Vietnam Journal of Mechanics, VAST, Vol. 30, No. 4 (2008), pp. 337 – 346
Special Issue of the $30^{\text{th}}$ Anniversary

# A $C^1$ BENDING ELEMENT FOR COMPOSITE PLATES BASED ON A HIGH-ORDER SHEAR DEFORMATION THEORY

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**Abstract.** A new $C^1$ rectangular element is proposed and the finite element formulation based on Reddy's higher-order shear deformation plate theory is developed. Although the plate theory is quite attractive but it could not be exploited as expected in finite-element analysis. This is due to the dificulties associated with satisfaction of inter-elemental continuity requirement and satisfy zero shear stress boundary conditions of the plate theory. In this paper, the proposed element is developed where Reddy's plate theory is successfully implemented. It has nine nodes and each node contains 7 degrees of freedom. The performance of the element is tested with different numerical examples, which show its precision and range of applicability.

## 1. INTRODUCTION

Laminated composite plates are finding extensive usage in the aeronautical, ship building and aerospace industries as well as in other fields of modern technology.

The problem of shear deformation has got a good amount of attention after the popularity of fibre reinforced laminated composites, which is now one of the major areas of research in recent times. Actually, the role of transverse shear is very important in composites, as the material is weak in shear due to its low shear modulus compared to extensional rigidity. In this context, a number of plate theories have been developed where the major emphasis is to model the shear deformation in a refined manner. Amongst these plate theories (higher-order shear deformation theories-HSDT) only a representative selection is made in reference [1–17].

In single layer displacement-based theories, the plate theory proposed by Reddy [7] is most simple, elegant and useful in the context of present problem. It allows parabolic variation of transverse shear stress along the plate thickness and satisfy zero shear stress boundary conditions at the top and bottom of the plate. This has helped to eliminate the necessity of any arbitrary shear correction factor like that, which is required in FSDT. Moreover Reddy's plate theory does not involve any unknown, which does not have any physical meaning like that found in some plate theories (e.g. [10,11]). Reddy's plate theory has all positive features except one drawback, which is found in a situation when finite element is applied to this plate model. The problem is concerned with the continuity requirement of $w$ at the common edges between two elements. It requires $C^1$ continuity of $w$ as the strain terms contain second-order derivatives of $w$. This problem is identical to that experienced in the development of thin-plate elements as mentioned earlier. This has rather put the main constrain in exploiting such an elegant plate theory in finite element analysis as expected [4].

In this context, Sheikh and Chakrabarti [4] have developed a triangular element. This basic element has six nodes (three corner nodes and three mid-side nodes). Recently, we have developed a $C^1$ rectangular element [1]. This element has four nodes, each node contains $u^\circ$; $v^\circ$; $w^\circ$; $\theta_x$, $\theta_y$, $\gamma_x$ and $\gamma_y$ as the degrees of freedom. However, this element was not really good in rate of convergence for the bending analysis of composite plates.

Keeping all the aspects in view, an attempt has been made to develop a nine-node rectangular element with high accuracy based on higher-order shear deformation theory of Reddy, where each node contains seven degrees of freedom: $u^\circ$; $v^\circ$; $w^\circ$; $\theta_x$; $\theta_y$; $\gamma_x$ and $\gamma_y$. In this element, the field variables: $u^\circ$; $v^\circ$; $\gamma_x$ and $\gamma_y$ are approximated by a complete quadratic polynomial having four unknowns; the transverse displacement $w$ and $\theta_x$, $\theta_y$ are approximated by a truncated quintic polynomial having 12 unknowns. With all these efforts it is found that the element does not satisfy the continuity requirement of normal slope. Thus, the proposed element is non-conforming but it is found that the performance of the element is excellent in a wide range of problems, which include different boundary condition, plate geometry, aspect ratio, stacking sequence, load distribution and so on.

## 2. ELASTICITY EQUATIONS

According to Reddy's plate theory [7], the displacement components of a point at a distance of z from the reference plane may be expressed in terms field variables (displacement parameters at the reference plane) as:

$$\begin{aligned} u(x,y,z) &= u^0(x,y) - z\left[\left(\frac{\partial w}{\partial x} + \gamma_x\right) - \frac{4}{3}\left(\frac{z}{h}\right)^2 \gamma_x\right] \\ v(x,y,z) &= v^0(x,y) - z\left[\left(\frac{\partial w}{\partial y} + \gamma_y\right) - \frac{4}{3}\left(\frac{z}{h}\right)^2 \gamma_y\right] \\ w(x,y,z) &= w^0(x,y) \end{aligned} \tag{1}$$

The strain vector $\{\varepsilon\}$ may be expressed as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \\ 0 \\ 0 \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \chi_{yz} \\ \chi_{xz} \end{Bmatrix} + z^3 \begin{Bmatrix} \eta_x \\ \eta_y \\ \eta_{xy} \\ 0 \\ 0 \end{Bmatrix}, \tag{2}$$

where

$$\left\{\begin{array}{c}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{array}\right\}=\left\{\begin{array}{c}u^0_{'x}\\ v^0_{'y}\\ u^0_{'y}+v^0_{'x}\end{array}\right\};\quad \left\{\begin{array}{c}k_x\\ k_y\\ k_{xy}\end{array}\right\}=-\left\{\begin{array}{c}w_{'xx}+\gamma_{x'x}\\ w_{'yy}+\gamma_{y'y}\\ 2w_{'xy}+\gamma_{x'y}+\gamma_{y'x}\end{array}\right\};$$
$$\left\{\begin{array}{c}\eta_x\\ \eta_x\\ \eta_{xy}\end{array}\right\}=\frac{4}{3h^2}\left\{\begin{array}{c}\gamma_{x'x}\\ \gamma_{y'y}\\ \gamma_{x'y}+\gamma_{y'x}\end{array}\right\};\ \left\{\begin{array}{c}\gamma_{yz}^0\\ \gamma_{xz}^0\end{array}\right\}=-\left\{\begin{array}{c}\gamma_y\\ \gamma_x\end{array}\right\};\ \left\{\begin{array}{c}\chi_{yz}\\ \chi_{xz}\end{array}\right\}=\frac{4}{h^2}\left\{\begin{array}{c}\gamma_y\\ \gamma_x\end{array}\right\} \tag{3}$$

## 3. FINITE ELEMENT MODELING OF EQUATIONS

The formulation is based on the assumptions followed in Reddy's plate theory. The middle plane of the plate is taken as the reference plane.

According to (1), there are 7 displacement components on a node. The nodal displacement vector is expressed by:

$$d_i=\left\{\begin{array}{ccccccc} u_i^0 & v_i^0 & w_i^0 & \left(\dfrac{\partial w}{\partial x}\right)_i & \left(\dfrac{\partial w}{\partial y}\right)_i & (\gamma_y)_i & (\gamma_x)_i\end{array}\right\}^T \tag{4}$$

or

$$d=\left\{\begin{array}{ccccccc} u^0 & v^0 & w^0 & \theta_x & \theta_y & \gamma_x & \gamma_y\end{array}\right\}^T \tag{5}$$

Seven components are 7 degrees of freedom of a node, respectively:

$$d_i=\{q_i, q_{i+1}, q_{i+2}, q_{i+3}, q_{i+4}, q_{i+5}, q_{i+6}\}^T$$

Therefore, the element's nodal displacement vector is presented by:

$$a=\left\{\begin{array}{ccccc} d_1^T & d_2^T\cdots & d_i^T & d_n^T\end{array}\right\}^T \tag{6}$$

According to the discussions made in the previous section, the field variables i.e., the independent displacement components at the reference plane may be expressed as follows:

$$\left[\begin{array}{cccc} u^0 & v^0 & \gamma_x & \gamma_y\end{array}\right]^T=\left[\begin{array}{cccc}\sum\limits_{i=1}^{N}N_iu_i^0 & \sum\limits_{i=1}^{N}N_iv_i^0 & \sum\limits_{i=1}^{N}N_i\gamma_{xi} & \sum\limits_{i=1}^{N}N_i\gamma_{yi}\end{array}\right]^T \tag{7}$$

$$w=H_1w_1^0+H_2\left(\frac{\partial w}{\partial x}\right)_1+H_3\left(\frac{\partial w}{\partial y}\right)_1+\ldots+H_{3N-2}w_N^0+H_{3N-1}\left(\frac{\partial w}{\partial x}\right)_N+H_{3N}\left(\frac{\partial w}{\partial y}\right)_N \tag{8}$$

$$\theta_x=\frac{\partial w}{\partial x}+\gamma_x=\frac{\partial}{\partial x}\sum_{i=1}^{N}\left(H_{3i-2}w_i^0+H_{3i-1}\left(\frac{\partial w}{\partial x}\right)_i+H_{3i}\left(\frac{\partial w}{\partial y}\right)_i\right)+\sum_{i=1}^{N}N_i\gamma_{xi} \tag{9}$$

$$\theta_y=\frac{\partial w}{\partial y}+\gamma_y=\frac{\partial}{\partial y}\sum_{i=1}^{N}\left(H_{3i-2}w_i^0+H_{3i-1}\left(\frac{\partial w}{\partial x}\right)_i+H_{3i}\left(\frac{\partial w}{\partial y}\right)_i\right)+\sum_{i=1}^{N}N_i\gamma_{yi} \tag{10}$$

where $N$ is number of nodes of element, $N_i$ are the Lagrange interpolations functions and $H_i$ are the Hermite interpolation functions.

The displacement vector is interpolated through element's nodal displacement vector as:

$$\boldsymbol{d}=\boldsymbol{B}\boldsymbol{a} \tag{11}$$

where, $B$ is interpolation matrix and is presented by:

$$B=\begin{bmatrix} N_1 & 0 & 0 & 0 & 0 & 0 & 0 & N_2 & \ldots \\ 0 & N_1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & H_1 & H_2 & H_3 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \frac{\partial}{\partial x}H_1 & \frac{\partial}{\partial x}H_2 & \frac{\partial}{\partial x}H_3 & 0 & N_1 & 0 & \ldots \\ 0 & 0 & \frac{\partial}{\partial y}H_1 & \frac{\partial}{\partial y}H_2 & \frac{\partial}{\partial y}H_3 & N_1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & N_1 & 0 & 0 & \ldots \end{bmatrix.$$
$$\begin{matrix} N_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{3N-2} & H_{3N-1} & H_{3N} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x}H_{3N-2} & \frac{\partial}{\partial x}H_{3N-1} & \frac{\partial}{\partial x}H_{3N} & 0 & N_N \\ 0 & 0 & \frac{\partial}{\partial y}H_{3N-2} & \frac{\partial}{\partial y}H_{3N-1} & \frac{\partial}{\partial y}H_{3N} & N_N & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_N \\ 0 & 0 & 0 & 0 & 0 & N_N & 0 \end{matrix}\right]$$

Now the field variables (3) is substituted in Eqs. (4), (5) and (11) to express the strain vector $\{\varepsilon\}$ in terms of element's nodal displacement vector as:

$$\left\{\varepsilon^0\right\}=L_1d=L_1Ba=B_1a;\ \{\kappa\}=L_2d=L_2Ba=B_2a;\ \{\eta\}=L_3d=L_3Ba=B_3a, \quad (12)$$
$$\left\{\gamma^0\right\}=L_1'd=L_1'Ba=B_1'a;\ \{\chi\}=L_2'd=L_2'Ba=B_2'a, \quad (13)$$

where, $L$ and $L'$ are operator matrices and can be presented by:

$$L_1'=-\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};\quad L_2'=\frac{4}{h^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

and

$$L_1=\begin{bmatrix} \frac{\partial}{\partial_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial_y} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial_y} & \frac{\partial}{\partial_x} & 0 & 0 & 0 & 0 & 0 \end{bmatrix};\quad L_2=-\begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial_y} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_x} & 0 & 0 \end{bmatrix};$$
$$L_3=\frac{4}{3h^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial_x} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} \end{bmatrix}$$

Similar to strain vector $\{\varepsilon\}$, the transverse displacement $w$may be expressed in terms of nodal displacement vector {a} with the help of Eqs. (7), (8) and (11) as

$$w(x,y) = B_p a = \{a\}^T \left[0\ 0\ H_1\ H_2\ H_3\ 0\ 0\ \cdots\ 0\ 0\ H_{3*N-2}\ H_{3*N-1}\ H_{3*N}\ 0\ 0\right]^T \tag{15}$$

The present element may have any rectangular shape. This is mapped in a different plane $(\xi - \eta)$, which gives a rectangular shape. The relationship between these two axes system is as follows:

$$\xi = \frac{2\,(x - x_c)}{a};\ \eta = \frac{2\,(y - y_c)}{b} \quad \text{and} \quad x = \frac{a}{2}\xi + x_c;\ y = \frac{b}{2}\eta + y_c. \tag{16}$$

where, $a \times b$ is dimension of rectangular element; $(x_c,\, y_c)$ are the global coordinates of the center of element.

The interpolation functions of nine-noded rectangular element are determined: An alternative derivation uses Hermitian polynomials which permit the writing down of suitable function directly. A Hermitian polynomial of $5^{th}$ order was determined.

$$H_1 = (\xi - \xi^2)(\eta - \eta^2)\left(-2 - 4\xi - 4\eta + \xi^2 + \eta^2 + 3\xi^3 + 3\eta^3\right)/8, \tag{17}$$

$$H_2 = (\xi - \xi^2)(\eta - \eta^2)\left(-\xi + \xi^3\right)/8;\quad H_3 = (\xi - \xi^2)(\eta - \eta^2)\left(-\eta + \eta^3\right)/8$$

$$H_4 = -(\xi + \xi^2)(\eta - \eta^2)\left(-2 + 4\xi - 4\eta + \xi^2 + \eta^2 - 3\xi^3 + 3\eta^3\right)/8, \tag{18}$$

$$H_5 = -(\xi + \xi^2)(\eta - \eta^2)\left(-\xi + \xi^3\right)/8;\quad H_6 = -(\xi + \xi^2)(\eta - \eta^2)\left(-\eta + \eta^3\right)/8$$

$$H_7 = (\xi + \xi^2)(\eta + \eta^2)\left(-2 + 4\xi + 4\eta + \xi^2 + \eta^2 - 3\xi^3 - 3\eta^3\right)/8 \tag{19}$$

$$H_8 = (\xi + \xi^2)(\eta + \eta^2)\left(-\xi + \xi^3\right)/8;\quad H_9 = (\xi + \xi^2)(\eta + \eta^2)\left(-\eta + \eta^3\right)/8$$

$$H_{10} = -(\xi - \xi^2)(\eta + \eta^2)\left(-2 - 4\xi + 4\eta + \xi^2 + \eta^2 + 3\xi^3 - 3\eta^3\right)/8 \tag{20}$$

$$H_{11} = -(\xi - \xi^2)(\eta + \eta^2)\left(-\xi + \xi^3\right)/8;\quad H_{12} = -(\xi - \xi^2)(\eta + \eta^2)\left(-\eta + \eta^3\right)/8 \tag{21}$$

$$H_{13} = (1 - \xi^2)(\eta^2 - \eta)\left(-4\eta - 2\xi^2 + \eta^2 + 3\eta^3\right)/4 \tag{22}$$

$$H_{14} = (1 - \xi^2)(\eta^2 - \eta)\left(\xi - \xi^3\right)/2;\quad H_{15} = (1 - \xi^2)(\eta^2 - \eta)\left(-\eta + \eta^3\right)/4$$

$$H_{16} = (\xi + \xi^2)(1 - \eta^2)\left(4\xi + \xi^2 - 2\eta^2 - 3\xi^3\right)/4 \tag{23}$$

$$H_{17} = (\xi + \xi^2)(1 - \eta^2)\left(-\xi + \xi^3\right)/4;\quad H_{18} = (\xi + \xi^2)(1 - \eta^2)\left(\eta - \eta^3\right)/2 \tag{24}$$

$$H_{19} = (1 - \xi^2)(\eta + \eta^2)\left(4\eta - 2\xi^2 + \eta^2 - 3\eta^3\right)/4 \tag{25}$$

$$H_{20} = (1 - \xi^2)(\eta + \eta^2)\left(\xi - \xi^3\right)/2;\quad H_{21} = (1 - \xi^2)(\eta + \eta^2)\left(-\eta + \eta^3\right)/4$$

$$H_{22} = (\xi^2 - \xi)(1 - \eta^2)\left(-4\xi + \xi^2 - 2\eta^2 + 3\xi^3\right)/4 \tag{26}$$

$$H_{23} = (\xi^2 - \xi)(1 - \eta^2)\left(-\xi + \xi^3\right)/4;\quad H_{24} = (\xi^2 - \xi)(1 - \eta^2)\left(\eta - \eta^3\right)/2$$

$$H_{25} = (1 - \xi^2)(1 - \eta^2)\left(1 - \xi^2 - \eta^2\right) \tag{27}$$

$$H_{26} = (1 - \xi^2)(1 - \eta^2)\left(\xi - \xi^3\right);\quad H_{27} = (1 - \xi^2)(1 - \eta^2)\left(\eta - \eta^3\right)$$

and

$$N_1 = 0.25(1 - \xi)(1 - \eta)\xi\eta;\ N_2 = -0.5(1 - \xi^2)(1 - \eta)\eta;\ N_3 = -0.25(1 + \xi)(1 - \eta)\xi\eta$$

$$N_4 = 0.5(1 + \xi)(1 - \eta^2)\xi;\ N_5 = 0.25(1 + \xi)(1 + \eta)\xi\eta;\ N_6 = 0.5(1 - \xi^2)(1 + \eta)\eta;$$

$$N_7 = -0,25(1-\xi)(1+\eta)\xi\eta; \quad N_8 = -0,5(1-\xi)(1-\eta^2)\xi; \quad N_9 = (1-\xi^2)(1-\eta^2) \quad (28)$$

Finally, using the finite element analysis to study the static behavior of the plate, the global stiffness matrix $[\boldsymbol{K}]$ and the global force vector $\{\boldsymbol{F}\}$ are needed and the equilibrium equations of them can be determined through the element stiffness matrix, $[K_e]$, and the nodal load vector, $\{P\}$. Where $[K_e]$ and $\{P\}$ are presented in compact form as the following:

$$[K_e] = \int\limits_{S_e} \begin{bmatrix} B_1^T A B_1 + B_1^T B B_2 + B_1^T E B_3 + B_2^T B B_1 + B_2^T D B_2 \\ +B_2^T F B_3 + B_3^T E B_1 + B_3^T F B_2 + B_3^T H B_3 + B_1'^T A' B_1' \\ +B_1'^T D' B_2' + B_2'^T D' B_1' + B_2'^T F' B_2' \end{bmatrix} dS \quad (29)$$

$$\iint\limits_{S_e} p(x,y)w(x,y)dS = a^T \iint\limits_{S_e} [B_p]^{\,T} \{p(x,y)\} dS = \{a\}^T \{P\} \quad (30)$$

The integration in the above equations is performed numerically following Gauss quadrature technique.

## 4. NUMERICAL RESULTS

Numerical examples of composite plates having different features are solved by the proposed Hermitian nine-noded rectangular element and the results obtained are presented with the published results for necessary comparison.

**Example 1**. The problem of a three ply $(0^0/90^0/0^0)$ square laminate; the material properties of each ply is assumed as: $E_1 = 175$ GPa; $E_2 = E_3 = 7$ GPa; $G_{12} = G_{13} = 3.5$ GPa; $G_{23} = 1.4$ GPa; $\nu_{12} = \nu_{13} = 0.25$; $\nu_{23} = 0.01$; simply supported at all the edges and subjected to uniformly distributed load, is studied for different thickness ratios $(h/a)$ ranging from 0.5 to 0.01. The following nondimensionalized quantities at specific points

*Table 1.* Deflection $(\overline{w})$ at the centre of a simply supported square laminate $(0^0/90^0/0^0)$ under uniform distributed load of intensity $q$

| **References** | **Theory** | **Thickness ratio (h/a)** | | | | | |
|---|---|---|---|---|---|---|---|
| | | **0.01** | **0.02** | **0.05** | **0.1** | **0.25** | **0.5** |
| Present (4x4 mesh) | HSDT | 0.7101 | 0.7259 | 0.8236 | 1.1076 | 2.4774 | 5.1557 |
| Present (6x6 mesh) | HSDT | 0.7036 | 0.7210 | 0.8217 | 1.106 | 2.4795 | 5.155 |
| Present (8x8 mesh) | HSDT | 0.7010 | 0.7190 | 0.8185 | 1.1011 | 2.4766 | 5.1525 |
| Present (10x10 mesh) | HSDT | 0.6998 | 0.7179 | 0.8152 | 1.0964 | 2.4725 | 5.1488 |
| Present (12x12 mesh) | HSDT | **0.6991** | **0.7170** | **0.8124** | **1.0926** | **2.4690** | **5.1457** |
| Sheikh et al. [4]. Mesh: (16x16) | HSDT | 0.6708 | 0.6841 | 0.7763 | 1.0910 | 2.9093 | 7.7670 |
| | FSDT | 0.6713 | 0.6823 | 0.7568 | 1.0235 | 2.6608 | 7.7068 |
| Reddy [7] | HSDT | 0.6705 | 0.6838 | 0.7760 | 1.0900 | 2.9091 | 7.7671 |
| | FSDT | 0.6697 | 0.6807 | 0.7573 | 1.0219 | 2.6596 | 7.7062 |
| Ghosh and Dey [12] | HSDT | 0.6823 | - | 0.7572 | 0.9650 | - | - |

are presented in Tables.

$$\overline{w} = 100w_0\left(\frac{a}{2},\frac{b}{2}\right)\left(\frac{E_2h^3}{qa^4}\right);\ \overline{\sigma}_{xx} = \sigma_{xx}\left(\frac{a}{2},\frac{b}{2},\frac{h}{2}\right)\left(\frac{h^2}{|q|\,b^2}\right);\ \overline{\sigma}_{yy} = \sigma_{yy}\left(\frac{a}{2},\frac{b}{2},\frac{h}{6}\right)\left(\frac{h^2}{|q|\,b^2}\right);$$

$$\overline{\sigma}_{xy} = \sigma_{xy}\left(0,0,\frac{h}{2}\right)\left(\frac{h^2}{qb^2}\right);\quad \overline{\sigma}_{yz} = \sigma_{yz}\left(\frac{a}{2},0,0\right)\left(\frac{h}{qb}\right);\quad \overline{\sigma}_{xz} = \sigma_{xz}\left(0,\frac{b}{2},0\right)\left(\frac{h}{qb}\right)$$

The plate is analysed with different mesh divisions and the deflection obtained at the plate centre is presented with the analytical solution of Reddy [7], finite element solutions of Sheikh & Chakrabarti [4], Ghosh and Dey [12] in Table 1. The results clearly show that the deflection values obtained using proposed $C^1$ element are in close agreement with the other results for $a/h$ ratios equal to 100, 50, 20 and 10. For very thick plates ($a/h$ ratio equal to 4 and 2), the present element underpredicts deflection by 15%-33% compared to the results of Reddy [7] and Sheikh [4].

**Example 2**. A simply supported four layered square antisymmetric angle-ply ($\theta$/-$\theta$/$\theta$/-$\theta$) composite plates under sinusoidal transverse load is considered for different thickness ratios ($h/a$) ranging from 0.25 to 0.01. The material properties of each ply is assumed as: $E_1 = 276$ GPa; $E_2 = E_3 = 6.895$ GPa; $G_{12} = G_{13} = 3.45$ GPa; $G_{23} = 4.12$ GPa; $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. The following nondimensionalized quantities at specific points are presented in Tables 2.

*Table 2.* ($\overline{w}$) at the important points of a simply supported square angle-ply ($\theta$/-$\theta$/$\theta$/-$\theta$) under sinusoidal load of amplitude $q$

| $\theta$ | **Theory** | **Thickness ratio ($h/a$)** | | |
|---|---|---|---|---|
| | | **0.01** | **0.1** | **0.25** |
| **15°** | Present, 4 node rectangular element (16×16 mesh) | 0.2537 | 0.4237 | 1.0818 |
| | Present, 4 node rectangular element (24×24 mesh) | 0.2534 | 0.4253 | 1.0880 |
| | **Present, 9 node rectangular element (8×8 mesh)** | **0.2651** | **0.4509** | **1.1219** |
| | Ren [17] | 0.2668 | 0.4505 | 1.3050 |
| | Swaminathan [16] (Model-1, full third order displacement theory-TSDT) | 0.2662 | 0.4423 | 1.2608 |
| | Swaminathan [16] (Model-3, High order displacement theory-HSDT) | 0.2666 | 0.4329 | 1.1903 |
| **30°** | Present, 4 node rectangular element (16×16 mesh) | 0.1944 | 0.3506 | 0.9171 |
| | Present, 4 node rectangular element (24×24 mesh) | 0.1951 | 0.3694 | 0.9284 |
| | **Present, 9 node rectangular element (12×12 mesh)** | **0.2001** | **0.3694** | **0.9554** |
| | Ren [17] | 0.2049 | 0.3543 | 1.0943 |
| | Swaminathan [16] (Model-1, full third order displacement theory-TSDT) | 0.2046 | 0.3439 | 1.0399 |
| | Swaminathan [16] (Model-3, High order displacement theory-HSDT) | 0.2046 | 0.3291 | 0.9494 |
| **45°** | Present, 4 node rectangular element (16×16 mesh) | 0.1727 | 0.3250 | 0.8454 |
| | **Present, 9 node rectangular element (8×8 mesh)** | **0.1769** | **0.3411** | **0.8826** |
| | Ren [17] | 0.1821 | 0.3201 | 1.0160 |
| | Swaminathan [16] (Model-1, full third order displacement theory-TSDT) | 0.1818 | 0.3101 | 0.9626 |
| | Swaminathan [16] (Model-3, High order displacement theory-HSDT) | 0.1818 | 0.2956 | 0.8747 |

There is a good agreement between the results obtained by using the proposed element and the analytical results of Swaminathan et al., [16] and the results reported by Ren, [17]. To assess the improvement of new element over 4 node rectangular element [1], [2] the size of mesh and the rate of convergence are also presented in these tables. We see that for all the fibre orientation ($\theta = 15^\circ$, $30^\circ$ and $45^\circ$) and $a/h$ ratio equal to 100 and 10, the values of transverse deflection predicted by the present model for 4 layered antisymmetric angle-ply ($\theta/\text{-}\theta/\theta/\text{-}\theta$) plates under sinusoidal transverse load are in good agreement with the analytical results of Swaminathan [16] and $C^\circ$ isoparametric finite element result of Ren [17]. In the case of thick plates with $a/h$ ratio equal 4, the values of deflection predicted by our model are 10%-15% lower for any given value of fibre orientation as compared to the results of Swaminathan [16] and Ren [17].

**Example 3.** A symmetric $(0^0/90^0/90^0/0^0)$ square plate with equal thickness layers has been subjected to a sinusoidal transverse load on top plane and the results are presented in Table 3. The material properties and boundary conditions are as Example 1.

*Table 3.* Deflection ($\overline{w}$) and Stresses ($\overline{\sigma}_{xx}, \overline{\sigma}_{yy}, \overline{\sigma}_{xz}, \overline{\sigma}_{yz}$ and $\overline{\sigma}_{xy}$) at the important points of a simply supported square $(0^0/90^0/90^0/0^0)$ plate under sinusoidal load of amplitude $q$

| h/a | References | Theory | $\overline{w}$ | $\overline{\sigma}_{xx}$ | $\overline{\sigma}_{yy}$ | $\overline{\sigma}_{xz}$ | $\overline{\sigma}_{yz}$ | $\overline{\sigma}_{xy}$ |
|---|---|---|---|---|---|---|---|---|
| **0.25** | **Present (9-noded element)** | **HSDT** | **1.9248** | **0.7264** | **0.4264** | **0.1971** | **0.2530** | **0.0453** |
| | High order displacement theory (4-noded element) [1] | HSDT | 1.8966 | 0.6797 | 0.4138 | 0.2127 | 0.2429 | 0.0409 |
| | Full third displacement theory (4-noded element) [2] | FTDT | 1.8787 | 0.7050 | 0.4034 | 0.2066 | 0.2377 | 0.0452 |
| | Pagano [13] | ESL | 1.9500 | 0.7200 | 0.6630 | 0.2190 | 0.2920 | 0.0467 |
| | Aagaah et al. [5] | TSDT | 1.9000 | 0.6810 | 0.6470 | 0.2190 | 0.2440 | 0.0451 |
| **0.1** | **Present (9-noded element)** | **HSDT** | **0.7506** | **0.5681** | **0.2870** | **0.2652** | **0.1868** | **0.0260** |
| | High order displacement theory (4-noded element) [1] | HSDT | 0.7145 | 0.5562 | 0.2618 | 0.2722 | 0.1585 | 0.0233 |
| | Full third displacement theory (4-noded element) [2] | FTDT | 0.7143 | 0.5518 | 0.2582 | 0.2626 | 0.1458 | 0.0265 |
| | Pagano [13] | ESL | 0.7430 | 0.5590 | 0.4010 | 0.3010 | 0.1960 | 0.0275 |
| | Aagaah et al. [5] | TSDT | 0.7320 | 0.5510 | 0.3940 | 0.2110 | 0.1630 | 0.0451 |
| **0.01** | **Present (9-noded element)** | **HSDT** | **0.4528** | **0.5392** | **0.2066** | **0.2636** | **0.1486** | **0.0210** |
| | High order displacement theory (4-nodedf element) [1] | HSDT | 0.4295 | 0.5439 | 0.1822 | 0.2927 | 0.1132 | 0.0209 |
| | Full third displacement theory (4-noded element) [2] | FTDT | 0.4339 | 0.5401 | 0.1804 | 0.2410 | 0.1425 | 0.0213 |
| | Pagano [13] | ESL | 0.4370 | 0.5390 | 0.2760 | 0.3370 | 0.1410 | 0.0216 |
| | Aagaah et al. [5] | TSDT | 0.4350 | 0.5390 | 0.2750 | 0.3080 | 0.1290 | 0.0216 |

There is also a good agreement between the results obtained by using the proposed element and the three-dimensional elasticity solution of Pagano [13] and the results of finite element solution of Aagaah et al. [5] based on the TSDT.

## 5. CONCLUSIONS

A new $C^1$ rectangular element is proposed and the finite element formulation based on Reddy's higher-order shear deformation plate theory is presented. This element has nine nodes, each node contains 7 degrees of freedom: $u^\circ$; $v^\circ$; $w^\circ$; $\theta_x$, $\theta_y$, $\gamma_x$ and $\gamma_y$. Thus the element is quite elegant from computational point of view. The formulation is based on displacement approach where $u^\circ$; $v^\circ$; $w$; $\gamma_x$ and $\gamma_y$ are taken as the independent displacement components. The element is tested numerically in a wide range of problems covering different loading, material property, stacking sequence and so on. It shows the performance of the element in terms of accuracy, rate of convergence, applicability and so on. The element is free from shear locking problem and it does not possess any spurious modes.

Based on these observations the element can be recommended for the analysis of any composite plate structures having moderate thickness to predict the deflection and stress with sufficient accuracy.

## ACKNOWLEDGEMENTS

*This work is sponsored by Ministry of Science and Technology and Project QGTD 08.07.*

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*Received May 25, 2009*

## PHẦN TỬ C$^1$ MỚI TRONG BÀI TOÁN UỐN TẤM COMPOSITE LỚP DỰA TRÊN LÝ THUYẾT BIẾN DẠNG CẮT BẬC CAO

Nghiên cứu tập trung vào xây dựng phần tử C1 dạng tứ giác chín nút (7 bậc tự do/nút), không tương thích dựa trên lý thuyết tấm biến dạng cắt bậc cao của Reddy (HSDT). Khó khăn nằm ở chỗ phần tử được xây dựng phải đảm bảo được tính liên tục của đạo hàm của độ võng tại các nút chung và điều kiện biên về ứng suất cắt ngang tại mặt trên và dưới của tấm. Vì vậy, mục đích được đặt ra trong bài báo này là xây dựng một kiểu phần tử Hermite chịu uốn. Sau đó, xây dựng thuật toán và chương trình máy tính để phân tích tĩnh các tấm composite lớp (dày và mỏng) chịu tải trọng uốn. Kết quả số của chương trình đã được so sánh với một số kết quả đã công bố khác để kiểm tra độ chính xác và phạm vi áp dụng của phần tử đề xuất.