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THE ERROR ESTIMATE AND THE CONVERGENCE RATE FOR h,p - REFINEMENT IN THE FINITE ELEMENT ANALYSIS

Nguyen Hoai Son
University of Technical Education, Hochiminh City

Abstract. The goal of this study is to further investigate and to develop a more efficient way in the error estimate and the rate of the convergence for the mesh h, p-refinement procedure in the finite element analysis for two-dimensional and three-dimensional elastostatic mechanics problems. The oscillation of the stress field around singularity points is also considered in the refinement process. In this paper, a-refinement procedure with a h-uniform and p-uniform analysis capability is called a h, p-refinement procedure. We also establish the refinement criterion for the adaptive strategy with h-refinement based on the ratio of error indicator or the enrichment indicator.
Keywords *Estimate, refinement, error indicator.*

1. INTRODUCTION

In this paper, a-*refinement* procedure with a h-uniform and p-uniform analysis capability is called *h, p-refinement procedure.* We have established a criterion for refinement procedure in the error estimate and the rate of convergence based on the ratio of error indicator. Several error estimates have been developed for engineering use. However, they are computationally intensive and still need to be validated for accuracy in practical engineering analysis [1, 3, 6]. Besides, the mentioned error estimates are in the form of an *energy norm.* This quantity differs from the quantities of interest in solid mechanics analysis, which are usually displacements or stresses at particular points inside a problem domain. Also, the energy convergence may not guarantee stress convergence. The development of an accurate error estimate for stress will result in a more efficient way of controlling the refinement process [2, 4, 7] and will provide valuable information for checking the convergence of final stress results.

2. THE ENERGY FUNCTIONAL AND THE ERROR EQUATION IN THE ENERGY NORM

Find $u \in V$ such that the Dirichlet's boundary condition satisfies the following equations:

$$\begin{aligned} &B(u,v) = L(v) \qquad \forall v \in V \\ &J(u) = \frac{1}{2}B(u,u) - L(u) \end{aligned} \tag{1}$$

$$B(e,v) = B(u_{EX},v) - B(u_{FE},v) = L(v) - B(u_{FE},v), \tag{2}$$

where

$$\begin{aligned} e &= u_{EX} - u_{FE} \\ \|e\|_{E(\Omega)} &= \|u_{EX} - u_{FE}\|_{E(\Omega)} \end{aligned} \tag{3}$$

$J(u)$, $\|e\|_{E(\Omega)}$, u_{EX}, u_{FE}: the energy norm, the error energy norm, the exact energy and the approximation element finite energy, respectively.

3. THE CRITERION OF THE CONVERGENCE

- The algebraic rate of convergence:

$$\|e\|_{E(\Omega)} = \|u_{EX} - u_{FE}\|_{E(\Omega)} \le \frac{k}{N^{\beta}} \tag{4}$$

- The exponential rate of convergence:

$$\|e\|_{E(\Omega)} = \|u_{EX} - u_{FE}\|_{E(\Omega)} \le \frac{k}{\exp(\gamma N^{\theta})} \tag{5}$$

where k, β, γ, θ, N: the positive constants and the degree of freedom, respectively. These constants are defined to depend on the energy norm error (4) and (5) by using the least three finite element meshes.

4. A STRATEGY OF THE ENRICHMENT INDICATOR IN *h*-REFINEMENT

We have

$$\eta_e \ge \lambda_R \max_{j=1,\ldots,N_\Delta} \eta_j \tag{6}$$

By an other way in the computation for the enrichment indicator, we have

$$\begin{aligned} &\frac{\tau_\ell}{\|\eta_{es}\|_{\max}} \approx \left(\frac{h_{new}}{h}\right)^p \ge \lambda_R \\ &\Leftrightarrow \left\|e_{es}^i\right\| \le \tau_\ell \left(\frac{\|U_h\|^2 + \|e_{es}\|^2}{N_\Delta}\right)^{1/2} = \bar{e}_{N\Delta} \to \lambda_R \le \frac{\bar{e}_{N\Delta}}{\|e_{es}^i\|} \end{aligned} \tag{7}$$

$0 < \lambda_R \le 1,\ \tau_\ell = 4\%,\ \lambda_R = 0.82\ \to\ \lambda_R < 0.82,\ h_{new} > 0.67$ (h-refinement is necessary) (see Table 1)

Table 1. A new size of the element and the enrichement indicator in computing of the error criteria for the plate with holes 2-D by (7) and (6) [5], [8]

#Element	h	h_{new}	λ_R
75	5.0000	0.21389655791820	0.24213368200584
192	3.1250	0.29742892449463	0.34700835422613
300	2.5000	0.34221505046118	0.40865949469253
675	1.6667	0.44306729695172	0.55090348171000
1200	1.2500	0.33296603779371	0.55140470348212
1875	1.0000	0.61536514766244	0.80372930397042
3352	0.8929	0.66194147175037	0.82500021777001

5. APPLICATION: PLATE WITH HOLES [8]

The numerical model here consists of a three-hole plate under plane-stress. The geometry and boundary conditions, as well as some numerical parameters used in this example, are shown in Fig. 1. We used triangular mesh with hierarchical basis function.

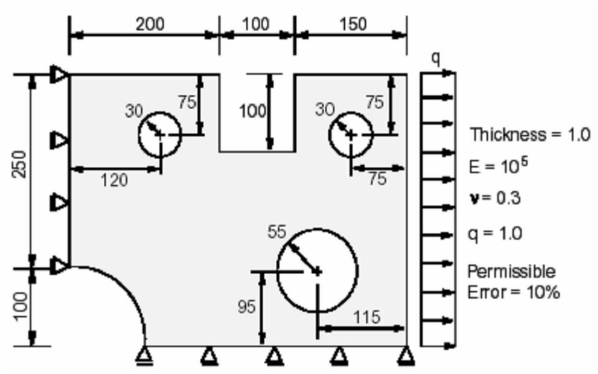
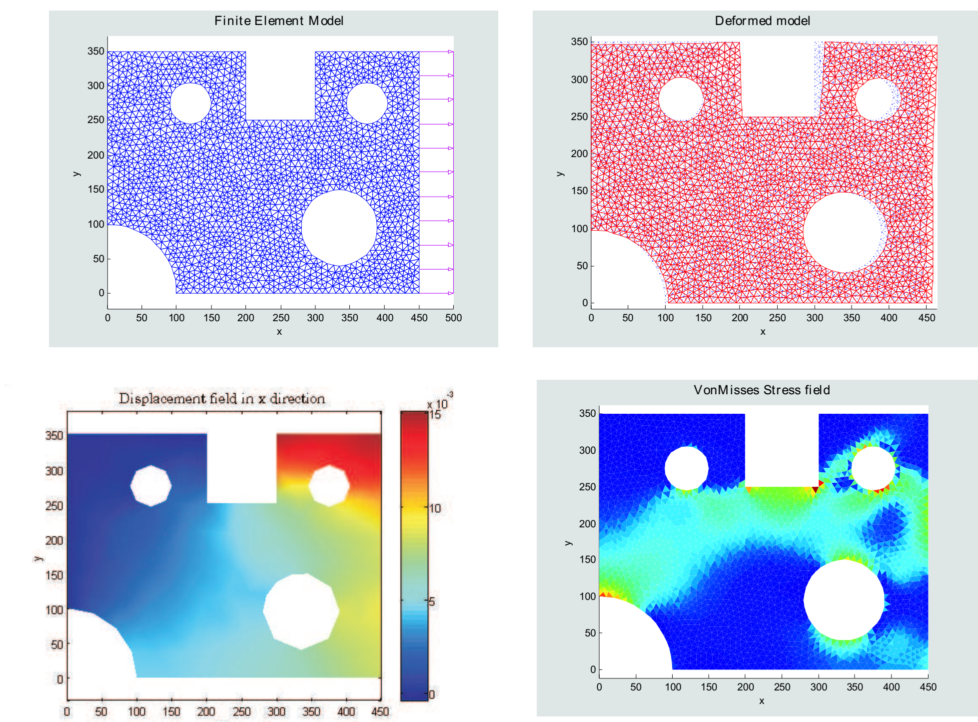


Fig. 1. Plate with Holes



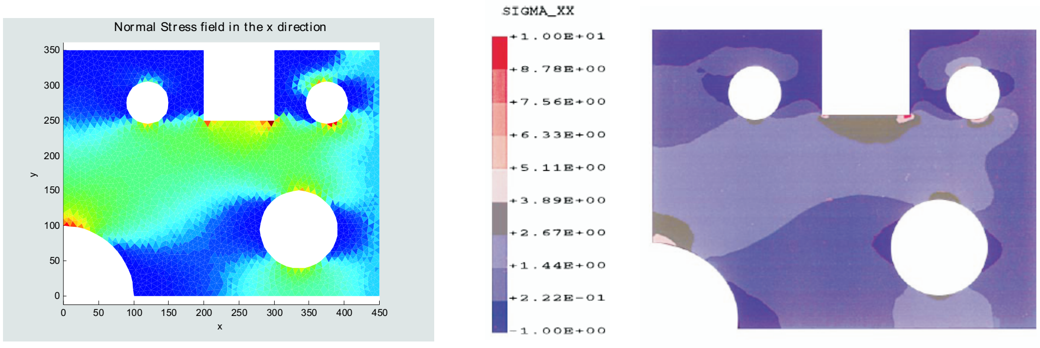


Fig. 2. Finite Element model (left-top) Deformed Field (right-top) Displacement field in x-direction (left-middle) Von mises stress field (right-middle) Normal stress field in the x direction (left-bottom) Normal stress field in the x direction [G. H. Paulino with FESTA Software] (right-bottom)

Energy Norm-Relative Error Estimations are given in Table 2 and Table 3 for h and p-refinements, recpectively.

- ***h-refinement***

Table 2. Energy Norm - Relative Error Estimation - Effectivity index - CPU time versus DOF

#dof	**Energy Norm** $\frac{1}{2}\|U\|^2$	$\|e_{es}\|$	η_{es}	θ	**CPU time(s)**
324	*1.53896741*	*0.427945*	*0.305137*	0.8219	1.672
382	*1.57066927*	*0.389146*	*0.277975*	0.7954	2.219
510	*1.59908889*	*0.350735*	*0.251178*	0.7636	3.907
560	*1.60910783*	0.336149	*0.241015*	0.7499	4.765
774	*1.63780383*	0.290345	0.209095	0.6996	10.828
1174	*1.67180703*	*0.224270*	*0.162871*	0.6032	33.172
2028	*1.692116327*	*0.173170*	0.126795	0.5043	169.28
4504	*1.716601025*	0.074192	0.055441	0.2427	521.234

- ***p-refinement***

Table 3. Energy Norm - Relative Error Estimation - Effectivity index - CPU time versus DOF

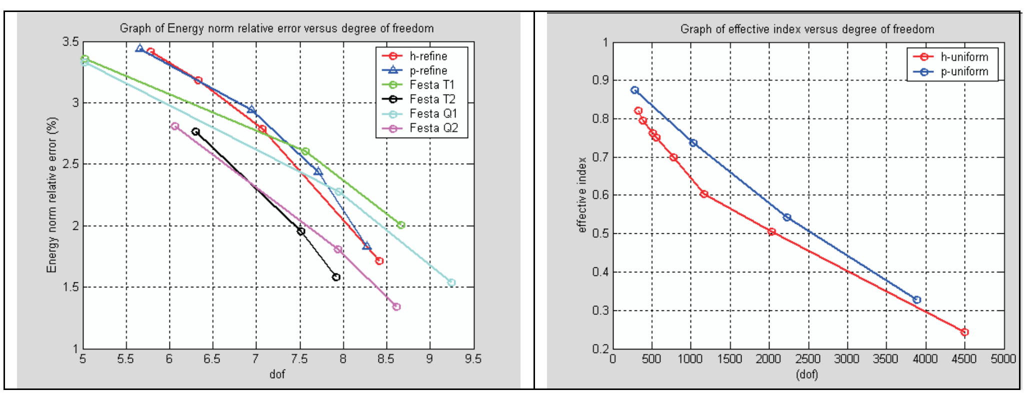
p	**#dof**	**Element number**	**Energy Norm** $\frac{1}{2}\|U\|^2$	$\|e_{es}\|$	η_{es}	θ	**CPU (times)**
1	286	228	1.5585	*0.438976*	0.3105	0.8752	3.765
2	1032	228	1.6814	*0.264196*	0.1894	0.7366	7.438
3	2234	228	1.7265	*0.157162*	0.1145	0.5437	25.375
4	3892	228	1.7441	0.084261	0.0623	0.3281	80.359

Table 4. The comparison relative error estimation between the computation h, p-refinement and FESTA software adaptive with triangular element versus DOF

Computation				*FESTA Software Adaptive [8]*			
h-refinement		*p-refinement*		*Linear triangular* (T1)		*Quadratic triangular* (T2)	
#*dof*	$\eta_{es}(\%)$	#*dof*	$\eta_{es}(\%)$	#*dof*	$\eta_{es}(\%)$	#*dof*	$\eta_{es}(\%)$
324	30.51	286	31.05	152	28.71	546	15.91
1174	16.28	2234	11.45	1925	13.56	1842	7.07
4504	5.54	3892	6.23	5782	7.45	2758	4.85

Table 5. The comparison relative error estimation between the computation h, p-refinement and FESTA software adaptive with quadrilateral element versus DOF

Computation				*FESTA Software Adaptive [8]*			
h-refinement		*p-refinement*		*Linear quadrilateral(Q1)*		*Quadratic quadrilateral (Q2)*	
#*dof*	$\eta_{es}(\%)$	#*dof*	$\eta_{es}(\%)$	#*dof*	$\eta_{es}(\%)$	#*dof*	$\eta_{es}(\%)$
324	30.51	286	31.05	152	27.79	428	16.56
1174	16.28	2234	11.45	2846	9.72	2808	6.10
4504	5.54	3892	6.23	10346	4.66	5494	3.83



(a) (b)

Fig. 3. Log-log Error estimation versus DOF (a) the effectivity index versus DOF for h and p refinement (b)

- Table 4 and 5 represent the comparison relative error estimation between the modified computation and FESTA software with the SPR (super-convergence patch recovery [2]) and REP (recovery Equilibrium patch [8]) Technique *give the results reliability with the permissible relative error* $\eta_{\max} = 10\%$ *for all elements.*

- Fig. 3 shows that the element types T2 and Q2 are computed from Festa software and modified computation with p and h-refinement giving the better than the element types T1, Q1 on the rate of convergence and relative error at the same degree of freedom, respectively.

- From Table 1 and the equations (6), (7) we observe that $1 \leq \frac{\eta_{esp}}{\eta_{esp-1}}$, the error estimate decreases when increasing polynomial degree. If enrichment wase indicated on element e, h-refinement would be the preferred strategy and if $\frac{\eta_{esp}}{\eta_{esp-1}} < 1$ this would suggest p-refinement as the preferred strategy.

6. CONCLUSION

- The convergence of the direct calculated stresses depends mainly on the element order. Indeed, their high order p-elements reduce the oscillation in the stress error and error indicator. The reliability of p-refinement finite element method is thus improved for a practical engineering analysis.

- In order to upgrade the evaluation of the energy norm, we implemented a modification for the Richardson's extrapolation technique. It added the higher order terms to the residual error. The issued results are then compared against those obtained with its least square method keeping the same order.

- The p-refinement and the h-p-refinement mesh seem to be the best methods to reduce the approximate error for the problems without singular points. The h-refinement method shows better efficiency when the mesh embeds singular points.

REFERENCES

1. B. A. Szabo, Mesh design for the p-refinement of the finite element method, *Computer Methods in Applied Mechanics and Engineering* **55** (1986) 181-197.
2. O. C. Zienkiewicz, and J. Z. Zhu, Adaptive techniques in the finite element method, *Communications in Applied Numerical Methods* **4** (1998) 197-204.
3. F. Cugnon, and P. Beckers, Error estimation for h and p methods, *8^{th} Mechanical Engineering Chilean Congress*, Concepcion, 27-30 October 2004, pp.737-744.
4. N. H. Son, D. M. Dai, The error estimate for finite element analysis with h-p refinement in the linear elasticity 2-D, 3-D, *International conference 8-2004 French-Vietnam.*
5. J. E. Flaherty, *Finite element analysis*, Troy, New York, 2002.
6. M. Aiffa, Adaptive hp-refinement methods for singularly-perturbed elliptic and parabolic systems, *PhD thesis*, Rensselaer Polytechnic Institute, Troy, 1997.
7. F. Cugnon, *Automatisation des calculs elements finis dans le cadre de la methode-p*, Thesis de doctorat, 2000, ULG.
8. G. H. Paulino, et al, A Methodology for adaptive finite element analysis towards an integrated computational environment, *Computational Mechanics* **23** (1999) 361-388 @ Springer-Verlag 1999.

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ĐÁNH GIÁ SAI SỐ VÀ TỐC ĐỘ HỘI TỤ THEO KIỂU LÀM MỊN *h-p* TRONG PHÂN TÍCH PHẦN TỬ HỮU HẠN

Mục đích chính của việc nghiên cứu này là đưa ra hướng phát triển giải thuật làm mịn có hiệu quả trong đánh giá sai số và tốc độ hội tụ cho các bài toán cơ học trong phân tích phần tử hữu hạn. Đặc biệt quan tâm phân tích tính nhạy cảm của trường phân bố ứng suất tại các điểm suy biến trong quá trình làm mịn. Trong bài báo này, một thủ tục làm mịn h-p được đề nghị. Qua nghiên cứu này, chúng tôi cũng thiết lập một tiêu chuẩn làm mịn theo kiểu h dựa trên chỉ số sai số hay chỉ số làm giàu theo kích thước lưới một cách có hiệu quả với độ tin cậy cao.