Vietnam Journal of Mechanics, VAST, Vol. 30, No. 4 (2008), pp. 347 – 358
Special Issue of the 30^{th} Anniversary

IMPROVED APPROXIMATIONS FOR THE RAYLEIGH WAVE VELOCITY IN [-1, 0.5]

Pham Chi Vinh
Hanoi University of Science, Vietnam
Peter G. Malischewsky
Institute for Geosciences, Friedrich-Schiller University
Jena Burgweg 11, 07749 Jena, Germany

Abstract. In the present paper we derive improved approximations for the Rayleigh wave velocity in the interval $\nu \in [-1, 0.5]$ using the method of least squares. In particular: (i) We create approximate polynomials of order 4, 5, 6 whose maximum percentage errors are 0.035 %, 0.015 %, 0.0083 %, respectively. (2i) Improved approximations in the form of the inverse of polynomials of order 3, 5 are also established. They are approximations with very high accuracy. (3i) By using the best approximate second-order polynomial of the cubic power in the space $C[0.474572, 0.912622]$, we derive an approximation that is the best, so far, of the approximations obtained by approximating the secular equation.

1. INTRODUCTION

Elastic surface waves in isotropic elastic solids, discovered by Lord Rayleigh [1] more than 120 years ago, have been studied extensively and exploited in a wide range of applications in seismology, acoustics, geophysics, telecommunications industry and materials science, for example. It would not be far-fetched to say that Rayleigh's study of surface waves upon an elastic half-space has had fundamental and far-reaching effects upon modern life and many things that we take for granted today, stretching from mobile phones through to the study of earthquakes, as noted by Samuel [2].

For the Rayleigh wave, its velocity is a fundamental quantity which is significance in practical applications, so researchers have attempted to find its analytical approximate expressions which are of simple forms and accurate enough for practical purposes.

Let c be the Rayleigh wave velocity in isotropic elastic solids and $x(\nu) = c/\beta$, where β is the velocity of shear waves and ν is Poisson's ratio. The earliest known approximate formula of $x(\nu)$ was proposed by Bergmann [3], namely:

$$x_b(\nu) = \frac{0.87 + 1.12\nu}{1 + \nu}\,,\ \nu \in [0,\, 0.5]. \tag{1}$$

Since Bergmann's approximation has a simple form (also is very well-known) it has a wide range of applications. However, its accuracy is not so high, so it is very significant to improve the accuracy of it. An empirical correction of its numerical coefficients was proposed by Klerk [4], but the accuracy is changed inconsiderably. Recently, Vinh & Malischewsky [5] have found an improved approximation of Bergmann's form, namely:

$$x_{vmb}(\nu) = \frac{1.68522 + 1.27223\nu}{1.92899 + \nu}\,,\ \nu \in [0,\, 0.5], \tag{2}$$

which is 10 times more accurate than $x_b(\nu)$, in the sense of maximum percentage error (defined by (15)). For the range $\nu \in [0,\, 0.5]$, some other improved approximations of the Rayleigh wave velocity have also been established (see [6], [7]).

Materials with negative values of Poisson's ratio, so-called auxetic materials, really exist (see e. g. a new recent review by Yang et al. [8]), and their applications are numerous, including their use as core material of sandwich panel, minimization of creep buckling failure, drug-eluting stents, anti-vibration glove, textiles, ect., as noted by Lim [9]. Thus approximations of the Rayleigh wave velocity for the range $[-1,\, 0.5]$ become significant for practical applications. The first approximation for the range $[-1,\, 0.5]$ was proposed by Malischewsky [10], namely:

$$x_m(\nu) = 0.874 + 0.196\nu - 0.043\nu^2 - 0.055\nu^3\,,\ \nu \in [-1,\, 0.5], \tag{3}$$

and it is shown by Vinh & Malischewsky [11] that $x_m(\nu)$ can be considered as the best approximation of the Rayleigh wave velocity $x(\nu)$ in the interval $[-1,\, 0.5]$, in the sense of least squares, with respect to the class of Taylor expansions of $x(\nu)$ up to the third power at the values $y \in [-1,\, 0.5]$. An improved third-order approximate polynomial for the interval $[-1,\, 0.5]$ has been found recently by Vinh & Malischewsky [6], namely:

$$x_{vm}(\nu) = 0.87384 + 0.192422\nu - 0.0350168\nu^2 - 0.0439059\nu^3\,,\ \nu \in [-1,\, 0.5], \tag{4}$$

which is the best approximate polynomial of order 3 of the Rayleigh wave velocity $x(\nu)$ in the interval $[-1,\, 0.5]$, in the sense of least squares.

By using Lanczos's approximation [12], Rahmann and Michelitsch [13] has obtained the approximate formula:

$$x_{rm} = \sqrt{\frac{30.876 - 14.876\nu - \sqrt{224.545376\nu^2 - 93.122752\nu + 124.577376}}{26(1-\nu)}}\,, \tag{5}$$

$\nu \in [-1,\, 0.5]$. Independentlly, Vinh & Malischewsky [6] and Li [14] have obtained the following approximation

$$x_{vml} = \sqrt{\frac{15.4 - 7.4\nu - \sqrt{56.06\nu^2 - 22.52\nu + 30.46}}{13(1-\nu)}}\,,\ \nu \in [-1,\, 0.5], \tag{6}$$

by using the best approximate second-order polynomials of the cubic power in the interval $[0,\, 1]$. Using the intervals $[0.47,\, 1]$ and $[0.474572,\, 0.912622]$, instead of the one $[0,\, 1]$, Li [14] and Vinh & Malischewsky [6] have found, respectively, the following approximations:

$$x_{l2} = \sqrt{\frac{28.84 - 12.84\nu - \sqrt{198.89\nu^2 - 66.98\nu + 124.1}}{23.18(1-\nu)}}\,,\ \nu \in [-1,\, 0.5], \tag{7}$$

and

$$x_{vm2} = \sqrt{\frac{29.171 - 13.171\nu - \sqrt{203.188\nu^2 - 70.194\nu + 123}}{23.677(1-\nu)}}, \ \nu \in [-1,\, 0.5]. \tag{8}$$

The accuracies of the obtained approximations of the Rayleigh wave velocity in the interval $[-1,\, 0.5]$ in the sense of maximum percentage error are shown in Table 1.

Table 1. Maximum percentage error $\bar{I}$ of the approximations in the interval $\nu \in [-1,\, 0.5]$: $\bar{I} = \max_{[-1,\, 0.5]} |1 - g(\nu)/x(\nu)| \times 100\%$, $g(\nu)$ is an approximation of $x(\nu)$.

Appr.	$\bar{I}(\%)$	Appr.	$\bar{I}(\%)$	Appr.	$\bar{I}(\%)$
x_{rm}	0.418	x_{vml}	0.312	x_{l2}	0.16
x_m	0.4111	x_{vm}	0.21	x_{vm2}	0.09

It is shown, from Table 1, that among existing approximations for the range $[-1,\, 0.5]$, the approximation $x_{vm2}(\nu)$ is the best. However, its accuracy is not so high. As stressed by Nesvijski [15], nondestructive testing of composites is a complex problem because components of materials may have very similar physical-mechanical properties. In order to distinguish one component from another we need highly accurate approximations of the Rayleigh wave velocity. Some recent experimental results cannot be explained unambiguously by existing approximate formulas. This motivates the authors to improve previously proposed approximations for the values $\nu \in [-1,\, 0.5]$. In particular: (i) We obtain approximate polynomials of order 4, 5, 6 which are 2.57 times, 6 times, 10.8 times more accurate than $x_{vm2}(\nu)$, respectively. (2i) Improved approximations in the form of the inverse of polynomials of order 3, 5 are also established. They are 1.8 times, 15.5 times, respectively, better than the approximation $x_{vm2}(\nu)$. (3i) By replacing the cubic power by its best approximate second-order polynomial in the space $C[0.474572, 0.912622]$, we derive an approximation that is the best, so far, of the approximations obtained by approximating the secular equation.

2. APPROACH OF LEAST SQUARES

In order to obtain the improved approximations of the Rayleigh wave velocity we will use the least-square method which was presented in detail in [6]. Here we recall it shortly.

Let V be a subset of the space $L^2[a, b]$ (that consists of all functions measurable in (a, b), whose squared values are integrable on $[a, b]$ in the sense of Lebesgue), and f is a given function of $L^2[a, b]$. A function $g \in V$ is called the best approximation of f with respect to V, in the sense of least squares, if it satisfies the equation

$$\int_a^b [f(\nu) - g(\nu)]^2 d\nu = \min_{h \in V} I(h), \tag{9}$$

where

$$I(h) = \int_a^b [f(\nu) - h(\nu)]^2 d\nu. \tag{10}$$

If V is a finite dimensional linear subspace (a compact set) of $L^2[a,b]$, then the problem (9) has a unique solution (a solution) (see [16]).

Since polynomials are considered as the simplest functions, V is normally taken as the set of polynomials of order not bigger than n, denoted by P_{n+1}, which is a linear subspace of $L^2[a,b]$, has dimension $n+1$, and its basic functions can be chosen as:

$$1,\ \nu, ..., \nu^n. \tag{11}$$

For this case, in order to solve problem (9) we represent $h(\nu)$ as a linear combination of $1,\ \nu, ..., \nu^n$:

$$h(\nu) = \sum_{i=0}^{n} a_i \nu^i. \tag{12}$$

Then the functional $I(h)$ becomes a function of the $n+1$ variables $a_0, a_1, ..., a_n$, and from the conditions: $\partial I/\partial a_i = 0,\ i = 0,\, 1,\, ..., n$, problem (9) is leaded to a system of $n+1$ linear equations for $a_0, a_1, ..., a_n$:

$$\begin{cases} \sum_{i=0}^{n}(1/(i+1))(b^{i+1} - a^{i+1})a_i = b_0 \\ \sum_{i=0}^{n}(1/(i+2))(b^{i+2} - a^{i+2})a_i = b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{i=0}^{n}(1/(i+1+n))(b^{i+1+n} - a^{i+1+n})a_i = b_n \end{cases} \tag{13}$$

which has a unique solution, where:

$$b_i = \int_a^b \nu^i f(\nu) d\nu,\ i = 0, 1, ..., n. \tag{14}$$

In order to evaluate an approximation's accuracy we use the maximum percentage (relative) error $\bar{I}$ defined as follows:

$$\bar{I} = \max_{[a,\,b]} |1 - g(\nu)/f(\nu)| \times 100\%\,, \tag{15}$$

where $g(\nu)$ is an approximation of $f(\nu)$ in the interval $[a,\, b]$.

3. HIGHLY ACCURATE POLYNOMIAL APPROXIMATIONS

Now we find the best approximate polynomial of order 5 of $x(\nu)$ in the interval $[-1,\, 0.5]$ in the sense of least squares. That means we have to solve problem (9) in which $a = -1,\ b = 0.5$, $h(\nu)$ is presented by (12) with $n = 5$, $f(\nu) = x(\nu)$ and $x(\nu)$ is given by (see [17]):

$$x(\nu) = c/\beta = \sqrt{\bar{x}(\nu)},\ \bar{x}(\nu) = \frac{2}{3}\left[4 - \sqrt[3]{h_3(\gamma)} + \frac{2(1-6\gamma)}{\sqrt[3]{h_3(\gamma)}}\right], \tag{16}$$

where:

$$\gamma = \frac{1-2\nu}{2(1-\nu)} = (\beta/\alpha)^2, \tag{17}$$

and:

$$h_1(\gamma) = 3\sqrt{33 - 186\gamma + 321\gamma^2 - 192\gamma^3},\ h_3(\gamma) = 17 - 45\gamma + h_1(\gamma). \tag{18}$$

Here α is the velocity of longitudinal waves, and the main values of the cubic roots are to be used in (16). The problem is then leaded to the system (13), (14) with $n = 5,\ a = -1,\ b = 0.5,\ f(\nu) = x(\nu)$, whose unique solution is:

$$a_0 = 0.873985,\ a_1 = 0.19489,\ a_2 = -0.038276,\ a_3 = -0.059703,$$

$$a_4 = 0.004023,\ a_5 = 0.01586. \tag{19}$$

Thus, the desired polynomial is:

$$p_5(\nu) = 0.873985 + 0.19489\nu - 0.038276\nu^2 - 0.059703\nu^3 + 0.004023\nu^4 + 0.01586\nu^5, \tag{20}$$

whose the maximum percentage error is 0.03% (see Fig. 1).

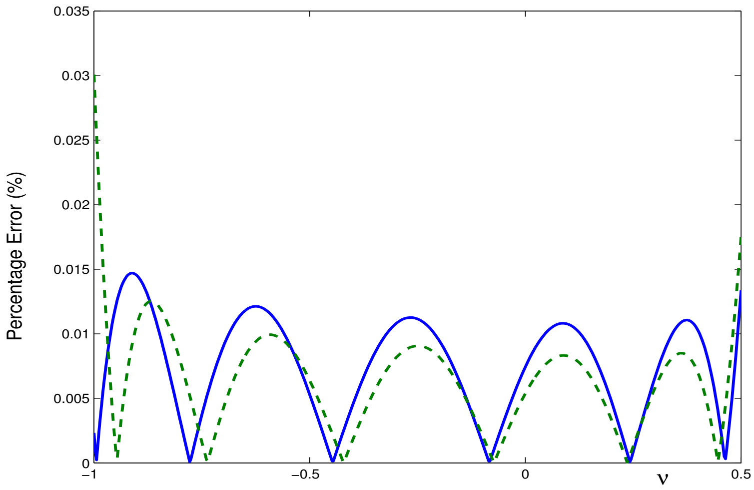

Fig. 1. Percentage error of approximations: $p_5(\nu)$ (dashed line), $p_{5*}(\nu)$ (solid line).

It is shown, from Fig. 1, that the percentage error of $p_5(\nu)$ is small at every value of ν of the interval $[-1,\ 0.5]$ except the values belong to a very small neighbourhood of -1 and 0.5, and it very quickly increases when ν tends to these values. Egorov's theorem [18] suggests that, in order to decrease the percentage error of $p_5(\nu)$ in the interval $[-1,\ 0.5]$ we should find the best approximate polynomial of order 5 in the interval $[-1-\epsilon_1,\ 0.5+\epsilon_2]$, where ϵ_1, ϵ_2 are positive and appropriately small.

By choosing $\epsilon_1 = 0.05$, $\epsilon_2 = 0.02$ we obtain:

$$p_{5*}(\nu) = 0.873967{+}0.194795\nu{-}0.037806\nu^2{-}0.058746\nu^3{+}0.0028151\nu^4{+}0.014051\nu^5 \tag{21}$$

whose the maximum percentage error in the interval $[-1,\ 0.5]$ is only 0.015% (see Fig. 1), less than the one of $p_5(\nu)$. It is clear that the approximation $p_{5*}(\nu)$ is a very highly accurate approximation of $x(\nu)$ in the interval $[-1,\ 0.5]$ in the sense of maximum percentage error. It is 6 times better than x_{vm2}, the best of the obtained approximations of $x(\nu)$ in the interval $\nu \in [-1,\ 0.5]$.

Analogously, by solving the system (13), (14) with $n = 6,\ a = -1,\ b = 0.5,\ f(\nu) = x(\nu)$ (defined by (16)), we obtain the best approximate polynomial of order 6 of $x(\nu)$ in the interval $[-1,\, 0.5]$, namely:

$$\begin{aligned} p_6(\nu) &= 0.874057 + 0.195792\nu - 0.040855\nu^2 - 0.07346\nu^3 + 0.009183\nu^4 \\ &+ 0.061261\nu^5 + 0.030267\nu^6, \end{aligned} \tag{22}$$

whose the maximum percentage error is 0.024% (see Fig. 2).

It is clear, from Fig. 2, that the percentage error of $p_6(\nu)$ very quickly increases when ν tends to the boundary values -1 and 0.5. According to Egorov's theorem [18], in order to decrease the the percentage error of $p_6(\nu)$ in the interval $[-1,\, 0.5]$ we should find the best approximate polynomial of order 6 in the interval $[-1-\epsilon_1,\, 0.5+\epsilon_2]]$, where ϵ_1, ϵ_2 are positive and appropriately small.

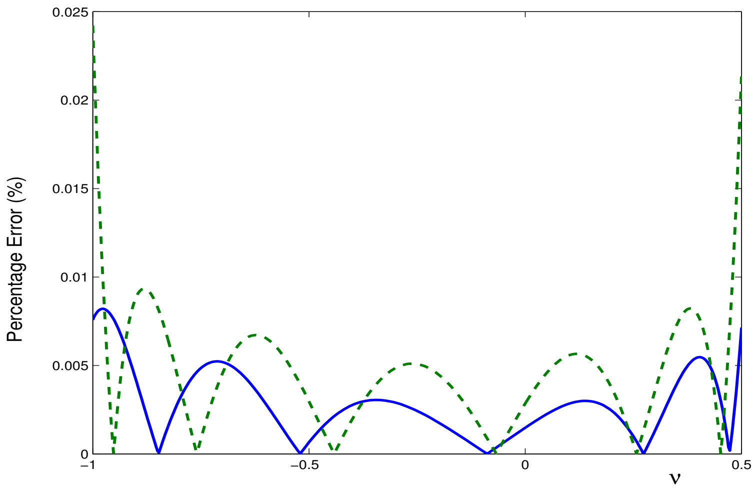

Fig. 2. Percentage error of approximations: $p_6(\nu)$ (dashed line), $p_{6*}(\nu)$ (solid line).

By choosing $\epsilon_1 = 0.1$, $\epsilon_2 = 0.02$ we obtain an approximation being better than $p_6(\nu)$ in the sense of maximum percentage error, namely:

$$\begin{aligned} p_{6*}(\nu) &= 0.874045 + 0.195586\nu - 0.040371\nu^2 - 0.069955\nu^3 + 0.008382\nu^4 \\ &+ 0.04879\nu^5 + 0.021205\nu^6, \end{aligned} \tag{23}$$

whose the maximum percentage error (in the interval $[-1,\, 0.5]$) is only 0.0083%, less than the one of $p_6(\nu)$. The approximation $p_{6*}(\nu)$ is also a very highly accurate approximation of $x(\nu)$ in the interval $[-1,\, 0.5]$. It is 10.8 times better than x_{vm2} in the sense of maximum percentage error.

Remark 1:

i) Doing analogously as above, i. e. employing the method of least squares and taking into account Egorov's theorem, we obtain the following approximation:

$$p_{3*}(\nu) = 0.873776 + 0.192001\nu - 0.034105\nu^2 - 0.041932\nu^3\,, \tag{24}$$

that is the best approximate polynomial of order 3 of $x(\nu)$ in the interval $[-1.05, 0.5]$ in the sense of least squares, and its maximum percentage error in the interval $[-1, 0.5]$ is 0.1%. Since the maximum percentage errors of the approximations $x_m(\nu)$ (defined by (3)) and $x_{vm}(\nu)$ (defined by (4)) in the interval $[-1, 0.5]$ are 0.411% and 0.21%, respectively, $p_{3*}(\nu)$ is the best approximate polynomial of order 3 of $x(\nu)$ in the interval $[-1, 0.5]$, so far, in the sense of the maximum percentage error.

2i) Approximations $p_{3*}(\nu)$ and $x_{vm2}(\nu)$ have almost the same accuracy, but $p_{3*}(\nu)$ has a simpler form, so it may be better than $x_{vm2}(\nu)$ for practical purposes.

3i) By finding the best approximate polynomial of order 4, in the sense of least squares, in the interval $[-1.02, 0.5325]$ we obtain the following approximation:

$$p_{4*}(\nu) = 0.873785 + 0.195024\nu - 0.032795\nu^2 - 0.057681\nu^3 - 0.014513\nu^4\,, \qquad (25)$$

whose the maximum percentage error (in the interval $[-1, 0.5]$) is 0.035%. It is 2.57 times more accurate than x_{vm2}. It should be noted that, in the sense of maximum percentage error the approximation p_{4*} is better than best approximate polynomial of order 4 of $x(\nu)$ in the interval $[-1, 0.5]$.

4. HIGHLY ACCURATE APPROXIMATIONS OF THE FORM OF THE INVERSE OF POLYNOMIALS

For the interval [0, 0.5], an approximation of the form of the inverse of a polynomial was first proposed by Sinclair (see [19], [10]), namely:

$$x_{sc}(\nu) = \frac{1}{1.14418 - 0.25771\nu + 0.12661\nu^2}\,,\ \nu \in [0,\ 0.5]. \qquad (26)$$

It was published without the derivation procedure. Interestingly, it was proved recently by Vinh & Malischewsky [7] that the inverse of Sinclair's approximation is the best approximation of $s(\nu)$ in the interval [0, 0.5], in the sense of least squares, with respect to the set of all Taylor expansions of $s(\nu)$ up to the second power at the values $y \in [0,\ 0.5]$.

Although Sinclair's expression (26) approximates rather well $x(\nu)$ in [0, 0.5] (its maximum percentage error is 0.02%) but it is not the best one of this form. Recently, Vinh & Malischewsky [7] have found an inverse of a second-order polynomial, namely:

$$x_{vms2}(\nu) = \frac{1}{1.14413 - 0.25689\nu + 0.12457\nu^2}\,, \qquad (27)$$

which is 4 times more accurate than Sinclair's approximation. Approximations of Sinclair type with higher accuracy for this range were also obtained recently by Vinh & Malischewsky [7] using the method of least squares. In this section approximations of $x(\nu)$ in the form of the inverse of polynomials of order 3 and 5, denoted by $x_{vms3}(\nu)$ and $x_{vms5}(\nu)$, respectively, are created for the interval $[-1, 0.5]$. They are approximations with high accuracies.

Because $0 < x(\nu) < 1\ \forall\, \nu \in\ [-1, 0.5]$, it follows that $s(\nu) = 1/x(\nu) > 1\ \forall\, \nu \in [-1, 0.5]$. This leads to:

$$\parallel x - \frac{1}{g} \parallel < \parallel s - g \parallel\ \ \forall g \in L^2[0,\ 0.5] : g(\nu) > 1\ \forall\, \nu \in\ [-1,\ 0.5]. \qquad (28)$$

The inequality (28) is valid for both $L^2[-1,\, 0.5]$-norm and $C[-1,\, 0.5]$-norm. Here $C[a,\, b]$ denotes the space of continuous functions in $[a,\, b]$. It follows form inequality (28) that if $g(\nu)$ is a good approximation of $s(\nu)$ then $1/g(\nu)$ is a good one of $x(\nu)$. A more accurate approximation of $s(\nu)$ likely leads to a corresponding more accurate one of $x(\nu)$ by this way. Following this idea, in order to obtain the approximations of the form of the inverse of polynomials with high accuracy we should find highly accurate approximations of $s(\nu)$in the interval $[-1,\, 0.5]$, where $s(\nu)$ (dimensionless slowness) is given by ([20]):

$$s(\nu) = \frac{1}{x(\nu)} = \sqrt{\bar{s}},\;\; \bar{s} = \frac{1}{4(1-\gamma)}\left[2 - \frac{4}{3}\gamma + \sqrt[3]{v(\gamma)} + \frac{3+(4\gamma-3)^2}{9\sqrt[3]{v(\gamma)}}\right], \tag{29}$$

where:

$$v(\nu) = \frac{2}{27}(27 - 90\gamma + 99\gamma^2 - 32\gamma^3) + \frac{2}{3\sqrt{3}}(1-\gamma)\sqrt{11 - 62\gamma + 107\gamma^2 - 64\gamma^3}. \tag{30}$$

Now we find the best approximate polynomial of order 5 of $s(\nu)$ in the interval $[-1,\, 0.5]$ in the sense of least squares. That means we have to solve the system (13), (14) with $n = 5,\; a = -1,\; b = 0.5,\; f(\nu) = s(\nu)$. It is not difficult see that its unique solution is:

$$\begin{aligned} a_0 &= 1.14416,\, a_1 = -0.25557,\, a_2 = 0.10791,\, a_3 = 0.04916, \\ a_4 &= -0.03049,\, a_5 = -0.02378. \end{aligned} \tag{31}$$

The desired polynomial is:

$$q_5(\nu) = 1.14416 - 0.25557\nu + 0.10791\nu^2 + 0.04916\nu^3 - 0.03049\nu^4 - 0.02378\nu^5, \tag{32}$$

From (32) we obtain a very good approximation, namely $1/q_5$, of $x(\nu)$ in the interval $[-1,\, 0.5]$ whose maximum percentage error is 0.012% (see Fig. 3). It is clear, however, from Fig. 3, that the percentage error of $1/q_5(\nu)$ very quickly increases when ν tends to the boundary points -1 and 0.5, especially the point -1. According to Egorov's theorem [18] and inequality (28), in order to decrease the the percentage error of $1/q_5(\nu)$ in the interval $[-1,\, 0.5]$ we should find the best approximate polynomial of order 5 of $s(\nu)$ in the interval $[-1-\epsilon_1,\, 0.5+\epsilon_2]]$, where ϵ_1, ϵ_2 are positive and appropriately small. Taking $\epsilon_1 = 0.03,\; \epsilon_2 = 0.02$ we obtain the following approximate polynomial:

$$q_{5*}(\nu) = 1.14417 - 0.25552\nu + 0.10772\nu^2 + 0.04871\nu^3 - 0.03\nu^4 - 0.02297\nu^5\,. \tag{33}$$

Thus, the approximation $x_{vms5}(\nu)$ is given by:

$$x_{vms5}(\nu) = \frac{1}{1.14416 - 0.25557\nu + 0.10791\nu^2 + 0.04916\nu^3 - 0.03049\nu^4 - 0.02378\nu^5} \tag{34}$$

The maximum percentage error of $x_{vms5}(\nu)$ in the interval $[-1,\, 0.5]$ is only 0.0058%. This says that $x_{vms5}(\nu)$ is very highly accurate approximation of $x(\nu)$ in the interval $[-1,\, 0.5]$. It is 15.5 times more accurate than x_{vm2}.

Following the same procedure as above we obtain:

$$x_{vms3}(\nu) = \frac{1}{1.14447 - 0.25632\nu + 0.0993\nu^2 + 0.04888\nu^3}\,, \tag{35}$$

where the denominator of $x_{vms3}(\nu)$ is the best approximate third-order polynomial of $s(\nu)$ in the interval $[-1,\, 0.54]$. The maximum percentage error of $x_{vms3}(\nu)$ in the interval

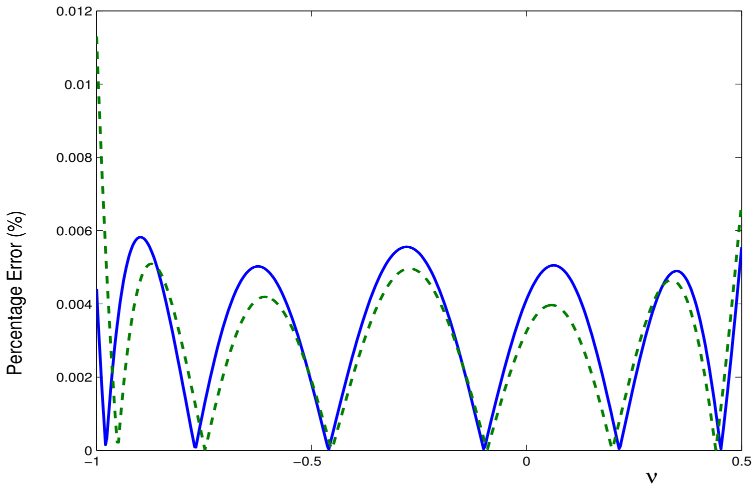

Fig. 3. Percentage error of approximations: $x_{vms5}(\nu)$ (solid line), $1/q_5(\nu)$ (dashed line).

$[-1,\ 0.5]$ is 0.046%, 1.8 times smaller than that of x_{vm2} , so it is a highly accurate approximation of $x(\nu)$ in the interval $[-1,\ 0.5]$. It should be noted that $x_{vms3}(\nu)$ is 2 times better than $p_{3*}(\nu)$ and $x_{vms5}(\nu)$ is 2.6 times better than $p_{5*}(\nu)$, even it is more accurate than $p_{6*}(\nu)$.

5. AN IMPROVED APPROXIMATION DERIVED BY APPROXIMATING THE SECULAR EQUATION

In order to establish approximate expressions of the Rayleigh wave speed one can replace the cubic secular equation by quadratic ones approximating the cubic power by the best approximate second-order polynomials (see [6], [13], [14]). The approximating can be carried out in the space $L^2[a,b]$ or in the one $C[a,b]$. Vinh & Malischewsky [6] and Li [14] used the space $L^2[a,b]$, and from obtained results it is shown that the best approximation of Rayleigh wave speed corresponds to the choice $a = \bar{x}(-1) = 0.474572$, $b = \bar{x}(0.5) = 0.912622$. Now we employ the space $C[a,b]$ to approximate the cubic power, and according to this conclusion, we choose $a = 0.474572$, $b = 0.912622$. It should be noted that, as shown by Vinh & Malischewsky [6], Lamczos's approximation used by Rahman & and Michelistch [13] is nothing but the best approximate second-order polynomial of the cubic power in $C[0,1]$.

As demonstrated in [6], *among all polynomials $q(t)$ of the n-th degree whose leading coefficient is unity, the Chebysev polynomial $T_n(t)/2^{n-1}$ (see* [12]*) deviates the least from zero in $C[-1,1]$*. By the transformation $t(z) = (2z-a-b)/(b-a)$, this observation leads to the conclusion: *among all polynomials of the n-th order whose leading coefficient is unity, the polynomial $(b-a)^n T_n(t(z))/2^{2n-1}$ deviates the least from zero in $C[a,b]$*. That means the following proposition is valid: *the polynomial $p_{n-1}(z) = z^n - (b-a)^n T_n(t(z))/2^{2n-1}$ deviates the least from z^n in $C[a,b]$*. Applying the proposition for $n = 3$, $a = 0.474572$, $b =$

0.912622 we have:

$$p_2(z) = 2.080791z^2 - 1.4072514z + 0.3087185. \tag{36}$$

that is the best approximate second-order polynomial of z^3 in $C[0.474572, 0.912622]$. Employing the proposition for $a = 0$, $b = 1$ and $n = 3$ produces Lanczos's approximation, namely: $1.5z^2 - 0.5625z + 0.03125$, that is the best approximate second-order polynomial of z^3 in $C[0, 1]$.

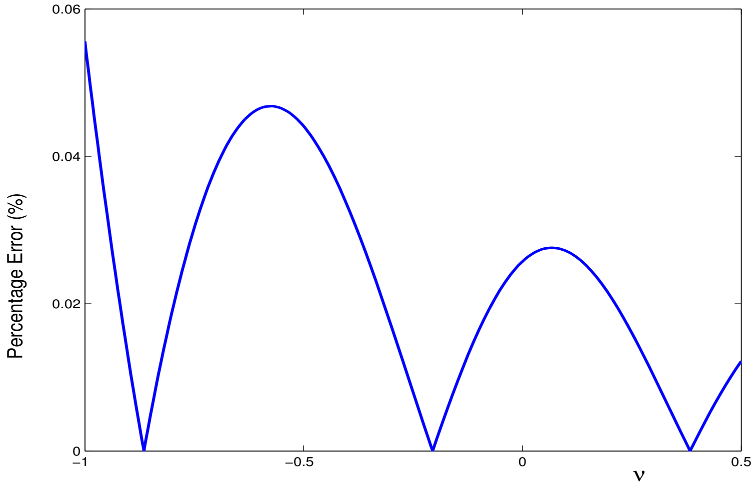

Fig. 4. Percentage error of approximation x_{vm2*}.

Replacing the cubic power in the secular equation by the approximation (36) leads to a quadratic equation. Its solution corresponding to Rayleigh wave speed is then easy obtained, namely:

$$x_{vm2*} = \sqrt{\frac{29.185498 - 13.185498\nu - \sqrt{203.095267\nu^2 - 70.467717\nu + 123.372451}}{23.676836(1-\nu)}} \tag{37}$$

Making use of (15)-(17) and (37), the maximum percentage error of x_{vm2*} in the interval $[-1, 0.5]$ is 0.056%. This can also be seen from Figure 4. From Table 1 and this fact it is concluded that among approximations of the Rayleigh wave velocity in the interval $[-1, 0.5]$ derived by approximating the cubic power, x_{vm2*} is the best, so far, in the sense of maximum percentage error, and it is 1.6 times, 2.8 times, 5.5 times, 7.5 times, respectively, more accurate than x_{vm2}, x_{l2}, x_{vml}, x_{rm}, the approximations previously obtained by this way.

6. CONCLUSIONS

In this paper we have derived improved approximations of the Rayleigh wave velocity for the range $\nu \in [-1, 0.5]$ using the method of least squares and taking account into Egogov's theorem. Some of them are very highly accurate approximations. That are approximations p_{5*}, p_{6*} and x_{vms5} whose maximum percentage error are 0.015%, 0.0083%

and 0.0056%, respectively. They are 6 times, 10.8 times and 15.5 times better than x_{vm2}, the best of the previously established approximations of $x(\nu)$ for the values $\nu \in [-1, 0.5]$. By approximating the cubic power in the space $C[0.474572, 0.912622]$ an approximation with high accuracy has been established that is the best, so far, of the approximations obtained by approximating the secular equation. It should be noted that the technique used here can be employed to create approximations with higher accuracy.

REFERENCES

[1] L. Rayleigh, On waves propagating along the plane surface of an elastic solid, Proc. R. Soc. Lond. A17 (1885) 4-11.

[2] D. Samue et al, Rayleigh waves guided by topography, Proc. R. Soc. Lond. A463 (2007) 531-550.

[3] L. Bergmann, Ultrasonics and their Scientific and Technical Applications, John Wiley Sons, New York, 1948.

[4] J. de Klerk, Ultrasonic Transducers: 3. Surface Wave Transducers, Ultrasonics 9 (1971) 35-48.

[5] Pham Chi Vinh and P. Malischewsky, An improved approximation of Bergmann's form for the Rayleigh wave velocity, Ultrasonic 47 (2007) 49-54.

[6] Pham Chi Vinh and P. Malischewsky, An approach for obtaining approximate formulas for the Rayleigh wave velocity, Wave Motion 44 (2007) 549-562.

[7] Pham Chi Vinh and P. Malischewsky, Improved Approximations of the Rayleigh Wave Velocity, J. Thermoplast. Comp. Mater. 21 (2008) 337-352.

[8] W. Yang, Z-M. Li, W. Shi, B-H. Xie, M-B. Yang, On auxetic materials, J. Mater. Sci. 39 (2004) 3269-3279.

[9] T-K. Lim, Out-of-plane modulus of semi-auxetic laminates, *Eur. J. Mech. A/Solids* **28** (2009) 752-756

[10] Malischewsky P, Comparison of approximated solutions for the phase velocity of Rayleigh waves (Comment on 'Characterization of surface damage via surface acoustic waves'), Nanotechnology 16 (2005) 995-996.

[11] Pham Chi Vinh and P. Malischewsky, Explanation for Malischewsky's approximate expression for the Rayleigh wave velocity, Ultrasonics 45 (2006) 77-81.

[12] C. Lanczos, Applied Analysis, Prentice-Hall Inc., New Jersy 1956.

[13] M. Rahman, T. Michelitsch, A note on the formula for the Rayleigh wave speed, Wave Motion 43 (2006) 272-276.

[14] X-F. Li, On approximate analytic expressions for the velocity of Rayleigh waves, Wave Motion 44 (2006) 120-127.

[15] E. G. Nesvijski, On Rayleigh Equation and Accuracy of Its Real Roots Calculations, J. Thermo. Plast. Compt. Mater. 14 (2001) 356-364.

[16] G. Meinardus, Approximation of functions: Theory and Numerical Methods, Springer-Verlag, Berlin Heidelberg New York 1967.

[17] Malischewsky Auning P, A note on Rayleigh-wave velocities as a function of the material parameters, Geofisica Internacional 43 (2004) 507-509.

[18] A. N. Kolmogorov and S. V. Fomin, Elements of the theory of functions and functional analysis, Volume 2: measure, the Lebesgue integral, Hilbert space, Graylock Press, Albany, New York, 1961.

[19] G. A. D. Briggs, Acoustic Microscopy, Clarendon Press, Oxford, 1992.

[20] Pham Chi Vinh and R. Ogden, On formulas for the Rayleigh wave speed, Wave Motion 39 (2004) 191-197.

Received April 9, 2009

CÁC XẤP XỈ VỚI ĐỘ CHÍNH XÁC CAO CỦA VẬN TỐC SÓNG RAYLEIGH TRÊN ĐOẠN [-1, 0.5]

Trong bài báo này, các xấp xỉ với độ chính xác cao của vận tốc sóng Rayleigh trên đoạn [-1, 0.5] đÓ được xây dựng dựa trên phương pháp bình phương tối thiểu. Cụ thể: (i) Đó là các đa thức xấp xỉ bậc 4, 5, 6 với các sai số tương đối là 0.035%, 0.015%, 0.0083%. (ii) Các xấp xỉ là nghịch đảo của các đa thức bậc 3, bậc 5 cũng đÓ được xây dựng. Chúng có độ chính xác rất cao. (iii) Sử dụng đa thức bậc hai xấp xỉ tốt nhất của lũy thừa bậc ba trong không gian C[0.474572, 0.912622], các tác giả đÓ thu được một xấp xỉ mà cho đến nay là tốt nhất trong các xấp xỉ tìm thấy bằng cách xấp xỉ trực tiếp phương trình tán sắc.