi}{\partial x_1^2}\Big\}\sin\frac{i\pi x_1}{L}\sin\frac{jx_2}{R}dx_1dx_2 = 0 \quad (i,j = 1,2,\dots,M). \tag{2.4}$$

For taking this integral, it needs to use the result

$$\int_0^L\int_0^{2\pi R}\sin\frac{m\pi x_1}{L}\sin\frac{i\pi x_1}{L}\sin\frac{nx_2}{R}\sin\frac{jx_2}{R}dx_1dx_2 = \begin{cases}0 & \text{with } m\neq i,\ n\neq j\\ \frac{1}{2}\pi RL & \text{with } m=i,\ n=j\,.\end{cases}$$

After series of calculations, the relation (2.4) gives us

$$\frac{\pi RL}{2}\Big\{\alpha_1\Big(\frac{m\pi}{L}\Big)^4+\alpha_3\Big(\frac{m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2+\alpha_5\Big(\frac{n}{R}\Big)^4-\frac{9}{h^2N}\Big[p\Big(\frac{m\pi}{L}\Big)^2+q\Big(\frac{n}{R}\Big)^2\Big]$$
$$+\frac{9}{h^2R^2}\Big(\frac{m\pi}{L}\Big)^4\Big[\beta_1\Big(\frac{m\pi}{L}\Big)^4+\beta_3\Big(\frac{m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2+\beta_5\Big(\frac{n}{R}\Big)^4\Big]^{-1}\Big\}A_{mn}=0. \qquad (2.5)$$

Taking into account the existence of non-trivial solution i.e. $A_{mn}\neq 0$, we receive the expression for determining critical loads

$$p\Big(\frac{m\pi}{L}\Big)^2+q\Big(\frac{n}{R}\Big)^2=\frac{h^2N}{9}\Big\{\alpha_1\Big(\frac{m\pi}{L}\Big)^4+\alpha_3\Big(\frac{m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2+\alpha_5\Big(\frac{n}{R}\Big)^4\Big\}$$
$$+\frac{N}{R^2}\Big(\frac{m\pi}{L}\Big)^4\Big\{\beta_1\Big(\frac{m\pi}{L}\Big)^4+\beta_3\Big(\frac{m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2+\beta_5\Big(\frac{n}{R}\Big)^4\Big\}^{-1}. \qquad (2.6)$$

Noticing that the relation (2.6) coincides with one established by another method in [1, 4].

By putting $\psi=n^2$, $\theta=\Big(\frac{m\pi R}{nL}\Big)^2$, $i=\frac{3R}{h}$, the relation (2.6) is written in the form

$$i^2=\frac{N\psi^2(\alpha_1\theta+\alpha_3+\frac{\alpha_5}{\theta})(\beta_1\theta+\beta_3+\frac{\beta_5}{\theta})}{(p+\frac{q}{\theta})(\beta_1\theta+\beta_3+\frac{\beta_5}{\theta})\psi-N}. \qquad (2.7)$$

Minimizing this expression, i.e. $\frac{\partial i^2}{\partial\psi}=0$, $\frac{\partial i^2}{\partial\theta}=0$, after some calculations we get

$$\psi=\frac{2N}{(p+\frac{q}{\theta})(\beta_1\theta+\beta_3+\frac{\beta_5}{\theta})}. \qquad (2.8)$$

$$\Big(\alpha_1-\frac{\alpha_5}{\theta^2}\Big)\Big(\beta_1\theta+\beta_3+\frac{\beta_5}{\theta}\Big)-\Big(\beta_1-\frac{\beta_5}{\theta^2}\Big)\Big(\alpha_1\theta+\alpha_3+\frac{\alpha_5}{\theta}\Big)+$$
$$\frac{2q}{\theta^2\Big(p+\frac{q}{\theta}\Big)}\Big(\alpha_1\theta+\alpha_3+\frac{\alpha_5}{\theta}\Big)\Big(\beta_1\theta+\beta_3+\frac{\beta_5}{\theta}\Big)=0. \qquad (2.9)$$

Substituting the values (2.8) and (2.9) into (2.7) we have

$$i^2=\frac{4N^2\theta^2}{(p\theta+q)^2}\Big\{\Big[1-\frac{3}{4}\Big(1-\frac{\phi'}{N}\Big)\frac{p^2}{p^2-pq+q^2}\Big]\theta^2+\Big[2-\frac{3}{2}\Big(1-\frac{\phi'}{N}\Big)\frac{pq}{p^2-pq+q^2}\Big]\theta+$$
$$1-\frac{3}{4}\Big(1-\frac{\phi'}{N}\Big)\frac{q^2}{p^2-pq+q^2}\Big\}\cdot\Big\{\Big[1+\frac{1}{4}\Big(\frac{N}{\phi'}-1\Big)\frac{(2q-p)^2}{p^2-pq+q^2}\Big]\theta^2+$$
$$\Big[2+\frac{1}{2}\Big(\frac{N}{\phi'}-1\Big)\frac{(2q-p)(2p-q)}{p^2-pq+q^2}\Big]\theta+1+\frac{1}{4}\Big(\frac{N}{\phi'}-1\Big)\frac{(2p-q)^2}{p^2-pq+q^2}\Big\}^{-1}. \qquad (2.10)$$

where θ is a solution of the equation (2.9).

Applying the loading parameter method [1], we solve simultaneously the equation (1.3) and (2.10). After finding the critical value t^*, we can determine the critical forces as follows

$$p^*=p(t^*),\quad q^*=q(t^*).$$

For long cylindrical shells, i.e. $\psi = 1$, $\theta \ll 1$, see [2], we deduce from (2.7)

$$i^2 = \frac{N\alpha_5\beta_5}{(p\theta + q)\beta_5 - N\theta^2}. \tag{2.11}$$

Minimizing the expression of i^2, i.e. $\dfrac{\partial i^2}{\partial \theta} = 0$, gives us

$$\theta = \frac{p\beta_5}{2N} = \theta_*.$$

Now consider the sufficient condition of extremum [5]

$$\frac{\partial^2 i^2}{\partial \theta^2}\Big|_{\theta=\theta_*} = \frac{2\alpha_5\beta_5 N^2}{\left(\dfrac{p^2\beta_5^2}{4N} + q\beta_5\right)^2}.$$

Because

$$\alpha_5 = 1 - \frac{3}{4}\Big(1 - \frac{\phi'}{N}\Big)\frac{q^2}{\sigma_u^2} = \frac{(2p-q)^2}{4\sigma_u^2} + \frac{3}{4}\frac{\phi'}{N}\frac{q^2}{\sigma_u^2} > 0,$$

$$\beta_5 = 1 + \frac{1}{4}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2p-q)^2}{\sigma_u^2} = \frac{3q^2}{4\sigma_u^2} + \frac{1}{4}\frac{N(2p-q)^2}{\phi'\sigma_u^2} > 0.$$

So $\dfrac{\partial^2 i^2}{\partial \theta^2}\Big|_{\theta=\theta_*} > 0$, the sufficient condition of minimum is satisfied.

Substituting the values of α_5, β_5 and $\theta = \theta_*$ into (2.11) we obtain

$$i^2 = \frac{4N^2\left[1 - \dfrac{3}{4}\Big(1 - \dfrac{\phi'}{N}\Big)\dfrac{q^2}{p^2 - pq + q^2}\right]}{p^2\left[1 + \dfrac{1}{4}\Big(\dfrac{N}{\phi'} - 1\Big)\dfrac{(2p-q)^2}{p^2 - pq + q^2}\right] + 4Nq}. \tag{2.12}$$

3 Solving the elastoplastic stability problem of clamped cylindrical shell

The kinematic boundary conditions of the clamped shell at the planes $x_1 = 0$ and $x_1 = L$ are satisfied completely by choosing

$$\delta w = \sum_{m=1}^{M}\sum_{n=1}^{M} D_{mn}\Big(1 - \cos\frac{2m\pi x_1}{L}\Big)\sin\frac{nx_2}{R}. \tag{3.1}$$

Using the expression of δw and the equation (1.4) we can find the particular solution φ in the form

$$\varphi = \sum_{m=1}^{M}\sum_{n=1}^{M} E_{mn}\cos\frac{2m\pi x_1}{L}\sin\frac{nx_2}{R}, \tag{3.2}$$

where

$$E_{mn} = -\frac{N}{R}\left(\frac{2m\pi}{L}\right)^2 D_{mn}\left[\beta_1\left(\frac{2m\pi}{L}\right)^4 + \beta_3\left(\frac{2m\pi}{L}\right)^2\left(\frac{n}{R}\right)^2 + \beta_5\left(\frac{n}{R}\right)^4\right]^{-1}. \tag{3.3}$$

In order to apply the Bubnov-Galerkin method, we need to verify the linearly independence of the system of functions in the (3.1).

Lemma. *The system of functions*

$$\delta w_{mn} = \left(1 - \cos\frac{2m\pi x_1}{L}\right)\sin\frac{nx_2}{R} \quad (m, n = 1, 2, \dots, M)$$

is linearly independence.

Proof. Let's consider a linear combination

$$\sum_{m=1}^{M}\sum_{n=1}^{M}\gamma_{mn}\delta w_{mn} = 0 \quad \forall x_1 \in [0, L],\ \forall x_2 \in [0, 2\pi R]. \tag{3.4}$$

Multiplying both sides of (3.4) by $\sin\frac{jx_2}{R}$ $(j = 1, 2, \dots, M)$ and integrating the received expression with respect to x_2 on the segment $[0, 2\pi R]$, we have

$$\int_0^{2\pi R}\sum_{m=1}^{M}\sum_{n=1}^{M}\gamma_{mn}\delta w_{mn}\sin\frac{jx_2}{R}dx_2 = 0. \tag{3.5}$$

Since

$$\int_0^{2\pi R}\delta w_{mn}\sin\frac{jx_2}{R}dx_2 = \begin{cases} 0 & \text{with } n \neq j \\ \pi R\left(1 - \cos\frac{2m\pi x_1}{L}\right) & \text{with } n = j\,. \end{cases}$$

Thus the relation (3.5) becomes

$$\pi R\sum_{m=1}^{M}\gamma_{mj}\left(1 - \cos\frac{2m\pi x_1}{L}\right) = 0 \quad \forall x_1 \in [0, L], \quad \forall j = 1, 2, \dots, M. \tag{3.6}$$

For demonstrate $\gamma_{mj} = 0 \ \forall n, j = 1, \dots, m$, we choose

$$x_1 = \frac{L}{2}, \quad x_1 = \frac{L}{4}, \dots, x_1 = \frac{L}{2M}.$$

Substituting in turn these values of x_1 into (3.6), we receive

$$\gamma_{1j} = 0, \quad \gamma_{2j} = 0, \dots, \gamma_{Mj} = 0 \quad \forall j = 1, 2, \dots, M.$$

This leads to $\gamma_{mj} = 0 \ \forall m, j = 1, 2, \dots, M$.

This result demonstrates that the system of functions δw_{mn} is linearly independent. So the lemma is proven.

From the chosen system of functions, we can use the Bubnov-Galerkin method to get

$$\int_0^L \int_0^{2\pi R} \Big\{ \alpha_1 \frac{\partial^4 \delta w}{\partial x_1^4} + \alpha_3 \frac{\partial^4 \delta w}{\partial x_1^2 \partial x_2^2} + \alpha_5 \frac{\partial^4 \delta w}{\partial x_2^4} + \frac{9}{h^2 N}\Big(p\frac{\partial^2 \delta w}{\partial x_1^2} + q\frac{\partial^2 \delta w}{\partial x_2^2}\Big) - \frac{9}{h^2 N R}\frac{\partial^2 \varphi}{\partial x_1^2}\Big\}\Big(1 - \cos\frac{2i\pi x_1}{L}\Big)\sin\frac{jx_2}{R}dx_1dx_2 = 0 \quad (i,j = 1,2,\ldots,M). \tag{3.7}$$

For taking this integral, first of all substituting δw and φ represented by (3.1) and (3.2) into (3.7), afterwards integrating that received expression, we will obtain a system of linear algebraic equations with the unknowns D_{ij} written in the matrix form as follows

$$[a_{ij}][D_{ij}] = 0, \quad i,j = 1,2,\ldots,M.$$

Because of the condition on the existence of non-trivial solution i.e. $D_{ij} \neq 0$ then the determinant of the coefficients of D_{ij} must be equal to zero

$$\det[a_{ij}] = 0, \quad i,j = 1,2,\ldots,M. \tag{3.7a}$$

Associating this expression with (1.3) and by using the parameter method, we can find the critical value t^* of the loading parameter and the critical forces p^*, q^*.

Note that the development of the determinant (3.7a) in general case is mathematically complicated, therefore we will take the solution in the first and second approximations

a) *The first approximated solution:* we choose δw and φ in the form

$$\delta w = D_{mn}\Big(1 - \cos\frac{2m\pi x_1}{L}\Big)\sin\frac{nx_2}{R}$$

$$\varphi = -\frac{\frac{N}{R}\Big(\frac{2m\pi}{L}\Big)^2 D_{mn}\cos\frac{2m\pi x_1}{L}\sin\frac{nx_2}{R}}{\beta_1\Big(\frac{2m\pi}{L}\Big)^4 + \beta_3\Big(\frac{2m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2 + \beta_5\Big(\frac{n}{R}\Big)^4}.$$

Substituting δw, φ into (3.7) and taking that integral, gives us

$$\frac{1}{2}\pi RL\Big\{\alpha_1\Big(\frac{2m\pi}{L}\Big)^4 + \alpha_3\Big(\frac{2m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2 + 3\alpha_5\Big(\frac{n}{R}\Big)^4 - \frac{9}{h^2N}\Big[p\Big(\frac{2m\pi}{L}\Big)^2 + 3q\Big(\frac{n}{R}\Big)^2\Big] + \frac{9}{h^2R^2}\Big(\frac{2m\pi}{L}\Big)^4\Big[\beta_1\Big(\frac{2m\pi}{L}\Big)^4 + \beta_3\Big(\frac{2m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2 + \beta_5\Big(\frac{n}{R}\Big)^4\Big]^{-1}\Big\}D_{mn} = 0.$$

Because of the existence of non-trivial solution i.e. $D_{mn} \neq 0$, yields a relation for finding critical loads

$$p\Big(\frac{2m\pi}{L}\Big)^2 + 3q\Big(\frac{n}{R}\Big)^2 = \frac{h^2N}{9}\Big\{\alpha_1\Big(\frac{2m\pi}{L}\Big)^4 + \alpha_3\Big(\frac{2m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2 + 3\alpha_5\Big(\frac{n}{R}\Big)^4\Big\} + \frac{N}{R^2}\Big(\frac{2m\pi}{L}\Big)^4\Big\{\beta_1\Big(\frac{2m\pi}{L}\Big)^4 + \beta_3\Big(\frac{2m\pi}{L}\Big)^2\Big(\frac{n}{R}\Big)^2 + \beta_5\Big(\frac{n}{R}\Big)^4\Big\}^{-1}. \tag{3.8}$$

Using notations $\xi = n^2$, $\eta = \left(\dfrac{2m\pi R}{Ln}\right)^2$, $i = \dfrac{3R}{h}$, the equation (3.8) is written in the form

$$i^2 = \frac{N\xi^2\Big(\alpha_1\eta + \alpha_3 + \dfrac{3\alpha_5}{\eta}\Big)\Big(\beta_1\eta + \beta_3 + \dfrac{\beta_5}{\eta}\Big)}{\xi\Big(p + \dfrac{3q}{\eta}\Big)\Big(\beta_1\eta + \beta_3 + \dfrac{\beta_5}{\eta}\Big) - N}. \tag{3.9}$$

Minimizing this relation i.e. $\dfrac{\partial i^2}{\partial \xi} = 0$, $\dfrac{\partial i^2}{\partial \eta} = 0$, after some calculations we have

$$\xi = \frac{2N}{\Big(p + \dfrac{3q}{\eta}\Big)\Big(\beta_1\eta + \beta_3 + \dfrac{\beta_5}{\eta}\Big)}. \tag{3.10}$$

$$\begin{aligned}&\Big(\alpha_1 - \frac{3\alpha_5}{\eta^2}\Big)\Big(\beta_1\eta + \beta_3 + \frac{\beta_5}{\eta}\Big) - \Big(\beta_1 - \frac{\beta_5}{\eta^2}\Big)\Big(\alpha_1\eta + \alpha_3 + \frac{3\alpha_5}{\eta}\Big) + \\ &\frac{6q}{\eta^2\Big(p + \dfrac{3q}{\eta}\Big)}\Big(\alpha_1\eta + \alpha_3 + \frac{3\alpha_5}{\eta}\Big)\Big(\beta_1\eta + \beta_3 + \frac{\beta_5}{\eta}\Big) = 0.\end{aligned} \tag{3.11}$$

Substituting the values of ξ and η in the expressions (3.10), (3.11) into (3.9), gives us

$$\begin{aligned} i^2 = \frac{4N^2\eta^2}{(p\eta + 3q)^2}&\Big\{\Big[1 - \frac{3}{4}\Big(1 - \frac{\phi'}{N}\Big)\frac{p^2}{p^2 - pq + q^2}\Big]\eta^2 + \Big[2 - \frac{3}{2}\Big(1 - \frac{\phi'}{N}\Big)\frac{pq}{p^2 - pq + q^2}\Big]\eta \\ &+ 3\Big[1 - \frac{3}{4}\Big(1 - \frac{\phi'}{N}\Big)\frac{q^2}{p^2 - pq + q^2}\Big]\Big\} \cdot \Big\{\Big[1 + \frac{1}{4}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2q - p)^2}{p^2 - pq + q^2}\Big]\eta^2 \\ &+ \Big[2 + \frac{1}{2}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2q - p)(2p - q)}{p^2 - pq + q^2}\Big]\eta + 1 + \frac{1}{4}\Big(\frac{N}{\phi'} - 1\Big)\frac{(2p - q)^2}{p^2 - pq + q^2}\Big\}^{-1}, \end{aligned} \tag{3.12}$$

where η is a solution of the equation (3.11).

In order to reach a values of critical loads, we need to solve simultaneously the equation (1.3) and (3.12) by applying the loading parameter method [1]. After determining the critical value t^*, we can find the critical forces as follows

$$p^* = p(t^*), \quad q^* = q(t^*).$$

Now consider an interesting case. It is a long cylindrical shell. Based on [2], we have

$$\xi = 1, \quad \eta \ll 1, \quad i^2 = \frac{3N\alpha_5\beta_5}{(p\eta + 3q)\beta_5 - N\eta^2}. \tag{3.13}$$

The minimization of the relation (3.13), i.e. $\dfrac{\partial i^2}{\partial \eta} = 0$, yields

$$\eta = \frac{p\beta_5}{2N} = \eta^*.$$

And again

$$\frac{\partial^2 i^2}{\partial \eta^2}\Big|_{\eta=\eta^*} = \frac{6N^2\alpha_5\beta_5}{\left(3q\beta_5 + \dfrac{p^2\beta_5^2}{4N}\right)^2}. \tag{3.14}$$

Since $\alpha_5 > 0$, $\beta_5 > 0$ then $\dfrac{\partial^2 i^2}{\partial \eta^2}\Big|_{\eta=\eta^*} > 0$; The sufficient condition of minimum is satisfied. Taking into account α_5, β_5, η^*, the relation (3.13) is written in the form

$$i^2 = \frac{12N^2\Big[1 - \dfrac{3}{4}\Big(1 - \dfrac{\phi'}{N}\Big)\dfrac{q^2}{p^2 - pq + q^2}\Big]}{p^2\Big[1 + \dfrac{1}{4}\Big(\dfrac{N}{\phi'} - 1\Big)\dfrac{(2p-q)^2}{p^2 - pq + q^2}\Big] + 12Nq}. \tag{3.15}$$

b) *The second approximated solution:* We take the solution as follows

$$\begin{aligned} \delta w &= D_{11}\Big(1 - \cos\frac{2\pi x_1}{L}\Big)\sin\frac{x_2}{R} + D_{21}\Big(1 - \cos\frac{4\pi x_1}{L}\Big)\sin\frac{x_2}{R}, \\ \varphi &= \overline{D}_{11}\cos\frac{2\pi x_1}{L}\sin\frac{x_2}{R} + \overline{D}_{21}\cos\frac{4\pi x_1}{L}\sin\frac{x_2}{R}, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} \overline{D}_{11} &= -\frac{N}{R}\Big(\frac{2\pi}{L}\Big)^2\Big[\beta_1\Big(\frac{2\pi}{L}\Big)^4 + \beta_3\Big(\frac{2\pi}{L}\Big)^2\Big(\frac{1}{R}\Big)^2 + \beta_5\Big(\frac{1}{R}\Big)^4\Big]^{-1}D_{11}, \\ \overline{D}_{21} &= -\frac{N}{R}\Big(\frac{4\pi}{L}\Big)^2\Big[\beta_1\Big(\frac{4\pi}{L}\Big)^4 + \beta_3\Big(\frac{4\pi}{L}\Big)^2\Big(\frac{1}{R}\Big)^2 + \beta_5\Big(\frac{1}{R}\Big)^4\Big]^{-1}D_{21}. \end{aligned} \tag{3.17}$$

Substituting (3.16) into (3.7) and taking this integral, we obtain a system of two linear algebraic equations with the unknowns D_{11}, D_{21}. From the condition $D_{11} \neq 0$, $D_{21} \neq 0$ we have the relation which permits to determining the critical loads

$$\begin{aligned} &\Big\{\Big(\frac{2\pi}{L}\Big)^4\alpha_1 + \Big(\frac{2\pi}{L}\Big)^2\Big(\frac{1}{R}\Big)^2\alpha_3 + 3\alpha_5\Big(\frac{1}{R}\Big)^4 - \frac{9}{h^2N}\Big[\Big(\frac{2\pi}{L}\Big)^2p + 3q\Big(\frac{1}{R}\Big)^2\Big] + \\ &\frac{\dfrac{9}{h^2R^2}\Big(\dfrac{2\pi}{L}\Big)^4}{\beta_1\Big(\dfrac{2\pi}{L}\Big)^4 + \beta_3\Big(\dfrac{2\pi}{L}\Big)^2\Big(\dfrac{1}{R}\Big)^2 + \beta_5\Big(\dfrac{1}{R}\Big)^4}\Big\} \cdot \Big\{\Big(\frac{4\pi}{L}\Big)^4\alpha_1 + \Big(\frac{4\pi}{L}\Big)^2\Big(\frac{1}{R}\Big)^2\alpha_3 + \\ &3\alpha_5\Big(\frac{1}{R}\Big)^4 - \frac{9}{h^2N}\Big[\Big(\frac{4\pi}{L}\Big)^2p + 3q\Big(\frac{1}{R}\Big)^2\Big] + \frac{\dfrac{9}{h^2R^2}\Big(\dfrac{4\pi}{L}\Big)^4}{\beta_1\Big(\dfrac{4\pi}{L}\Big)^4 + \beta_3\Big(\dfrac{4\pi}{L}\Big)^2\Big(\dfrac{1}{R}\Big)^2 + \beta_5\Big(\dfrac{1}{R}\Big)^4}\Big\} \\ &- \Big[2\Big(\frac{1}{R}\Big)^4\alpha_5 - \frac{18q}{h^2R^2N}\Big]^2 = 0. \end{aligned} \tag{3.18}$$

4 Some results of numerical calculation and discussion

We consider the long cylindrical shell made of the steel $30X\Gamma CA$ with an elastic modulus $3G = 2.6 \cdot 10^5$ MPa, an yield point $\sigma_s = 400$ MPa (see [1]).

The relations for determining the critical loads are given in the form
* formulae (2.12) and (1.3) for the part a) of the examples.
* Formulae (3.15) and (1.3) for the part b) of the examples.
The numerical results are realized by the MATLAB program.

Example 1. Suppose that the complex loading law is of the form

$$p \equiv p(t) = \frac{(p_0 + p_1 t)^2}{p_1}, \quad q \equiv q(t) = q_0 + q_1 t,$$

where $p_0 = 2\,\text{MPa}$, $p_1 = 0.1\,\text{MPa}$, $q_0 = 2\,\text{MPa}$, $q_1 = 0.1\,\text{MPa}$.

a) Numerical results for the simply supported cylindrical shell

Table 1

$\frac{R}{h}$	t^*	$s \cdot 10^3$	p^* MPa	q^* MPa	σ_u^* MPa
20	59.34	10.51	629.5	7.9	625.6
31	54.81	5.031	559.6	7.5	555.9
40	52.95	3.469	531.7	7.3	528.1
50	51.17	2.469	506.5	7.1	502.9
59	49.45	1.905	482.3	6.9	478.9
65	48.10	1.669	463.8	6.8	460.4
68	41.13	1.276	373.7	6.1	370.7
77	28.21	0.7346	232.4	4.8	230.0

b) Numerical results for the clamped cylindrical shell

Table 2

$\frac{R}{h}$	t^*	$s \cdot 10^3$	p^* MPa	q^* MPa	σ_u^* MPa
20	62.45	15.46	679.8	8.245	675.7
31	56.47	6.844	584.7	7.647	581.0
40	54.02	4.285	547.9	7.402	544.2
50	51.87	2.821	516.5	7.187	512.9
59	50.11	2.090	491.5	7.011	488.1
65	48.62	1.739	470.9	6.862	467.5
68	47.11	1.579	450.4	6.711	447.1
71	36.97	1.087	324.6	5.697	321.7
77	28.55	0.747	235.7	4.855	233.3

Example 2. The complex loading law is given in the form

$$p \equiv p(t) = p_0 + p_1 t^3, \quad p_0 = 2\,\text{MPa}, \quad p_1 = 0.1\,\text{MPa},$$
$$q \equiv q(t) = q_0 + q_1 t^2, \quad q_0 = 2\,\text{MPa}, \quad q_1 = 0.1\,\text{MPa}.$$

a) Results of numerical calculation for the simply supported cylindrical shell

Table 3

$\frac{R}{h}$	t^*	$s \cdot 10^3$	p^* MPa	q^* MPa	σ_u^* MPa
17	17.52	5.329	539.7	32.69	524.2
20	17.27	3.961	517.0	31.82	501.9
23	17.03	2.992	495.5	30.99	480.7
25	16.89	2.563	484.2	30.54	469.7
28	16.67	2.060	465.6	29.80	451.5
30	16.49	1.798	450.3	29.19	436.4
33	15.62	1.440	382.8	26.39	370.3
35	14.66	1.173	316.9	23.49	305.9

b) Results for the clamped cylindrical shell

Table 4

$\frac{R}{h}$	t^*	$s \cdot 10^3$	p^* MPa	q^* MPa	σ_u^* MPa
17	17.60	5.864	547.4	32.98	531.7
20	17.33	4.222	522.1	32.02	506.9
23	17.07	3.165	499.4	31.14	484.6
25	16.93	2.671	487.3	30.66	472.7
28	16.71	2.117	468.2	29.91	454.0
30	16.52	1.828	452.5	29.28	438.6
33	15.65	1.446	385.0	26.48	372.5
35	14.68	1.178	318.4	23.55	307.3

The above results lead us to some conclusions

1. We have used the Bubnov-Galerkin method for solving the elastoplastic stability problem of the cylindrical shells with two types of various kinematic boundary conditions. In this paper, the linearly independence of the systems of functions δw_{ij} are also investigated.

2. For long shells we have shown the necessary and sufficient conditions of minimum.

3. The more the shell is thin the more the value of critical stress intensity σ_u^* is small (see Tables 1, 2, 3, 4).

4. The critical loads of the simply supported cylindrical shells subjected to complex loading are always smaller than critical ones when the cylindrical shells are clamped. This result corresponds to the real property of material (see Tables 1, 2, 3, 4).

5. Theory of elastoplastic processes can be applied to the stability problem of cylindrical shells when both pre-buckling and post-buckling processes are complicated.

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VỀ BÀI TOÁN ỔN ĐỊNH ĐÀN DẺO CỦA VỎ TRỤ TRÒN MỎNG CHỊU TẢI PHỨC TẠP VỚI CÁC ĐIỀU KIỆN BIÊN ĐỘNG HỌC KHÁC NHAU

Bài báo trình bày bài toán ổn định của vỏ trụ chịu tác dụng đồng thời cả lực nén dọc đường sinh và áp lực ngoài. Đã xét hai dạng điều kiện biên động học là tựa bản lề và ngàm tại $x_1 = 0$; $x_1 = L$. Sử dụng phương pháp Bubnov-Galerkin đã thiết lập được hệ thức để tìm tải tới hạn. Điều kiện đủ của cực trị cho vỏ dài đã được xem xét. Một số kết quả tính toán bằng số cũng được trình bày và thảo luận.