Vietnam Journal of Mechanics, VAST, Vol. 30, No. 2 (2008), pp. 67 – 79

NON-LINEAR VIBRATION OF ECCENTRICALLY STIFFENED LAMINATED COMPOSITE SHELLS

Dao Huy Bich, Vu Do Long
Vietnam National University, Hanoi

Abstract. The present paper deals with a non-linear vibration of eccentrically stiffened laminated composite doubly curved shallow shells. The calculations of internal forces and displacements of the shell are based upon the thin shell theory considering the geometrical non- linearity and the Lekhnitsky's smeared stiffeners technique. From the deformation compatibility equation and the motion equation a system of partial differential equations for stress function and deflection of shell is obtained. The Bubnov-Galerkin's method and iterative procedure in conjunction with Newmark constant acceleration scheme are used for dynamical analysis of shells to give the frequency- amplitude relation of free non-linear vibration and non-linear transient responses. Numerical results show the influence of boundary conditions and Gauss curvature on the non-linear vibration of shells.

1. INTRODUCTION

Reinforced laminated structures like plates and shallow shells are widely used in air-industry and ship-industry. The stiffening member provides the benefit of added load-carrying static and dynamic capability. When structures subjected to external loads may be appear a large deflection then the geometrical non-linearity of shell must be considered; of course it meets with mathematics difficulty. To solve problem we are concerned with two aspects: to seek an approximated analytical solution which allows to investigate the motion characterictics and to seek solution by numerical methods. The research results for nonlinear vibration of composite plates have been represented in [4, 11] and for cylindrical shells in [2, 5, 7, 8]. Approximated analytical solutions for the vibration problem of doubly curved unstiffened composite shells were given in [1, 6, 9, 10]. In [3] the authors carried out the non-linear dynamic analysis of doubly curved stiffened composite shells by the displacement approach.

The aim of this paper is to search an approximated analytical solution for the dynamic problem of doubly curved eccentrically stiffened laminated shells with negative and positive Gauss curvature and different boundary conditions by using stress function and Bubnov-Galerkin methods to give non-linear vibration equations of shell. Numerical solutions are given by the iterative method and Newmark constant acceleration scheme. The frequency-amplitude relation in the non-linear free vibaration and the influence of boundary conditions and Gauss curvature on the solution of dynamic problem of shells are examined.

2. GOVERNING EQUATIONS

Consider a symmetrically laminated composite doubly curved shallow shells of thickness h and in- plane edges a and b. The shell is reinforced by eccentrically longitudinal and transversal composite stiffeners and subjected to the transverse load of intensity $q(x_1, x_2, t)$.

Using Kirchoff-Love theory non-linear strain-displacement relations for doubly curved shallow shells are formulated

$$\begin{aligned}
\varepsilon_1^o &= \frac{\partial u}{\partial x_1} - k_1 w + \frac{1}{2}\Big(\frac{\partial w}{\partial x_1}\Big)^2; \quad \phi_1 = -\frac{\partial^2 w}{\partial x_1^2};\\
\varepsilon_2^o &= \frac{\partial v}{\partial x_2} - k_2 w + \frac{1}{2}\Big(\frac{\partial w}{\partial x_2}\Big)^2; \quad \phi_2 = -\frac{\partial^2 w}{\partial x_2^2};\\
\varepsilon_6^o &= \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_1}\cdot\frac{\partial w}{\partial x_2}; \quad \phi_6 = -2\frac{\partial^2 w}{\partial x_1 \partial x_2},
\end{aligned} \tag{1}$$

where $k_1 = \dfrac{1}{R_1}$, $k_2 = \dfrac{1}{R_2}$ are principal curvatures of the shell; R_1, R_2 are radii of curvature; u, v, w are displacements of the midle surface point along x_1, x_2, $x_3 \equiv z$ directions recpectively; ε_i^o and $\phi_i\,(i = 1, 2, 6)$ are strain of the midle surface point and curvature variations satisfying the deformation compatibility equation:

$$\frac{\partial^2 \varepsilon_1^o}{\partial x_2^2} + \frac{\partial^2 \varepsilon_2^o}{\partial x_1^2} - \frac{\partial^2 \varepsilon_6^o}{\partial x_1 \partial x_2} = \Big(\frac{\partial^2 w}{\partial x_1 \partial x_2}\Big)^2 - \frac{\partial^2 w}{\partial x_1^2}\cdot\frac{\partial^2 w}{\partial x_2^2} - k_1\frac{\partial^2 w}{\partial x_2^2} - k_2\frac{\partial^2 w}{\partial x_1^2}. \tag{2}$$

Internal forces of an unstiffened composite shell are caculated by Reddy [10]. Note that in a symmetrically laminated shell the coupling stiffnesses B_{ij} are equal to zero and the extensional A_{16}, A_{26} and bending D_{16}, D_{26} stiffnesses are negligible compared to the other stiffnesses. Using Lekhnitsky's smeared stiffeners technique we get governing internal force resultants and moments as follows

$$\begin{aligned}
N_1 &= \Big(A_{11} + \frac{EA_1}{s_1}\Big)\varepsilon_1^o + A_{12}\varepsilon_2^o - \frac{EA_1 z_1}{s_1}\cdot\frac{\partial^2 w}{\partial x_1^2},\\
N_2 &= A_{12}\varepsilon_1^o + \Big(A_{22} + \frac{EA_2}{s_2}\Big)\varepsilon_2^o - \frac{EA_2 z_2}{s_2}\cdot\frac{\partial^2 w}{\partial x_2^2},\\
N_6 &= A_{66}\varepsilon_6^o,
\end{aligned} \tag{3}$$

$$\begin{aligned}
M_1 &= -\Big(D_{11} + \frac{EI_1}{s_1}\Big)\frac{\partial^2 w}{\partial x_1^2} - D_{12}\frac{\partial^2 w}{\partial x_2^2} + \frac{EA_1 z_1}{s_1}\varepsilon_1^o,\\
M_2 &= -D_{12}\frac{\partial^2 w}{\partial x_1^2} - \Big(D_{22} + \frac{EI_2}{s_2}\Big)\frac{\partial^2 w}{\partial x_2^2} + \frac{EA_2 z_2}{s_2}\varepsilon_2^o,\\
M_6 &= -2D_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2},
\end{aligned} \tag{4}$$

where

$$(A_{ij}\,,\, D_{ij}) = \sum_{k=1}^{n_1} \int_{h_{k-1}}^{h_k} \overline{Q}_{ij}^{(k)}\,(1, z^2)\,dz \quad , \quad (i, j = 1, 2, 6)$$

are stiffnesses of unreinforced composite shells, $\overline{Q}_{ij}^{(k)}$ are transfomed stiffnesses of k^{th}- layer, E - elastic modulus of stiffeners. Spacings of the longitudinal and transversal stiffeners are denoted by s_1, s_2. Values z_1, z_2 - eccentricities of the longitudinal and transversal stiffeners with respect to the middle surface of the shell, A_1, A_2 are cross section areas of the stiffeners and I_1, I_2 are inertia moments of stiffener cross sections.

The reverse relations are obtained from equations (3)

$$\begin{aligned}
\varepsilon_1^o &= A_{22}^* N_1 - A_{12}^* N_2 - A_{12}^* \frac{EA_2 z_2}{s_2} \cdot \frac{\partial^2 w}{\partial x_2^2} + A_{22}^* \frac{EA_1 z_1}{s_1} \cdot \frac{\partial^2 w}{\partial x_1^2}, \\
\varepsilon_2^o &= A_{11}^* N_2 - A_{12}^* N_1 - A_{12}^* \frac{EA_1 z_1}{s_1} \cdot \frac{\partial^2 w}{\partial x_1^2} + A_{11}^* \frac{EA_2 z_2}{s_2} \cdot \frac{\partial^2 w}{\partial x_2^2}, \\
\varepsilon_6^o &= A_{66}^* N_6,
\end{aligned} \tag{5}$$

where:

$$\begin{aligned}
A_{11}^* &= \frac{1}{\Delta}\Big(A_{11} + \frac{EA_1}{s_1}\Big), \quad A_{22}^* = \frac{1}{\Delta}\Big(A_{22} + \frac{EA_2}{s_2}\Big), \\
A_{12}^* &= \frac{A_{12}}{\Delta}, \quad A_{66}^* = \frac{1}{\Delta}, \quad \Delta = \Big(A_{11} + \frac{EA_1}{s_1}\Big)\Big(A_{22} + \frac{EA_2}{s_2}\Big) - A_{12}^2
\end{aligned} \tag{6}$$

Inserting (5) into (4) yields

$$\begin{aligned}
M_1 &= -\Big(D_{11}^* \frac{\partial^2 w}{\partial x_1^2} + D_{12}^* \frac{\partial^2 w}{\partial x_2^2}\Big) + A_{22}^* \frac{EA_1 z_1}{s_1} N_1 - A_{12}^* \frac{EA_1 z_1}{s_1} N_2, \\
M_2 &= -\Big(D_{12}^* \frac{\partial^2 w}{\partial x_1^2} + D_{22}^* \frac{\partial^2 w}{\partial x_2^2}\Big) + A_{11}^* \frac{EA_2 z_2}{s_2} N_2 - A_{12}^* \frac{EA_2 z_2}{s_2} N_1, \\
M_6 &= -2D_{66}^* \frac{\partial^2 w}{\partial x_1 \partial x_2},
\end{aligned} \tag{7}$$

where:

$$\begin{aligned}
D_{11}^* &= D_{11} + \frac{EI_1}{s_1} + A_{22}^* \Big(\frac{EA_1 z_1}{s_1}\Big)^2, \\
D_{22}^* &= D_{22} + \frac{EI_2}{s_2} + A_{11}^* \Big(\frac{EA_2 z_2}{s_2}\Big)^2, \\
D_{12}^* &= D_{12} + A_{12}^* \frac{E^2 A_1 A_2 z_1 z_2}{s_1 s_2}, \quad D_{66}^* = D_{66}, \\
D_{33}^* &= D_{12}^* + 2D_{66}^* = D_{12} + 2D_{66} + A_{12}^* \frac{E^2 A_1 A_2 z_1 z_2}{s_1 s_2}
\end{aligned} \tag{8}$$

Because the load is perpendicular to the middle surface, then following Volmir [12] the wave propagation in middle surface of the shell can be overpass, it reduces to overlook inertial forces along x_1 and x_2 directions. The motion equations of doubly curved shallow composite shell are of the form

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = 0, \tag{9}$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = 0, \tag{10}$$

$$\begin{aligned}&\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}+N_1\frac{\partial^2 w}{\partial x_1^2}+2N_6\frac{\partial^2 w}{\partial x_1\partial x_2}+N_2\frac{\partial^2 w}{\partial x_2^2}\\&+k_1N_1+k_2N_2+q=J_0\frac{\partial^2 w}{\partial t^2}+J_2\Big(\frac{\partial^4 w}{\partial x_1^2\partial t^2}+\frac{\partial^4 w}{\partial x_2^2\partial t^2}\Big).\end{aligned}\tag{11}$$

Introduce the stress function

$$N_1=h\sigma_{11}=\frac{\partial^2\varphi}{\partial x_2^2},\ N_2=h\sigma_{22}=\frac{\partial^2\varphi}{\partial x_1^2},\ N_{12}=h\sigma_{12}=-\frac{\partial^2\varphi}{\partial x_1\partial x_2}\tag{12}$$

to equations (9) and (10) which are satisfied identically.

Substituting (5) into the compatibility equation (2), and (7) into (11), the system of two motion equations for stress function φ and deflection w are obtained

$$\begin{aligned}&A_{11}^*\frac{\partial^4\varphi}{\partial x_1^4}+\big(A_{66}^*-2A_{12}^*\big)\frac{\partial^4\varphi}{\partial x_1^2\partial x_1^2}+A_{22}^*\frac{\partial^4\varphi}{\partial x_2^4}-\Big[A_{12}^*\frac{EA_1z_1}{s_1}\cdot\frac{\partial^4 w}{\partial x_1^4}\\&+A_{12}^*\frac{EA_2z_2}{s_2}\cdot\frac{\partial^4 w}{\partial x_2^4}-\Big(A_{11}^*\frac{EA_2z_2}{s_2}+A_{22}^*\frac{EA_1z_1}{s_1}\Big)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+\Big(\frac{\partial^2 w}{\partial x_1\partial x_2}\Big)^2-\\&-\frac{\partial^2 w}{\partial x_1^2}\cdot\frac{\partial^2 w}{\partial x_2^2}-k_1\frac{\partial^2 w}{\partial x_2^2}-k_2\frac{\partial^2 w}{\partial x_1^2}\Big]=0;\end{aligned}\tag{13}$$

$$\begin{aligned}&D_{11}^*\frac{\partial^4 w}{\partial x_1^4}+2D_{33}^*\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+D_{22}^*\frac{\partial^4 w}{\partial x_2^4}-\Big(A_{11}^*\frac{EA_2z_2}{s_2}+A_{22}^*\frac{EA_1z_1}{s_1}\Big)\frac{\partial^4\varphi}{\partial x_1^2\partial x_2^2}\\&+A_{12}^*\frac{EA_1z_1}{s_1}\frac{\partial^4\varphi}{\partial x_1^4}+A_{12}^*\frac{EA_2z_2}{s_2}\frac{\partial^4\varphi}{\partial x_2^4}-k_1\frac{\partial^2\varphi}{\partial x_2^2}-k_2\frac{\partial^2\varphi}{\partial x_1^2}+2\frac{\partial^2\varphi}{\partial x_1\partial x_2}\cdot\frac{\partial^2 w}{\partial x_1\partial x_2}\\&-\frac{\partial^2\varphi}{\partial x_2^2}\cdot\frac{\partial^2 w}{\partial x_1^2}-\frac{\partial^2\varphi}{\partial x_1^2}\cdot\frac{\partial^2 w}{\partial x_2^2}-q+J_0\frac{\partial^2 w}{\partial t^2}-J_2\Big(\frac{\partial^4 w}{\partial x_1^2\partial t^2}+\frac{\partial^4 w}{\partial x_2^2\partial t^2}\Big)=0.\end{aligned}\tag{14}$$

3. VIBRATION OF THE SHELL WITH SIMPLY SUPPORTED EDGES

Let consider a simply supported shell at all edges, then the boundary conditions are

$$\begin{aligned}&w=0,\ M_1=0,\ N_1=0,\ N_6=0,\ \text{in}\,x_1=0,\ a;\\&w=0,\ M_2=0,\ N_2=0,\ N_6=0,\ \text{in}\,x_2=0,\ b.\end{aligned}\tag{15}$$

Solutions of stress function and deflection are chosen as

$$\begin{aligned}\varphi&=g(t)\sin\frac{m\pi x_1}{a}\sin\frac{n\pi x_2}{b},\\w&=f(t)\sin\frac{m\pi x_1}{a}\sin\frac{n\pi x_2}{b}.\end{aligned}\tag{16}$$

Substitution of (16) into (1), (3), (7) shows that boundary conditions (15) are satisfied.

Using expressions (16) to equations (13), (14) and Bubnov-Galerkin method leads to

$$\begin{aligned}&\Big[A_{11}^*\big(\frac{m\pi}{a}\big)^4+(A_{66}^*-2A_{12}^*)\big(\frac{mn\pi^2}{ab}\big)^2+A_{22}^*\big(\frac{n\pi}{b}\big)^4\Big]\frac{ab}{4}g(t)=\Big[B_{11}^*\big(\frac{m\pi}{a}\big)^4\\&-B_{12}^*\big(\frac{mn\pi^2}{ab}\big)^2+B_{22}^*\big(\frac{n\pi}{b}\big)^4+k_1\big(\frac{n\pi}{b}\big)^2+k_2\big(\frac{m\pi}{a}\big)^2\Big]\frac{ab}{4}f(t)-\frac{4mn\pi^2}{3ab}f^2(t)\end{aligned}\tag{17}$$

and

$$\begin{aligned}
&\Big[D^*_{11}\big(\frac{m\pi}{a}\big)^4 + 2D^*_{33}\big(\frac{mn\pi^2}{ab}\big)^2 + D^*_{22}\big(\frac{n\pi}{b}\big)^4\Big]\frac{ab}{4}f(t)\\
&+\Big[B^*_{11}\big(\frac{m\pi}{a}\big)^4 - B^*_{12}\big(\frac{mn\pi^2}{ab}\big)^2 + B^*_{22}\big(\frac{n\pi}{b}\big)^4 + k_1\big(\frac{n\pi}{b}\big)^2 + k_2\big(\frac{m\pi}{a}\big)^2\Big]\frac{ab}{4}g(t)\\
&-\frac{8mn\pi^2}{3ab}f(t)g(t) - q\frac{4ab}{mn\pi^2} + \Big[J_0 + J_2\pi^2\big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\big)\Big]\frac{ab}{4}\frac{d^2f}{dt^2} = 0,
\end{aligned} \tag{18}$$

where denote the coefficients

$$B^*_{11} = A^*_{12}\frac{EA_1z_1}{s_1},\ B^*_{22} = A^*_{12}\frac{EA_2z_2}{s_2},\ B^*_{12} = A^*_{11}\frac{EA_2z_2}{s_2} + A^*_{22}\frac{EA_1z_1}{s_1}$$

with m , n are odd numbers.

Remark: When a shell is reinforced by centrically stiffeners then $B^*_{ij} = 0$; a plate is reinforced by eccentrically stiffeners then $k_1 = k_2 = 0$, $B^*_{ij} \neq 0$; a plate is reinforced plate by centrically stiffeners or unreinforced plate then $k_1 = k_2 = 0$, $B^*_{ij} = 0$.

The equations (17) and (18) can be rewritten as following

$$g(t) = Af(t) - Bf^2(t), \tag{19}$$

$$H_1f(t) + H_2g(t) - H_3f(t)g(t) - H_4(t) + H_5\ddot{f} = 0 \tag{20}$$

with the coefficient

$$\begin{aligned}
A &= \frac{B^*_{11}\big(\frac{m\pi}{a}\big)^4 - B^*_{12}\big(\frac{mn\pi^2}{ab}\big)^2 + B^*_{22}\big(\frac{n\pi}{b}\big)^4 + k_1\big(\frac{n\pi}{b}\big)^2 + k_2\big(\frac{m\pi}{a}\big)^2}{A^*_{11}\big(\frac{m\pi}{a}\big)^4 + (A^*_{66} - 2A^*_{12})\big(\frac{mn\pi^2}{ab}\big)^2 + A^*_{22}\big(\frac{n\pi}{b}\big)^4},\\
B &= \frac{16mn\pi^2}{3a^2b^2\Big[A^*_{11}\big(\frac{m\pi}{a}\big)^4 + (A^*_{66} - 2A^*_{12})\big(\frac{mn\pi^2}{ab}\big)^2 + A^*_{22}\big(\frac{n\pi}{b}\big)^4\Big]},\\
H_1 &= \Big[D^*_{11}\big(\frac{m\pi}{a}\big)^4 + 2D^*_{33}\big(\frac{mn\pi^2}{ab}\big)^2 + D^*_{22}\big(\frac{n\pi}{b}\big)^4\Big]\frac{ab}{4},\\
H_2 &= \Big[B^*_{11}\big(\frac{m\pi}{a}\big)^4 - B^*_{12}\big(\frac{mn\pi^2}{ab}\big)^2 + B^*_{22}\big(\frac{n\pi}{b}\big)^4 + k_1\big(\frac{n\pi}{b}\big)^2 + k_2\big(\frac{m\pi}{a}\big)^2\Big]\frac{ab}{4},\\
H_3 &= \frac{8mn\pi^2}{3ab},\\
H_4(t) &= q(t)\frac{4ab}{mn\pi^2},\\
H_5 &= \Big[J_0 + J_2\pi^2\big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\big)\Big]\frac{ab}{4}.
\end{aligned}$$

Inserting (19) into (20) yields

$$H_5\ddot{f}(t) + (H_1 + H_2.A)f(t) - (H_2B + H_3A)f^2(t) + H_3Bf^3(t) = H_4(t).$$

The vibration equation of a shell is of the form

$$\ddot{f}(t) + m_1f(t) - m_2f^2(t) + m_3f^3(t) = \overline{q}_{mn}(t), \tag{21}$$

where denote

$$m_1 = \frac{H_1 + H_2 A}{H_5}, \; m_2 = \frac{H_2 B + H_3 A}{H_5}, \; m_3 = \frac{H_3 B}{H_5}, \; \overline{q}_{mn}(t) = \frac{H_4(t)}{H_5}. \tag{22}$$

For linear free vibration the equation (22) gets form

$$\ddot{f}(t) + m_1 f(t) = 0, \tag{23}$$

one can determine the fundamental frequency of vibration

$$\omega_0 = \sqrt{m_1} = \sqrt{\frac{H_1 + H_2 A}{H_5}}. \tag{24}$$

4. VIBRATION OF THE SHELL WITH SIMPLY SUPPORTED AND CLAMPED EDGES

Suppose that a shallow shell is simply supported on edges $x_1 = 0$, $x_1 = a$ and clamped on edges $x_2 = 0$, $x_2 = b$. Then in edges the following conditions are taken

$$\begin{aligned} &w = 0,\ M_1 = 0,\ N_1 = 0,\ N_6 = 0,\ \text{in}\, x_1 = 0,\ a; \\ &w = 0\,,\ \frac{\partial w}{\partial x_2} = 0\,,\ N_2 = 0\,,\ N_6 = 0\,,\ \text{in}\, x_2 = 0,\ b, \end{aligned} \tag{25}$$

The boundary conditions (25) are satisfied when the stress function and deflection are chosen as follows

$$\begin{aligned} \varphi &= g(t) \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}, \\ w &= f(t) \sin \frac{m\pi x_1}{a} \Big(1 - \cos \frac{2n\pi x_2}{b}\Big). \end{aligned} \tag{26}$$

Substituting expressions (26) into equations (13), (14) and using Bubnov-Galerkin method we get

$$\begin{aligned} &\Big[A_{11}^* \frac{m^4\pi^4 b}{4a^3} + (A_{66}^* - 2A_{12}^*) \frac{m^2 n^2 \pi^4}{4ab} + A_{22}^* \frac{n^4 \pi^4 a}{4b^3}\Big] g(t) = \Big[B_{11}^* \frac{4m^4\pi^3 b}{3na^3} \\ &- B_{12}^* \frac{4m^2 n\pi^3}{3ab} + B_{22}^* \frac{16 n^3 \pi^3 a}{3b^3} + k_1 \frac{4n\pi a}{3b} + k_2 \frac{4m^2 \pi b}{3na}\Big] f(t) - \frac{256 mn\pi^2}{45ab} f^2(t) \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\Big[D_{11}^* \frac{3m^4\pi^4 b}{4a^3} + 2D_{33}^* \frac{2m^2 n^2 \pi^4}{a^2 b^2} + D_{22}^* \frac{4n^4\pi^4 a}{b^3}\Big] f(t) + \\ &+ \Big[B_{11}^* \frac{4m^4\pi^3 b}{3na^3} - B_{12}^* \frac{4m^2 n\pi^3}{3ab} + B_{22}^* \frac{4n^3\pi^3 a}{3b^3} + k_1 \frac{4n\pi a}{3b} + k_2 \frac{4m^2\pi b}{3na}\Big] g(t) \\ &- \frac{512 mn\pi^2}{45ab} f(t) g(t) - q\frac{2ab}{m\pi} + \Big[J_0 \frac{3ab}{4} + J_2 \pi^2 \big(\frac{3m^2 b}{4a} + \frac{n^2 a}{b}\big)\Big] \frac{d^2 f}{dt^2} = 0, \end{aligned} \tag{28}$$

where denote the coefficients

$$B_{11}^* = A_{12}^* \frac{EA_1 z_1}{s_1}, \; B_{22}^* = A_{12}^* \frac{EA_2 z_2}{s_2}, \; B_{12}^* = A_{11}^* \frac{EA_2 z_2}{s_2} + A_{22}^* \frac{EA_1 z_1}{s_1}$$

and m, n are odd numbers.
Similarly we get the vibration equation of the shell in the form (21), where the terms m_i, $(i = 1, 2, 3)$ can be calculated by (22) but with coefficients

$$A = \frac{B_{11}^* \dfrac{4m^4\pi^3 b}{3na^3} - B_{12}^* \dfrac{4m^2 n\pi^3}{3ab} + B_{22}^* \dfrac{16n^3\pi^3 a}{3b^3} + k_1 \dfrac{4n\pi a}{3b} + k_2 \dfrac{4m^2\pi b}{3na}}{A_{11}^* \dfrac{m^4\pi^4 b}{4a^3} + (A_{66}^* - 2A_{12}^*) \dfrac{m^2n^2\pi^4}{4ab} + A_{22}^* \dfrac{n^4\pi^4 a}{4b^3}},$$

$$B = \frac{256mn\pi^2}{45ab\Big[A_{11}^* \dfrac{m^4\pi^4 b}{4a^3} + (A_{66}^* - 2A_{12}^*) \dfrac{m^2n^2\pi^4}{4ab} + A_{22}^* \dfrac{n^4\pi^4 a}{4b^3}\Big]},$$

$$H_1 = D_{11}^* \frac{3m^4\pi^4 b}{4a^3} + 2D_{33}^* \frac{2m^2n^2\pi^4}{a^2b^2} + D_{22}^* \frac{4n^4\pi^4 a}{b^3},$$

$$H_2 = B_{11}^* \frac{4m^4\pi^3 b}{3na^3} - B_{12}^* \frac{4m^2 n\pi^3}{3ab} + B_{22}^* \frac{4n^3\pi^3 a}{3b^3} + k_1 \frac{4n\pi a}{3b} + k_2 \frac{4m^2\pi b}{3na},$$

$$H_3 = \frac{512mn\pi^2}{45ab}, \quad H_4(t) = q(t)\frac{2ab}{m\pi}, \quad H_5 = J_0 \frac{3ab}{4} + J_2\pi^2\Big(\frac{3m^2 b}{4a} + \frac{n^2 a}{b}\Big).$$

5. NON-LINEAR DYNAMICAL ANALYSIS OF DOUBLY CURVED SHALLOW COMPOSITE SHELLS

5.1. Frequency-amplitude relation of non-linear free vibration

The equation of non-linear free vibration can be obtained from (21)

$$\ddot{f}(t) + m_1 f(t) - m_2 f^2(t) + m_3 f^3(t) = 0. \tag{29}$$

Using harmonic equilibrium method with seeking solution as $f(t) = A\cos(\omega t)$, after some transformations we get the frequency-amplitude relation of non-linear free vibration

$$\nu^2 = \frac{\omega^2}{\omega_0^2} = 1 - \frac{8\Omega_0}{3\pi} A + \frac{3K}{4} A^2, \tag{30}$$

where $\Omega_0 = m_2/m_1$, $K = m_3/m_1$, terms m_1, m_2, m_3 are determined from (22), ν is ratio of non-linear vibration frequency and fundamental frequency, A - amplitude of non-linear vibration.
In particular case for an unreinforced plate or a centrically reinforced plate we have $\Omega_0 = \dfrac{m_2}{m_1} = 0$, the expression (30) gets form

$$\nu^2 = 1 + \frac{3K}{4} A^2.$$

5.2. Dynamic analysis of composite shell in non-linear vibration

Let's consider non-linear vibration of a shallow composite shell subjected to excited load $q \equiv q(x_1, x_2, t) = q_0 \sin(\Omega t)$. Now the equation (21) can be represented in the form of Duffing equation

$$\ddot{f}(t) + \big[m_1 - m_2 f(t) + m_3 f^2(t)\big] f(t) = \overline{q}_{mn}(t), \tag{31}$$

where:

a) For a shell with simply supported edges

$$\overline{q}_{mn}(t) = \frac{16q_0 \sin(\Omega t)}{mn\pi^2\Big[J_0 + J_2\pi^2\big(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\big)\Big]}.$$

b) For a shell with simply supported edges and clamped edges

$$\overline{q}_{mn}(t) = \frac{8q_0 \sin(\Omega t)}{m\pi\Big[3J_0 + J_2\pi^2\big(\dfrac{3m^2}{a^2} + \dfrac{4n^2}{b^2}\big)\Big]}.$$

Put $K(f) = m_1 - m_2 f(t) + m_3 f^2(t)$, $\overline{q}_{mn}(t) = F(t)$, equation (31) can be rewritten in the form

$$\ddot{f}(t) + K(f)f(t) = F(t). \tag{32}$$

Equation (32) can be solved by iterative method with Newmark scheme. Dividing calculated process by time-step Δt, corresponding $(n+1)^{th}$ step we have $t_{n+1} = (n+1)\Delta t$. Using Newmark scheme we can rewrite (32) as follows

$$K^*(f)_{n+1} \cdot f_{n+1} = F^*_{n+1}, \tag{33}$$

where

$$\begin{aligned} K^*(f)_{n+1} &= K(f)_{n+1} + \frac{4}{\Delta t^2}, \\ F^*_{n+1} &= F_{n+1} + \Big(\frac{4}{\Delta t^2} f_n + \frac{4}{\Delta t}\dot{f}_n + \ddot{f}_n\Big). \end{aligned} \tag{34}$$

The solution f_{n+1} is determined at the time $t_{n+1} = (n+1)\Delta t$, then velocity and acceleration can be calculated respectively by formulas

$$\begin{aligned} \ddot{f}_{n+1} &= \frac{4}{\Delta t^2}\Big(f_{n+1} - f_n\Big) - \frac{4}{\Delta t}\dot{f}_n - \ddot{f}_n, \\ \dot{f}_{n+1} &= \dot{f}_n + \frac{\Delta t}{2}\Big(\ddot{f}_n + \ddot{f}_{n+1}\Big). \end{aligned} \tag{35}$$

Since the coefficient $K^*(f)_{n+1}$ in (33) is non-linear, we use an iterative method to solve the equation. Using here the direct iteration technique the equation (33) can be expressed as

$$K^*(f)^{(k)}_{n+1} \cdot f^{(k+1)}_{n+1} = F^*_{n+1}, \tag{36}$$

where k is the iteration number. At any fixed time for the $(k+1)^{th}$ iteration, the stiffness $K^*(f)^{(k)}_{n+1}$ is calculated from the k^{th} iteration. Calculating process stops when the convergence criterion is satisfied

$$\left|\frac{f_i^{(k)} - f_i^{(k+1)}}{f_i^{(k)}}\right| \leqslant \varepsilon, \tag{37}$$

where $\varepsilon > 0$ is a given small value.

6. NUMERICAL EXAMPLES

The shallow shell considered here is a panel with in-plane edges $a = b$=2 m, curvatures $k_1 = 1/R_1$, $k_2 = 1/R_2$. The shell is simply supported at all its edges or simply supported-clamped. The skin of the shell had 4 plies $[45/-45/-45/45]$, each ply being 1.5 mm. The material of the shell is $AS4/3501$ graphite/epoxy with parameters E_1=144.8 GPa, E_2=9.67 GPa

$$G_{12} = G_{13} = 4,14\ \text{GPa}, \quad G_{23} = 3.45\ \text{GPa}, \quad \nu_{12} = 0.3,\ \rho = 1389.23\ \text{kg/m}^3,$$

where E_1 , E_2 are the elastic modulus in directions x_1 and x_2 respectively, ν_{12} - the Poisson coefficient, G_{12}, G_{13} and G_{23} are the shear modulii in $12, 13$ and 23 planes.
Material of reinforced stiffener has elastic modulus $E = 600GPa$. The height of the stiffener is equal to $12mm$, while their width $4mm$. The spacings of longitudinal stiffener and tranversal stiffeners s_1=50 mm, and s_2=50 mm respectively.

The time-step Δt is taken as $T/300$ where period $T = 2\pi/\Omega$, Ω is a frequency of excited load and $t_n = n.\Delta t$ are used for numerically solving non-linear differential equation by Newmark method. The applied harmonic uniform load is of the form $q(x_1, x_2, t) = p \sin \Omega t$, where the magnitude p may be taken as 7,500 N/m^2, 15,000 N/m^2, 30,000 N/m^2. Two cases of Gauss curvature are taken into consideration: $k = k_1.k_2 > 0$ when $R_1 = R_2 = 5m$ and $k = k_1.k_2 < 0$ when R_1 = 3 m, R_2 = -10 m, we take respectively Ω = 1400 and $\Omega = 800(s^{-1})$, these frequency values of excited load are near to ones of fundamental frequency values when $k > 0$: $\omega_0 = 1416$ (in case SS-SS), $\omega_0 = 1390$ (in case SS-CC); when $k < 0$: $\omega_0 = 780$ (in case SS-SS), $\omega_0 = 749$ (in case SS-CC) respectively (calculated data according to (3.10)).

The Figs. 1, 2, 3 show graphs of maximum deflection $w_{max} = f$ by 25 period, that means the non - linear transient responses, for cases of boundary condition with simply supported edges (SS-SS) and simply supported-clamped edges (SS-CC) when Gauss curvature $k = k_1.k_2 > 0$ and magnitude p of excited load as 7,500 N/m^2, 15,000 N/m^2, 30,000 N/m^2 respectively. The Fig. 4 shows relation of maximum deflection and velocity of maximum deflection when Gauss cuvature $k = k_1.k_2 > 0$, p=30,000 N/m^2.

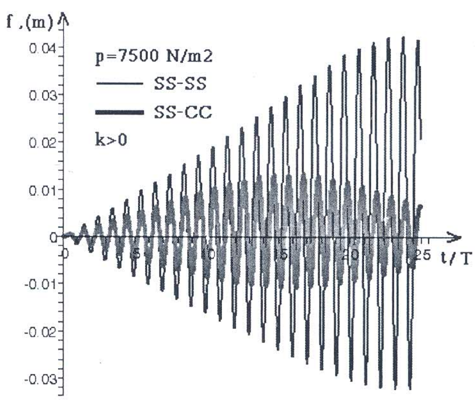

Fig. 1. Non-linear transient responses with p=7,500 N/m^2

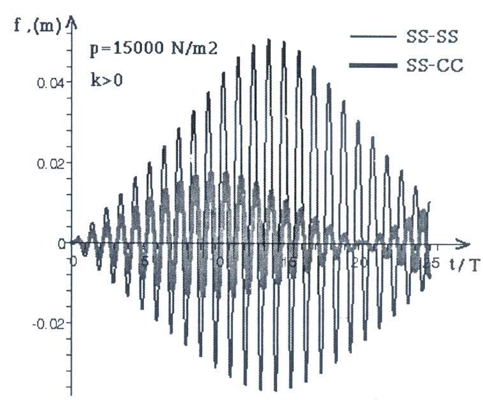

Fig. 2. Non-linear transient responses with p=1,5000 N/m^2

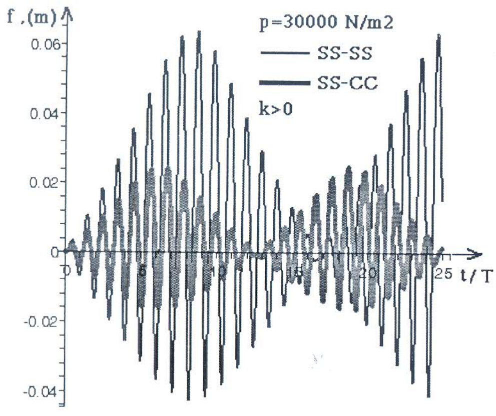

Fig. 3. Non-linear transient responses with p=30,000 N/m^2

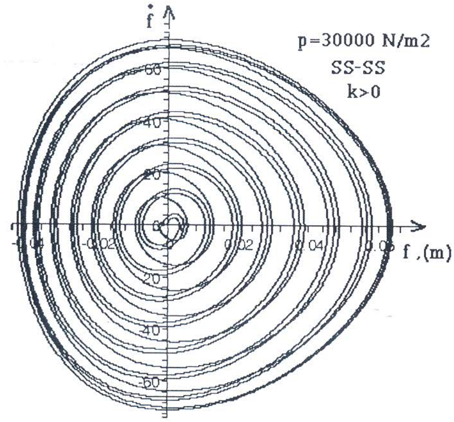

Fig. 4. Deflection-velocity relation

Fig. 5 shows the graph of maximum deflection $w_{max} = f$ by 25 period (non-linear transient responses) for cases of boundary condition with simply supported edges (SS-SS) and simply supported-clamped edges (SS-CC) when Gauss curvature $k = k_1.k_2 < 0$. Relation for maximum deflection and velocity of maximum deflection when p=7,500 N/m^2 is expressed in Fig. 6.

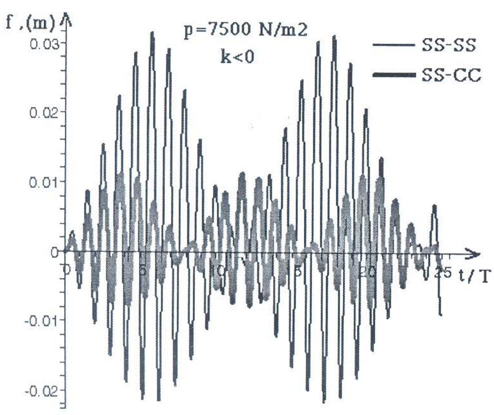

Fig. 5. Non-linear transient responses

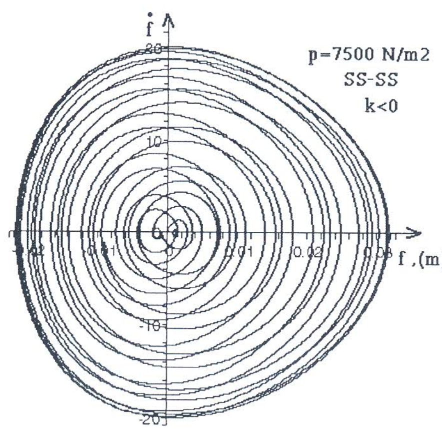

Fig. 6. Deflection-velocity relation

The Figs. 7, 8, 9 show non-linear transient responses and graphs $f - \dot{f}$ by 25 period respectively for cases with Gauss curvature $k < 0$, when p=1,5000 N/m^2, and Figs. 10, 11, 12 for cases with Gauss curvature $k < 0$, p=30,000 N/m^2.

Figs. 13, 14 show the non-linear transient responses for unstiffened and stiffened shell with simply supported at all edges, simply supported-clamped edges. The skin of the shell had 4 plies , each ply being 1 mm. The spacings s_1, s_2 of stiffeners vary differently, consequently the effect of stiffeners on the transient responses of shell is represented apparently.

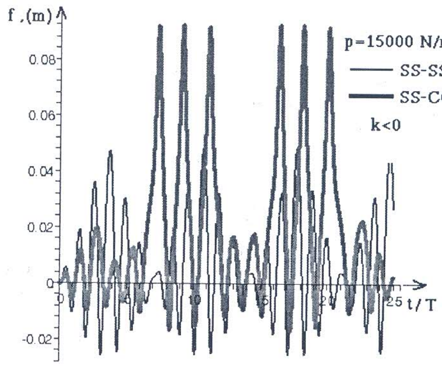

Fig. 7. Non-linear transient responses when p=1,500 N/m^2

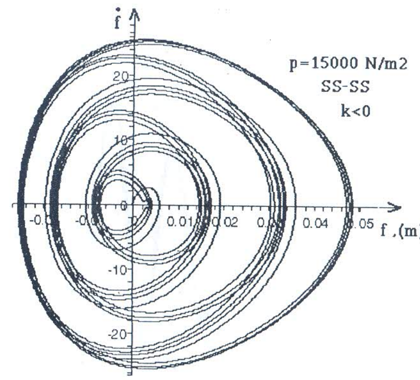

Fig. 8. Deflection-velocity relation (SS-SS)

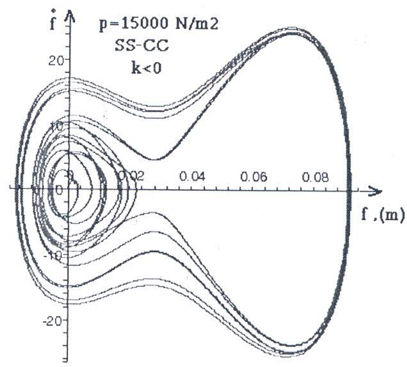

Fig. 9. Deflection-velocity relation (SS-CC)

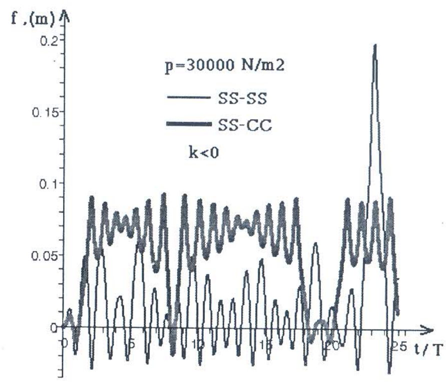

Fig. 10. Non-linear transient responses

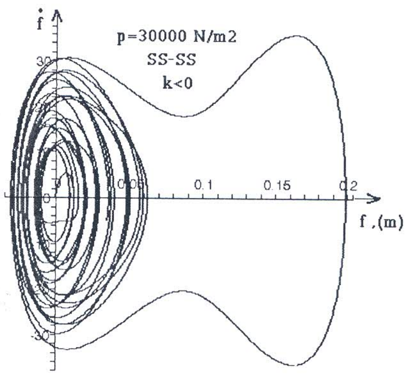

Fig. 11. Relation of deflection-velocity (SS-SS)

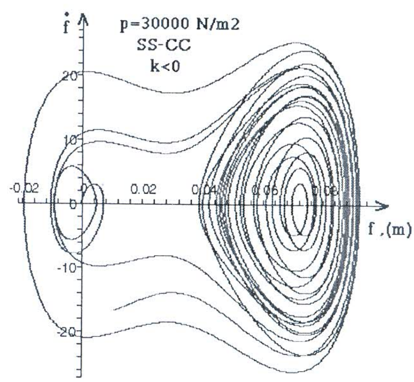

Fig. 12. Relation of deflection-velocity (SS-CC)

Discussion

- Although the excited load is harmonic, non - linear responses of shell are not harmonic by timing, however having a specific cycle for the shell of positive Gauss cuvature $k > 0$. For non-linear vibration of a shell when fundamental frequency is equal to frequency of excited load may occur the phenomenon of resonance, but in this considered example the

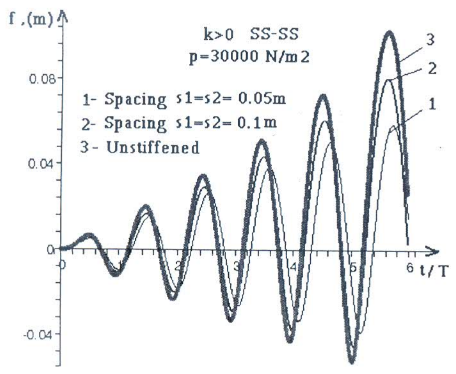

Fig. 13. Non-linear transient responses (SS-SS)

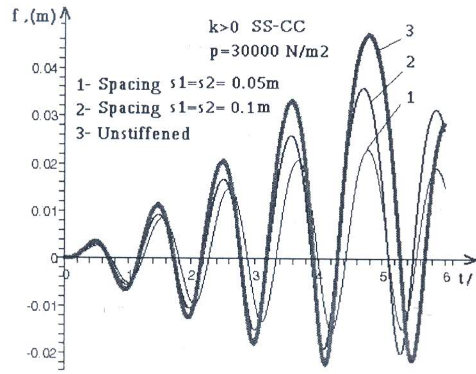

Fig. 14. Non-linear transient responses (SS-CC)

frequency of excited load is near to the fundamental frequency we observe the phenomenon like harmonic beat of a linear vibration and the amplitude of external load is smaller, then the cycle of non-linear transient responses is larger.

- For the shell with negative Gauss curvature $k < 0$, the phenomenon of cycle is not clear and its turn is rather complicated and needed to be examined more carefully.
- When $k > 0$ the amplitude of non-linear vibration for the shell with SS-SS edges is greater than with SS-CC edges, when $k < 0$ the above rule is not changed completely.
- Non-linear responses and relation of declection f-velocity $\dot{f}$ for a shell with $k > 0$ is stable, a sudden phenomenon does not happen but for a shell with $k < 0$ this rule is not true and needed to be examined more carefully.

7. CONCLUSION

The governing equations for dynamic problem of eccentrically stiffened laminated composite shallow shell are given and analysed by use of the thin shell theory cosidering geometrical non-linearity and the Lekhnitsky's smeared stiffeners technique. Using stress function and Bubnov-Galerkin methods the equation of non-linear vibration for shallow shell in the form of Duffing equation is obtained. Numerical solution is carried out by iterative method using Newmark calculated scheme. Obtained results show the influence of boundary conditions and Gauss curvature on the non-linear vibration of eccentrically stiffened laminated shallow shell.

This publication is supported by the National Council for Natural Science and the Vietnam National University, Hanoi.

REFERENCES

1. D. H. Bich, Nonlinear analysis of laminated composite doubly curved shallow shells, *Vietnam Journal of Mechanics* **28** (1) (2006) 43-55.
2. D. H. Bich, Tran Thanh Tuan. Dynamical analysis of multilayered reinforced composite cylindrical shells, *Vietnam Journal of Mechanics* **27** (4) (2005) 220-228.
3. D. H. Bich, V. D. Long, Nonlinear dynamic analysis of laminated reinforced composite doubly curved shallow shells, *Vietnam Journal of Mechanics* **29** (3) (2007) 257-269.
4. C. Y. Chia, *Non-linear analysis of plates*, McGraw-Hill. Inc. 1980.

5. A. A. Khdeir, J. N. Reddy, D.Frederich, A study of bending, vibration and buckling of cross-ply circular cylindrical shells with various shell theories, *Int. J. of Eng. Sci.* **27** (1989) 1337-1351.
6. Vu Do Long, Non-linear investigation for laminated composite shallow shell, *Proceeding of* 8^{th} *National Conference on solid mechanics*, VAST. Publishers 2006, 463-472.
7. C. T. Loy, K.Y.Lam, J. N. Reddy. Vibration of functionally graded cylindrical shells, *Int. J. of Mech. Sci.* **41** (1999) 309-324.
8. A. Noisier, J. N. Reddy, Vibration and stability of cross-ply laminated circular cylindrical shells, *J. of Sound and Vibration* **157** (1992) 139-159.
9. J. N. Reddy, K. Chandrashekhara. Geometrically nonlinear transient analysis of laminated doubly curved shells. *Int. J. Nonlinear Mech.* **20** (1985) 79
10. J. N. Reddy, *Mechanics of laminated composite plates and shells, Theory and analysis*, 2^{nd} edition, CRC Press, 2004.
11. M. Sathyamoorthy, Non-linear vibration analysis of plates: a review and survey of current developments, *Applied Mechanics Review* **40** (1987) 1553-1561.
12. A. S. Volmir, *Non-linear dynamics of plates and shells*, Science Edition, M.1972.

Received January 02, 2008

DAO ĐỘNG PHI TUYẾN CỦA VỎ COMPOSITE LỚP CÓ GÂN GIA CƯỜNG

Bài báo trình bầy dao động phi tuyến của vỏ thoải composite lớp hai độ cong có gân gia cường lệch tâm. Tính toán nội lực và chuyển vị dựa trên lý thuyết vỏ mỏng có tính đến yếu tố phi tuyến hình học và kỹ thuật tính gân gia cường theo Lekhnitoky. Dựa trên phương trình tương thích biến dạng và phương trình chuyển động của vỏ để thiết lập hệ phương trình đạo hàm riêng theo hàm ứng suất và độ võng của của vỏ. Sử dụng phương pháp Bubnov-Gaberkin và phương pháp bước lặp có dùng sơ đồ Newmark để phân tích động lực của vỏ. Đã nhận được liên hệ tần số-biên độ của dao động phi tuyến tự do và đáp ứng tức thời phi tuyến của vỏ. Các kết quả tính toán số cho thấy ảnh hưởng rõ rệt của điều kiện biên và độ cong Gauss đến dao động phi tuyến của vỏ.