

SLIP-SHAKEDOWN ANALYSIS AND THE ASSUMPTION OF SMALL COUPLING IN FRICTIONAL CONTACT

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Abstract. In the frictional contact of solids under cyclic loads, the shakedown behaviour of the relative displacement is of interest in the same spirit as the plastic deformation in plasticity. Cumulative slips may lead to the failure due to large relative displacements of the components of an assembly while cyclic slips are often undesired because of wear and fretting fatigue problems. Under Coulomb friction, it is well known that Melan and Koiter theorems are generally not available, except in certain particular cases. In this discussion, the particular case of *small coupling* between the contact pressures and the slip-displacements is considered. This assumption means that the tangent displacements have small or no influence on the contact pressures which can be then computed from the elastic response as in the *uncoupling* case. The pressure is thus a given time-dependent function and the Coulomb criterion is reduced to a Mises-like standard law of friction. It is shown here that Melan and Koiter theorems can be applied again as in standard plasticity. The dependence of the yield limit on the loading amplitude is however not classical and the extension of the static and kinematic approaches is discussed to obtain the critical shakedown load or the limit load. The validity of the assumption of small coupling is also explored by numerical simulation in an practice example.

1. INTRODUCTION

The mechanical behaviour of a system of solids in frictional contact under cyclic loads and pre-stresses is considered here. In particular, the response may lead to undesired effects such as the failure by unbounded slips of the solids in contact or to the fatigue damage by alternating slips. The slips may also shakedown in the sense that the slip vector at any point of the contact surface tends to a finite limit at large time. The conditions ensuring the slip-shakedown of the system are of interest in many engineering structures.

Some interesting discussions on the subject can be found in the literature, cf. [2], [1], [5]. The problem of slip-shakedown has been discussed in several examples of engine components under periodic thermo-mechanical loadings, cf. [2], and the analogy of this phenomenon with the elastic-plastic response of structures has been pointed out, especially

concerning the possibility of shakedown and the validity of Melan and Koiter theorems when the friction criterion does not depend on the normal pressure (Mises-like friction). For Coulomb friction, the validity of Melan theorem has been recently discussed, cf. [3] [5], under the assumptions of *full contact* and of *uncoupling* of the normal pressures versus the tangent slips. Since such a condition is rather restrictive for most common structures, it is interesting to analyze the magnitude of the coupling. In particular, if the coupling effect is small enough, these results can be applied as an approximation. Since the dependence of the yield force on the loading amplitude is not classical, the objective of this paper is to discuss in this case the validity of Melan and Koiter theorems and to show how the static and kinematic approaches can be extended in this framework to obtain the critical shakedown load or the limit load. The validity of the assumption of small coupling is also explored by numerical simulation in the example of a car-engine conrod submitted to a rotating load.

2. SOLIDS IN FRICTIONAL CONTACT UNDER CYCLIC LOAD

2.1. Governing equations

A system of two elastic solids is considered to describe the relative displacement problem. A solid V_2 is assumed to be elastic (or rigid) and maintained by prescribed displacements $\mathbf{u}_d = \mathbf{0}$ on a portion S_u of its boundary. A second elastic solid of volume V_1 is maintained only by an unilateral and frictional contact on a common portion S_c of the boundaries. The two solids are submitted to a mechanical loadings composed of the fields of volume force $\mathbf{f}(t)$ and surface force $\mathbf{F}(t)$ on the complementary portions of the boundaries ($S_1^f = S_1 - S_c$ and $S_2^f = S_2 - S_u - S_c$) for $0 \leq t \leq +\infty$. The existence of a pre-stress field σ_o in V_i is also admitted.

The normal and tangent relative displacements of V_2 with respect to V_1 on S_c is denoted as W and U and the friction force and the contact pressure acting by V_1 on V_2 are denoted as q and p respectively. On the contact surface S_c , the unilateral contact conditions and Coulomb law of friction must be satisfied. This means that

$$W \geq 0, p \geq 0, p W = 0 \quad (1)$$

$$\phi = \|q\| - k \leq 0, k = \varphi p, \dot{U} = \lambda Q, \lambda \geq 0, \phi \lambda = 0. \quad (2)$$

In a quasi-static transformation starting from a given initial value \mathbf{U}_o , the response of the system (in displacements $u(t)$, in stresses $\sigma(t)$) is governed by the system of equations:

$$\left\{ \begin{array}{l} \nabla \cdot \sigma + f = 0, \\ \sigma = \sigma_o + L : \nabla u \text{ in } V = V_1 \cup V_2, \\ \sigma \cdot n = F \text{ on } S^F, u = 0 \text{ on } S_1^u \\ W = [u] \cdot n, U = [u] - Wn, q = \sigma \cdot n - (n \cdot \sigma \cdot n)n, p = n \cdot \sigma \cdot n \\ + \text{Unilateral contact condition} \\ + \text{Coulomb law of friction} \\ + \text{Initial conditions in position.} \end{array} \right. \quad (3)$$

where $[u]$ is the relative displacement of the two solids and n denotes the outward normal vector to V_1 on S_c .

This cumbersome problem presents some similarity with incremental plasticity. Indeed, the slip-displacements can be understood as localized plastic strains on the contact surface S_c . The possibility of shakedown or alternating sliding or cumulative sliding under cyclic loadings can be observed and discussed as in plasticity. In particular, *under the assumption of no separation*, it has been pointed out that Melan and Koiter theorems hold as in classical plasticity [1] when the friction coefficient $\varphi = 0$ i.e. for a pressure-independent laws of friction. In particular, for an associated Mises-like law of friction of the form $\phi = \|q\| - k(x, t) \leq 0$, these classical results are available even if $k(x, t)$ is a given function of time.

2.2. The assumption of small coupling

Unfortunately, this possibility does not hold for a pressure-dependent law of friction such as Coulomb friction. However, it has been shown recently, cf. [5], [3], that Melan's theorem still holds if there is no coupling of the contact pressure with respect to the tangent displacement. This condition can be written from the classical decomposition of the response of the system into the elastic response and the coupling response:

- The elastic response is the response of the system to the loading when there is stick contact:

$$\begin{cases} \nabla \cdot \sigma_{el} + f(t) = 0, \\ \sigma_{el} = \sigma_o + L : \nabla u_{el}, \text{ in } V \\ \sigma_{el} \cdot n = F(t) \text{ on } S^F, u_{el} = 0 \text{ on } S_u \\ [u_{el}] = 0 \text{ on } S_c \text{ (stick contact)} \end{cases} \quad (4)$$

in which the data are $\mathbf{f}(t)$, $\mathbf{F}(t)$, σ_o

- The coupling response (contact pressure/tangent slip) is the response of the system, assumed to be free of load, under a prescribed slip:

$$\begin{cases} \nabla \cdot \sigma^* = 0, \\ \sigma^* = L : \nabla u^* \text{ in } V \\ \sigma^* \cdot n = 0 \text{ on } S^F, u^* = 0 \text{ on } S_u \\ [u^*] \cdot n = 0, [u^*] = U^* \text{ on } S_c \end{cases} \quad (5)$$

in which the data are \mathbf{U}^* and the response in displacement, stress and contact pressure are \mathbf{u}^* , σ^* , \mathbf{p}^* .

From these definitions, it is clear that the solution of (3) satisfies

$$u = u_{el} + u^*(U), \quad \sigma = \sigma_{el} + \sigma^*(U), \quad p = p_{el} + p^*(\mathbf{U}) \quad (6)$$

The local ratio of the coupling (contact pressure / tangent slip) associated with the solution of (3) can be defined as

$$r(x, t) = \frac{\|p^*(U)(x, t)\|}{p(x, t)}. \quad (7)$$

By definition, *there is uncoupling if $r = 0$ and small coupling if $r \ll 1$ for all t and all $x \in S_c$.*

In particular, there is uncoupling for the considered system if $p^* = 0$ for all U^* . Such assumption is satisfied for example in the problem of contact of a rigid punch on an elastic half-plane. The validity of Melan theorem and the analysis of shakedown have been discussed in [5], [3] for uncoupled systems. Since the assumption of uncoupling is generally not satisfied, the case of small coupling can also be of interest since this assumption enables us to consider the uncoupling case as an approximation and obtain a good estimate of the critical shakedown loads.

3. SLIP-SHAKEDOWN ANALYSIS

From (6) and from the assumption of small coupling (7), the Coulomb criterion of friction is modified and written as in the case of uncoupling

$$\|q(x, t)\| - \varphi p_{el}(x, t) \leq 0. \quad (8)$$

The modified criterion (8) is a Mises-like criterion and the Melan-Koiter framework can be thus applied for the approximate response. There is however a small difference with the classical description since in (8), *the yield limit depends on the amplitude of the loading*.

Let SF be the space of tangent self-forces defined on S_c . By definition, a field of tangent forces \mathbf{q} belongs to SF if there exists a self-stress field σ_s defined in V such that

$$\begin{cases} \nabla \cdot \sigma_s = 0 \text{ in } V \\ \sigma_s \cdot n = 0 \text{ on } S^F \\ [\sigma_s] \cdot n = 0 \text{ on } S_c \\ q = \sigma_s \cdot n - (n \cdot \sigma_s \cdot n)n \text{ on } S_c. \end{cases} \quad (9)$$

A relative rigid displacement by tangent slips of the solids is eventually possible when the contact surface is a spherical or a cylindrical or a plane surface. Let RS be the space of rigid slips i.e. of the slips due to relative rigid displacements of the two solids. By definition, $G \in RS$ if there exists at least a displacement field \mathbf{u} such that

$$\begin{cases} \epsilon(u) = 0 \ \forall x \in V, \ u = 0 \text{ on } S_u^1 \\ [u] \cdot n = 0, \ [u] = G \text{ on } S_c. \end{cases} \quad (10)$$

In particular, the following orthogonality relation holds:

$$\int_{S_c} q \cdot G da = 0, \ \forall q \in SF, \ \forall G \in SR. \quad (11)$$

3.1. Melan theorem

Let $Q_{EL}(x)$ be the set of values of $q_{el}(x, t)$ for all $t \geq 0$. Then a periodic loading path such that its associated elastic response $q_{el}(x, t)$ is of arbitrary period T , taking all the values in $Q_{EL}(x)$ for $0 \leq t \leq T$ can be introduced to obtain the following statement which is a sufficient condition of slip-shakedown for all standard pressure-independent laws of friction, in the spirit of Melan theorem:

If there exists a time-independent tangent force $\mathbf{q}^* \in SF$ and a coefficient $m > 1$ such that $\tilde{q}(x, t) = m(q^*(x) + q_{el}(x, t))$ satisfies for all $0 \leq t \leq T$ and for all $x \in S_c$ the modified criterion (8), then there is necessarily a shakedown of the slips whatever the initial conditions.

3.2. Safety coefficient and critical shakedown load

As in the theory of plasticity, cf. [4] [6], a safety coefficient m_s associated with a given loading (represented by the elastic response) for the slip-shakedown can be introduced:

$$\left\{ \begin{array}{l} m_s = \max_{q^* \in SF} m \\ \text{such that } \tilde{q}(x, t) = m(q^*(x) + q_{el}(x, t)) \text{ satisfies} \\ \text{the modified criterion } \forall 0 \leq t \leq T \text{ and } \forall x \in S_c. \end{array} \right. \quad (12)$$

It is clear that if $m_s > 1$, then, there is a slip-shakedown from Melan theorem. The definition (12) leads to the resolution of a problem of optimization under convex constraints. As in plastic shakedown theory [4] [7], [8] an associated lagrangean can be introduced and leads by min-max duality to the kinematic expression of the safety coefficient

$$\left\{ \begin{array}{l} m_s = \min_{\mathbf{g}} \int_0^T \int_{S_c} \varphi p_{el}(x, t) \|g(x, t)\| \, dx dt \\ \text{among the slip rates } g(x, t) \text{ such that} \\ \mathbf{G} = \int_0^T \mathbf{g} dt \in RS \\ \text{and that } \int_0^T \int_{S_c} q_{el}(x, t) \cdot g(x, t) \, dx dt = 1, \end{array} \right. \quad (13)$$

or equivalently

$$\left\{ \begin{array}{l} m_s = \min_{\mathbf{g}} \frac{\int_0^T \int_{S_c} \varphi p_{el} \|g\| \, dx dt}{\int_0^T \int_{S_c} q_{el} \cdot g \, dx dt} \\ \text{among the slip rates } \mathbf{g} \text{ such that} \\ \mathbf{G} = \int_0^T \mathbf{g} dt \in RS. \end{array} \right. \quad (14)$$

As in plasticity, the kinematic expression shows clearly two mechanisms of non-shakedown. The long-term slips may be alternative without final displacement $\mathbf{G} = 0$ or become cumulative and lead to a final rigid displacement $\mathbf{G} \neq 0$.

From these general results, the approximation (8) leads then to an estimate of the critical load of shakedown and of the limit load for a loading defined by a constant field of prestress σ^o and an external loading

$$(f(x, t), F(x, t)) = \mu(f^\ell(x, t), F^\ell(x, t)) \quad (15)$$

proportional to a load factor μ . In this case, the elastic leads to the following expressions

$$q_{el}(x, t) = q^o(x) + \mu q_{el}^\ell(x, t), \quad p_{el}(x, t) = p^o(x) + \mu p_{el}^\ell(x, t), \quad (16)$$

where p^o , q^o are due to the constant prestress and p_{el}^ℓ , q_{el}^ℓ are due to the external loading. It is assumed that

$$p^o(x) > 0, \quad p^o(x) + \mu p_{el}^\ell(x, t) > 0 \quad \forall x \in S_c, \quad \forall 0 \leq t \leq T. \quad (17)$$

Let $m_s(\mu)$ be the safety coefficient associated with the load factor μ . From (14), (16) and from (11), since $\mathbf{q}^o \in SF$ the following expression holds

$$\left\{ \begin{array}{l} m_s(\mu) = \min_{\mathbf{g}} \frac{\int_0^T \int_{S_c} \varphi(p_o(x) + \mu p_{el}^\ell(x, t)) \|g(x, t)\| \, dx dt}{\int_0^T \int_{S_c} \mu q_{el}^\ell(x, t) \cdot g(x, t) \, dx dt} \\ \text{among the slip rates } \mathbf{g} \text{ such that} \\ \mathbf{G} = \int_0^T \mathbf{g} dt \in RS. \end{array} \right. \quad (18)$$

The critical load factor for shakedown μ_{sh} is defined by condition

$$m_s(\mu_{sh}) = 1 \quad (19)$$

or equivalently

$$\left\{ \begin{array}{l} \mu_{sh} = \min_{\mathbf{g}} \int_0^T \int_{S_c} \varphi(p_o(x) + \mu_{sh} p_{el}^\ell(x, t)) \|g(x, t)\| \, dxdt \\ \text{among the slip rates } \mathbf{g} \text{ such that} \\ \mathbf{G} = \int_0^T \mathbf{g} dt \in RS \\ \text{and that } \int_0^T \int_{S_c} q_{el}^\ell(x, t) \cdot g(x, t) \, dxdt = 1, \end{array} \right. \quad (20)$$

or equivalently, by the condition

$$\left\{ \begin{array}{l} \mu_{sh} = \min_{\mathbf{g}} \frac{\int_0^T \int_{S_c} \varphi(p_o + \mu_{sh} p_{el}^\ell) \|g\| \, dxdt}{\int_0^T \int_{S_c} q_{el}^\ell \cdot g \, dxdt} \\ \text{among the slip rates } \mathbf{g} \text{ such that} \\ \mathbf{G} = \int_0^T \mathbf{g} dt \in RS. \end{array} \right. \quad (21)$$

Because of the presence of μ_{sh} in the right-hand side of (20) or of (21), the computation of μ_{sh} is slightly different from the classical analysis. Relation (19) defines a coefficient unique which is the maximum load factor ensuring the shakedown of the modified response. Indeed, for any load factor μ , from the fact that

$$\varphi(p_o + \mu p_{el}^\ell) \|g\| = \frac{\mu}{\mu_{sh}} (p_o + \mu_{sh} p_{el}^\ell) \|g\| + \frac{\mu_{sh} - \mu}{\mu_{sh}} p_o \|g\|$$

it follows that $m_s(\mu) > 1$ if $\mu < \mu_{sh}$ and $m_s(\mu) < 1$ if $\mu > \mu_{sh}$. From Melan theorem, there is thus shakedown for the modified response at load $\mu < \mu_{sh}$.

3.3. Kinematic approach and upper-bounds of the critical shakedown load

For any slip rate \mathbf{g} such that $\mathbf{G} = \int_0^T \mathbf{g} dt \in RS$, an associated factor μ_g can be computed by the relation

$$\mu_g = \frac{\int_0^T \int_{S_c} \varphi(p_o + \mu_g p_{el}^\ell) \|g\| \, dxdt}{\int_0^T \int_{S_c} q_{el}^\ell \cdot g \, dxdt}.$$

Since

$$m_s(\mu_g) \leq \frac{\int_0^T \int_{S_c} \varphi(p_o + \mu_g p_{el}^\ell) \|g\| \, dxdt}{\int_0^T \int_{S_c} \mu_g q_{el}^\ell \cdot g \, dxdt} = 1$$

it follows that necessarily the following estimate holds as in classical analysis

$$\mu_g \geq \mu_{sh} \quad \forall g \text{ such that } \mathbf{G} = \int_0^T \mathbf{g} dt \in RS \quad (22)$$

and that

$$\mu_{sh} = \min_{\{\mathbf{g} \mid \mathbf{G}(\mathbf{g}) \in RS\}} \frac{\int_0^T \int_{S_c} \varphi p_o \|g\| \, dxdt}{\int_0^T \int_{S_c} (q_{el}^\ell \cdot g - p_{el}^\ell \|g\|) \, dxdt}. \quad (23)$$

4. LIMIT LOAD IN SLIP-LIMIT ANALYSIS

It is well known that the shakedown analysis also includes the limit analysis. The limit load m_l deals with the set of Dirac-rates $g(x, t) = G(x)\delta_\tau(t)$ and thus, the limit load μ_l must satisfy

$$\begin{cases} \mu_l = \min_{(\mathbf{G}, \mathbf{0} \leq \tau \leq \mathbf{T})} \int_{S_c} \varphi(p_o(x) + \mu_l p_{el}^\ell(x, \tau)) \|G(x)\| dx \\ \text{among rigid displacements } G(x) \text{ such that} \\ \int_{S_c} q_{el}^\ell(x, \tau) \cdot G(x) dx = 1, \end{cases} \quad (24)$$

or equivalently

$$\mu_l = \min_{(\mathbf{G} \in \mathbf{RS}, \mathbf{0} \leq \tau \leq \mathbf{T})} \frac{\int_{S_c} \varphi(p_o(x) + \mu_l p_{el}^\ell(x, \tau)) \|G(x)\| dx}{\int_{S_c} q_{el}^\ell(x, \tau) \cdot G(x) dx} \quad (25)$$

or

$$\mu_l = \min_{(\mathbf{G} \in \mathbf{RS}, \mathbf{0} \leq \tau \leq \mathbf{T})} \frac{\int_{S_c} \varphi p_o \|G(x)\| dx}{\int_{S_c} (q_{el}^\ell(x, \tau) \cdot G(x) - p_{el}^\ell(x, \tau) \|G(x)\|) dx}. \quad (26)$$

Since this minimization is performed in a smaller set of slip rates, it is clear that

$$\mu_{sh} \leq \mu_l. \quad (27)$$

5. THE CONROD EXAMPLE

It is not clear that the assumption of small coupling can be available for engineering structures. For this, the example of an engine-conrod is considered to explore its validity. The conrod system is composed of a circular bearing shell V_1 in contact by pre-stress and friction with the conrod-end V_2 as shown in Fig. 1 and submitted to a cyclic rotating force



Fig. 1. A conrod bearing system of car engine under cyclic loading

F . In engineering design, the relative displacement of the bearing shell with respect to the conrod-end should be avoided in order to prevent possible failures of the assembly. In particular, cumulative slips might result in a failure by rigid-body displacements of the bearing shell (by translation in the axial direction or by rotation).

A simple modeling of the conrod end by two annular disks in plane strain is performed to obtain by a f.e.m analysis the response of the system under the cyclic loading, the inner solid (bearing shell) is maintained by pre-stresses and friction on the outer one

(conrod-end). The system is in small transformation under the action of the rotating force F defined by the loading path $\beta(t)$ taking the values of the whole interval $(0, 2\pi)$. Global rigid displacements of the system are prevented by homogeneous displacement conditions on the section $\theta = -\pi/2$ of the outer solid, cf. Fig. 2. Under the action of a strong rotat-

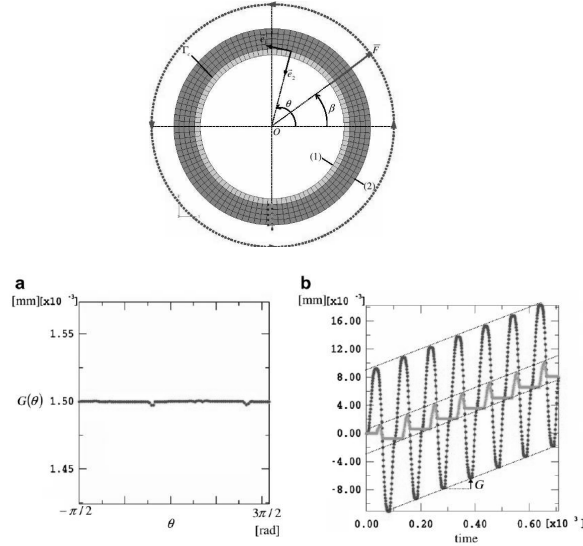


Fig. 2. 2D-simulation by f.e.m. and incremental response under a cyclic load. No shakedown at the considered load since a small rotation of the bearing shell results after each cycle

ing force, an example of non-shakedown is shown in Fig. 2a,b. Cumulative slips per cycle lead to a failure by rigid rotation of the bearing shell. The validity of the small-coupling assumption is shown in Fig.3 , the local ratio r is about 10%.

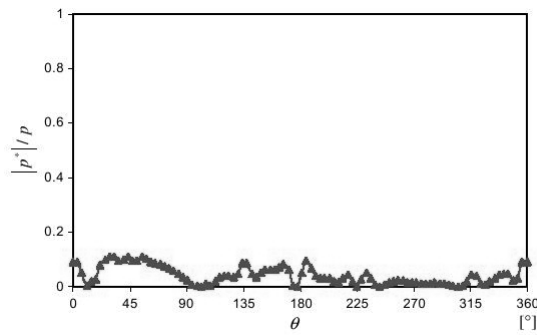


Fig. 3. Checking the small coupling assumption

6. CONCLUSION

In the considered problem, the dependence of the yield limit on the amplitude of the loading is not classical. Our discussion shows that well-known general theorems of shakedown analysis and of limit analysis remain available and that static and kinematic approaches can be applied within some minor modifications. New expressions of the critical shakedown load and of the limit load are derived. The considered problem can also be found in the shakedown analysis of elastic-plastic structures under cyclic thermal loadings when the yield limit depends on the temperature.

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PHÂN TÍCH THÍCH ỨNG-TRƯỢT VÀ GIẢ THIẾT LIÊN KẾT YẾU TRONG BÀI TOÁN TIẾP XÚC MA SÁT

Khi vật rắn tiếp xúc có ma sát dưới tác động của tải trọng chu kỳ, ứng xử thích ứng của chuyển dịch tương đối cũng được quan tâm như đối với biến dạng dẻo trong bài toán dẻo. Sự trượt tích lũy có thể dẫn đến phá hủy do dịch chuyển tương đối của các thành phần của cả hệ thống trở nên lớn trong khi đó sự trượt chu kỳ thường là không mong muốn do hiện tượng mòn và hư hỏng - là các vấn đề môi. Với ma sát Coulomb các định lý của Melan và Koiter nói chung là không áp dụng được trừ một số trường hợp đặc biệt. Trong bài báo này, xem xét trường hợp đặc biệt khi sự liên hợp giữa áp lực pháp tuyến và dịch chuyển trượt nhỏ. Giả thiết này có nghĩa dịch chuyển tiếp tuyến có ảnh hưởng nhỏ hoặc không ảnh hưởng đến áp lực pháp tuyến, mà ta có thể tính được từ ứng xử đàn hồi như trường hợp không liên hợp. áp lực này làm hàm của thời gian và tiêu chuẩn Coulomb dẫn đến luật về ma sát giống của Mises. Điều này chỉ ra rằng định lý của Melan và Koiter có thể áp dụng như trường hợp dẻo thông thường. Tuy nhiên, sự phụ thuộc của giới hạn chảy dẻo lên biên độ đặt tải không như thông thường và một mở rộng của cách tiếp cận tĩnh và động học được đưa ra để nhận được tải thích nghi tới hạn và tải tới hạn. Tính đúng đắn của giả thiết về sự liên hợp nhỏ được biểu diễn bằng các mô phỏng số trong các ví dụ thực tế.