Vietnam Journal of Mechanics, VAST, Vol. 30, No. 3 (2008), pp. 158 – 166

ON MODELLING AND CONTROL DESIGN FOR SELF-BALANCED TWO-WHEEL VEHICLE

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Abstract. In this paper, the modeling and control design of a self-balancing mobile robot are presented. The method of sub-structures is employed to derive the differential equations of motion of the robot. Based on the linearized equations of motion, a controller is designed to maintain a stable motion of the robot. Some numerical simulation results are shown to clarify the designed controller.

1. INTRODUCTION

A two-wheel vehicle with a type of inverted pendulum has a commercial name, so-called Segway, which was invented by an American named Dean Kamen. This vehicle has been investigated and designed for 10 years ago, and was introduced to public since 2001. Nowadays, thousands of such vehicles were sold in the USA and Europe. It is convenient to use this vehicle to travel within 10 km distance. These vehicles can reach a velocity of 20 km/h.

Main mechanical structures of the vehicle consist of a bodywork and two wheels those two axes coincides. Two wheels are driven by two independent electric motors. Rotors are fixed with the wheels, while stators with the bodywork, or in inverse order. Power supports for the vehicle is an accumulator on the bodywork. In order to maintain a stationary motion of the vehicle – bodywork is directed above -, a controller is necessary. In addition, there is a screen to display motion states such as velocity, some bottoms to set motion parameters.

In recent years, several authors have investigated this new vehicle type [2-5]. In this paper, a dynamic model of the vehicle moving on a straight line and the designing of a controller based on linearized model are presented. Based on differential equations of motion obtained by the method of substructures, a controller to ensure the stable motion of the vehicle is designed. Finally, some numerical simulations are given to verify the reliability of the control law.

2. DIFFERENTIAL EQUATION OF MOTION OF VEHICLE

Mechanically, this vehicle consists of three major members: a bodywork and two independent wheels. In case the vehicle moving on a straight line – the motion of two wheels is identical - we can combine two wheels together. So the vehicle is understood as a system with two bodies in a vertical plane: a bodywork and a wheel connecting with each other by a hinge (Fig. 1). In order to establish the differential equations of motion of the vehicle, the substructure method is exploited in this study. After resolving the vehicle into two substructures: the bodywork and the wheel, the Newton-Euler equations [1, 6] are applied to planar motion of bodies.

- The wheel together with the rotor:

$$\begin{aligned}
&(m_1 + m_{ro})\ddot{x}_{C1} = F - X_0 \\
&(m_1 + m_{ro})\ddot{y}_{C1} = N - (m_1 + m_{ro})g - Y_0 \\
&(J_1 + J_{ro})\ddot{\varphi}_1 = \tau - Fr - M_l
\end{aligned} \tag{1}$$

- The bodywork:

$$\begin{aligned}
&m_2\ddot{x}_{C2} = X_0 \\
&m_2\ddot{y}_{C2} = Y_0 - (m_2 + m_{sta})g \\
&J_2\ddot{\varphi}_2 = -\tau - X_0(y_{C2} - y_O) + Y_0(x_{C2} - x_O)
\end{aligned} \tag{2}$$

where m_1 is the mass of the wheel, r – wheel radius, m_{ro} – rotor mass, J_1 and J_{ro} are the mass inertias of the wheel and robot, respectively, M_l – damping moment at contact area between the wheel and a floor. The interactive moment between the rotor and the stator τ is created by an electric motor that is related to input current or voltage, and damping moment at the journal; m_2 – the mass of the bodywork, J_2 – mass inertia of the bodywork with respect to the axis past through the center of mass C_2.

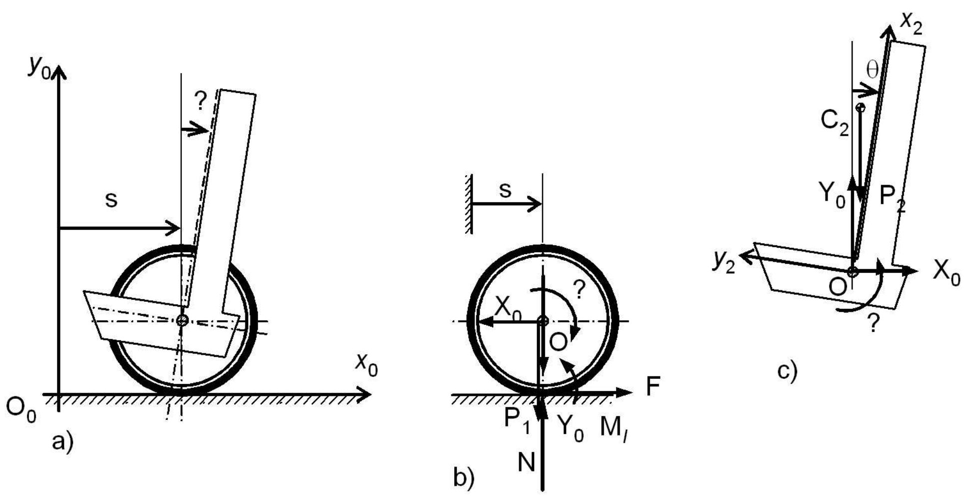

Fig. 1. Vehicle model and two substructures

From Fig. 1. a) and b) with an assumption that the wheel rolls without slipping we have some kinematic relations as following:

$$\begin{aligned}
&x_{C1} = s && \ddot{x}_{C1} = \ddot{s} && x_{C2} = s + u_{2x}\sin\theta - u_{2y}\cos\theta \\
&y_{C1} = r && \ddot{y}_{C1} = 0 && y_{C2} = r + u_{2x}\cos\theta + u_{2y}\sin\theta \\
&\varphi_1 = s/r && \ddot{\varphi}_1 = \ddot{s}/r && \varphi_2 = \theta
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\ddot{x}_{C2} = \ddot{s} + (u_{2x}\cos\theta + u_{2y}\sin\theta)\ddot{\theta} + (-u_{2x}\sin\theta + u_{2y}\cos\theta)\dot{\theta}^2 \\
&\ddot{y}_{C2} = (-u_{2x}\sin\theta + u_{2y}\cos\theta)\ddot{\theta} - (u_{2x}\cos\theta + u_{2y}\sin\theta)\dot{\theta}^2 \\
&\ddot{\varphi}_2 = \ddot{\theta}
\end{aligned} \tag{4}$$

In order to determine the moment acting between the wheel and the bodywork, a model of a permanentmagnet DC motor is considered (Fig. 2).

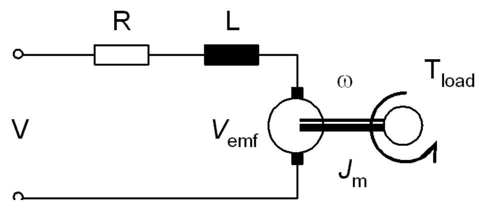

Fig. 2. Diagram of a DC motor

A torque of the motor is proportional to current passing the armature

$$M = K_m i. \tag{5}$$

A voltage V_{emf} is generated across its terminals that is proportional to the angular velocity of rotor is called back emf

$$V_{emf} = K_e \dot{\theta}. \tag{6}$$

By applying the theorem of momentum and Kirchoff law, the differential equation describing dynamics of DC motor is obtained as follows:

- mechanical equation

$$J_m \dot{\omega} + b\omega = K_m i - T_{load}, \tag{7}$$

- electrical equation

$$L\frac{di}{dt} + Ri = V - K_e \omega. \tag{8}$$

Frequently, it is assumed that the "electric time constant" R/L is much smaller than the "mechanical time constant" J_m/b. This is a reasonable assumption for many electromechnical systems and leads to a reduced order model of the actuator dynamics. From equation (8) with $L/R \approx 0$ we have

$$i = (V - K_e \dot{\theta})/R. \tag{9}$$

Substituting (9) into (7) one obtains the equation presenting the relation between input voltage V and motion of rotor $\dot{\theta}$

$$J_m \ddot{\theta} + (b + K_m K_e/R)\dot{\theta} = V/R - T_{load}, \tag{10}$$

and torque at the motor journal

$$M = K_m i = K_m (V - K_e \dot{\theta})/R = (K_m/R)V - (K_m K_e/R)\dot{\theta}. \tag{11}$$

Torque interacting between the bodywork and the wheel is a summation of the motor torque and friction torque at the journal. In this paper, it is assumed that a friction torque is proportional to the relative angular velocity of the bodywork and the wheel. So we have

$$\begin{aligned} \tau &= M_{dc} + M_{\text{damp}} = K_m i - b\omega \\ &= (K_m/R)V - (K_m K_e/R + b)(\dot{s}/r - \dot{\theta}) \end{aligned} \tag{12}$$

with $\omega = \dot{\varphi}_1 - \dot{\varphi}_2 = \dot{s}/r - \dot{\theta}$.

In all above equations the following quantities have been used: V – applied voltage of the motor, R - armature resistance, L - armature inductance, J_m - mass inertia of the rotor, V_{emf} - back emf voltage, b - damping coefficient, M – torque due to the current, ω

- relative angular velocity of the rotor respect to the stator, K_m - torque constant of the motor, K_e – back emf constant, b_1 - roll damping constant, $M_l = b_1\omega_1$.

Eliminating the constraint forces X_0, Y_0, and F from equation (1) and (2) one obtains the differential equations of motion of the vehicle in matrix form as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{L}V, \tag{13}$$

with $\mathbf{q} = [s, \theta]^T$ is a generalied coordinate vector of the vehicle.

To simplify the model one takes $u_{2y} = 0$, in this case the mass matrix $\mathbf{M}(\mathbf{q})$, vector $\boldsymbol{h}$ and matrix $\boldsymbol{L}$ are obtained as:

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} \dfrac{J_1 + J_{ro}}{r^2 + (m_1 + m_{ro} + m_2)} & m_2 u_{2x} \cos\theta \\ \dfrac{J_1 + J_{ro}}{r + r(m_1 + m_{ro} + m_2) + m_2 u_{2x} \cos\theta} & J_2 + m_2 u_{2x}^2 + r m_2 u_{2x} \cos\theta \end{bmatrix}, \tag{14}$$

$$\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -m_2 u_{2x} \sin\theta\dot{\theta}^2 + \frac{K_m K_e}{Rr}(\dot{s}/r - \dot{\theta}) + b/r(\dot{s}/r - \dot{\theta}) + b_1\dot{s}/r^2 \\ b_1\dot{s}/r - r m_2 u_{2x} \sin\theta\dot{\theta}^2 - m_2 g u_{2x} \sin\theta \end{bmatrix}, \tag{15}$$

$$\mathbf{L}_0 = \begin{bmatrix} K_m/Rr \\ 0 \end{bmatrix}. \tag{16}$$

Now, we will examine whether there exists a motion state with constant velocity and the bodywork direct above, i.e:

$$\dot{\theta} = 0, \dot{s} = v_d = \text{const},\ \theta = \text{const} \approx 0. \tag{17}$$

Substituting (17) into (13) one obtains a system of algebraic equations:

$$\begin{bmatrix} \frac{K_m K_e + R(b+b_1)}{Rr^2} v_d \\ b_1 v_d/r - m_2 g u_{2x} \sin\theta_0 \end{bmatrix} = \begin{bmatrix} \frac{K_m}{Rr} \\ 0 \end{bmatrix} u_0. \tag{18}$$

Solving the algebraic equations (18) one can determine a slope of the bodywork and an applied voltage corresponding to the constant velocity of the vehicle, $v_d = \text{const}$

$$\theta_0 = f(v_d) = \arcsin\frac{b_1 v_d}{m_2 g u_{2x} r}, \quad u_0 = g(v_d) = \frac{K_m K_e + R(b + b_1)}{K_m r} v_d \tag{19}$$

Linearilizing the differential equations of motion (13) about a desired motion (17), one obtains differential equations of disturbed motion. The equations of motion after linearilizing have a form as follows

$$\mathbf{M}_0\delta\ddot{\mathbf{q}} + \mathbf{D}_0\delta\dot{\mathbf{q}} + \mathbf{K}_0\delta\mathbf{q} + \mathbf{h}(\mathbf{q}_0, \dot{\mathbf{q}}_0) = \mathbf{L}(u_0 + \delta u), \tag{20}$$

with $\delta\mathbf{q} = [\delta s\ \ \delta\theta]^T$ is a vector containing disturbed motion,

$$\mathbf{h}_0(\mathbf{q}_0, \dot{\mathbf{q}}_0) = \begin{bmatrix} \frac{K_m K_e + R(b+b_1)}{Rr^2}\dot{s} \\ b_1\dot{s}/r - m_2 g u_{2x} \sin\theta_0 \end{bmatrix} = \mathbf{L}u_0, \quad \mathbf{M}_0 = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$

$$\begin{aligned} m_{11} &= (J_1 + J_{ro})/r^2 + (m_1 + m_{ro} + m_2) \\ m_{12} &= m_2 u_{2x} \cos\theta_0 \\ m_{21} &= \tfrac{1}{r}(J_1 + J_{ro}) + (m_1 + m_{ro} + m_2)r + m_2 u_{2x} \cos\theta_0 \\ m_{22} &= J_2 + m_2(u_{2x}^2 + r u_{2x} \cos\theta_0) \end{aligned}$$

$$\mathbf{D}_0 = \begin{bmatrix} \frac{K_m K_e + R(b+b_1)}{Rr^2} & -\frac{K_m K_e + bR}{Rr} \\ \frac{b_1}{r} & 0 \end{bmatrix},$$

$$\mathbf{K}_0 = \begin{bmatrix} 0 & 0 \\ 0 & -m_2 g u_{2x} \cos\theta_0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} K_m/Rr \\ 0 \end{bmatrix}.$$

Putting $\mathbf{x} = [\delta\mathbf{q} \;\; \delta\dot{\mathbf{q}}]^T$, equation (20) can be transformed in the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\delta u, \tag{21}$$

where two matrices $\mathbf{A}$ and $\mathbf{B}$ are given as following

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{E}_{2\times 2} \\ -\mathbf{M}_0^{-1}\mathbf{K}_0 & -\mathbf{M}_0^{-1}\mathbf{D}_0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ -\mathbf{M}_0^{-1}\mathbf{L} \end{bmatrix}. \tag{22}$$

If a solution $\mathbf{x}(t) = 0$ of equation (21) is stable, the motion (17) will be stable. It is clear that the equilibrium ($\theta = \theta_0 \approx 0$) of inverted pendulum without control is always instable. So we can conclude that the desired motion (17) is instable. The problem of stabilization of the vehicle will be presented in the following section.

3. CONTROL DESIGN

The objectives of a control problem is to find a law of input voltage of the motor to ensure a given desired motion $s = s_d(t)$ while the bodywork is directed above, $\theta =$ const ≈ 0. A feedback control needs the actual state of the vehicle, so sensors to collect actual motion $x(t)$, $\dot{x}(t)$, $\theta(t)$, $\dot{\theta}(t)$ of the vehicle are necessary. There are many methods to design a controller for a linear system (21), in this study a linear feedback control will be applied to stabilize the motion of the vehicle. The diagram of this control law is given in Fig. 3.

The control voltage $u = u_0 + \delta u$ depends on desired and actual motion of the vehicle. The nominal voltage $u_0 = u(v_0)$ coresponding to a constant velocity v_0 is determined from equation (19). The stabilization of desired motion is ensured by the voltage δu, which is proportional to state vector $\mathbf{x}$ with a gain matrix $\mathbf{K}$

$$\delta u = -\mathbf{K}\mathbf{x} = -[k_1 \; k_2 \; k_3 \; k_4]\mathbf{x}. \tag{23}$$

Substituting (23) into (21) one yields

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}. \tag{24}$$

The elements of gain matrix $\mathbf{K}$ are chosen so that matrix owns negative eigenvalues. Firstly, we can choose eigenvalues of the matrix , and then determine the gain matrix $\mathbf{K}$. Let $\boldsymbol{p}$ be a vector containing eigenvalues of , one can obtain the gain matrix $\mathbf{K}$ by mean of Matlab.

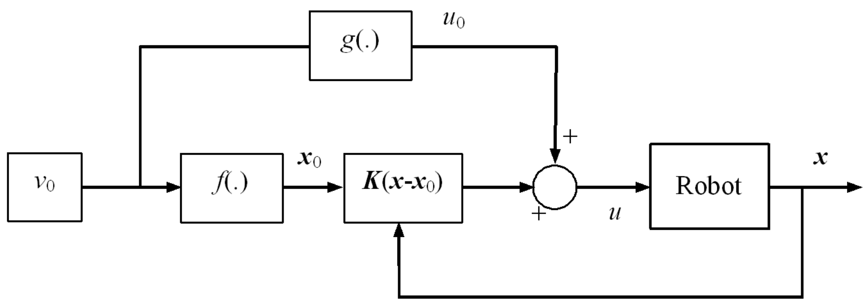

Fig. 3. Block-diagram of velocity control

Table 1. Vehicle's parameters

Parameters of bodywork and wheel			
	Notation	value	unit
Mass of wheel	m_1	0.51	kg
Mass inertia of wheel	J_1	5.1 x 10-4	kgm^2
Radius of wheel	r	0.062	m
Damping coefficient floor - wheel	b_1	5.73 x 10-3	Nm/(rad/s)
Mass of bodywork	m_2	9.01	kg
Mass inertia of bodywork	J_2	0.228	kgm^2
Center of mass	u_{2x}	0.138	m
Gravitational acceleration	g	9.81	m/s^2
Parameters of DC motor			
Mass inertia of rotor	J_{ro}	3.2 x 10-6	kgm^2
Damping coefficient journal - bodywork	b	4.25 x 10-3	Nm/(rad/s)
Armature resistance	R	1	Ω
Moment constant	K_m	1	Nm/A
back emf constant	K_e	1	Vs

4. NUMERICAL SIMULATION

In oder to simulate by mean of Matlab, the parameters of the vehicle from [2] are used (Table 1).

When the vehicle has a speed of 5 m/s (18 km/h), from equation (19) the orientation of the bodywork is calculated:

$$\theta_0 = 0.0379 \text{ rad } (= 2.17^0),$$

Because the angle θ_0 is small, instead of studying stability of the system at θ_0 we can consider the stability at the vertical position of the bodywork, $\theta_0 = 0$. Using parameters in table 1, two matrices **A** and **B** are given as:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -6.6997 & -55.2757 & 3.4108 \\ 0 & 51.3728 & 212.5356 & -13.1264 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -3.3963 \\ 13.0709 \end{bmatrix} \tag{25}$$

Eigenvalues of the matrix $\mathbf{A}$ are [0, -68.8401, -4.3213, 4.7592], therefore at the position where the bodywork is directed towards above, the vehicle without control is instable, one eigenvalue of matrix $\mathbf{A}$ is positive.

Controlling the vehicle can be divided into two problems: position control and velocity control. In position control, the vehicle need to be kept at a certain position (s, 0). To do this we have to choose all four eigenvalues of $\mathbf{A}_c$ to be negative. For example, here we choose

$\boldsymbol{p}$ = [-2 -3 -5 -6]^T

and a gain matrix $\boldsymbol{K}$ is obtained by mean of Matlab as follows

$\boldsymbol{K}$ = [-2.0711 -11.3372 -18.6895 -0.8472].

The simulation results with above gain matrix $\boldsymbol{K}$ are shown in Fig. 4. The vehicle reaches the desired position after about 4 s.

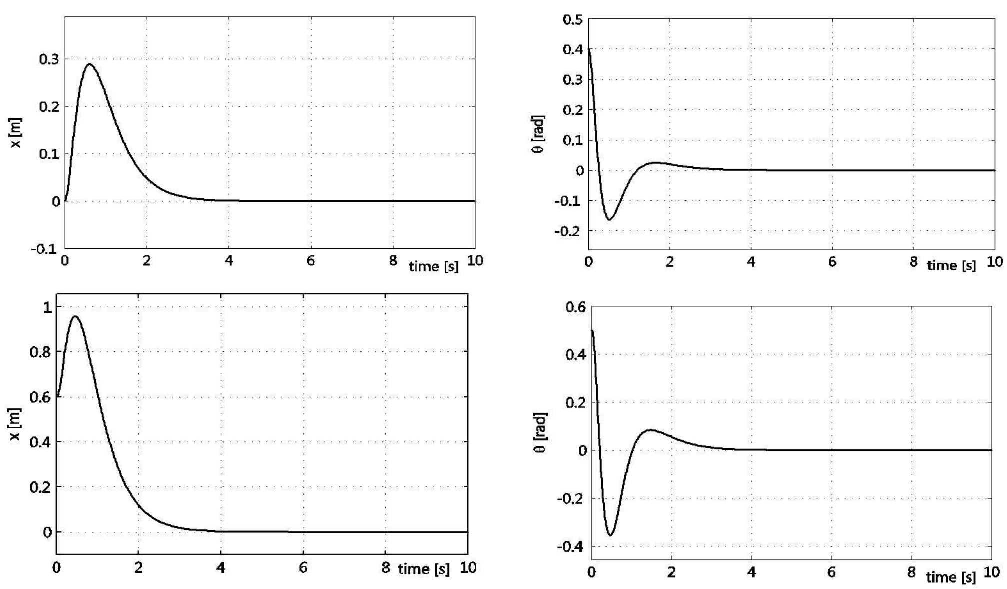

Fig. 4. Position control with different initial conditions

In velocity control, the vehicle need to be kept at a constant velocity, while its orientation is small. To do this we set the first eigenvalue of $\mathbf{A}_c$ to be zero and other three to be negative. For example, here we choose

$$p = [0 \;\; -3 \;\; -4 \;\; -3.5]^T$$

and a gain matrix $\boldsymbol{K}$ is obtained by mean of Matlab as follows

$$\mathbf{K} = [0 \;\; -6.6772 \;\; -16.7229 \; 0.0843].$$

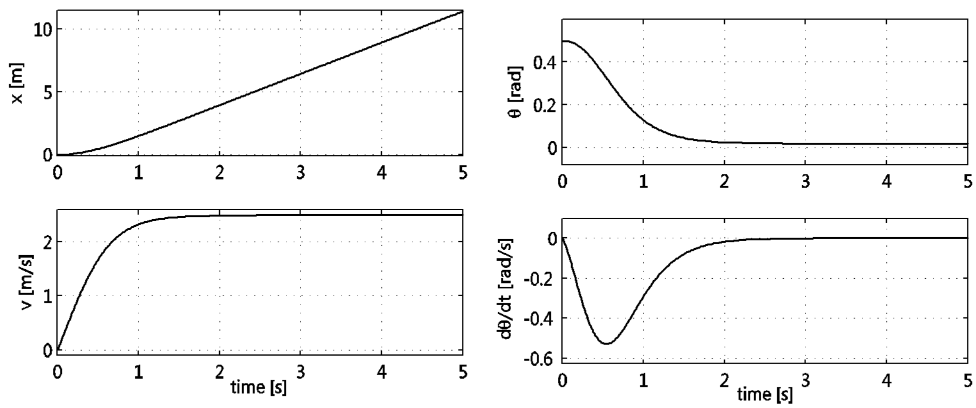

Fig. 5. Results of velocity control

The simulation results are shown in Fig. 5. After about 3 seconds the vehicle reaches the desired velocity.

5. CONSLUSION

In this study, deriving the differential equations of motion of self-balanced two-wheel vehicle and designing a controller to maintain vehicle motion are presented. The method of substructures has been exploited successfully in establishing equations of motion of the vehicle. Because the system has two degrees of freedom but only one actuator, a consideration of controllability become necessary. The disturbed motion is examined by using linearilzed equations, so it is not difficult to choose parameters of the feedback controller by mean of universal software Matlab. Some other problems should be considered further such as the uncertainties of vehicle parameters. The contact between the wheel and the surface of the road should be modeled in more details.

ACKNOWLEDGMENT

This paper was completed with the financial support of National Foundation for Science and Technology Development (NAFOSTED).

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Received September 25, 2008

VỀ VIỆC MÔ HÌNH HÓA VÀ ĐIỀU KHIỂN XE HAI BÁNH KIỂU CON LẮC NGƯỢC TỰ CÂN BẰNG

Trong bài báo này, bài toán mô hình hóa và thiết kế điều khiển cho rôbốt di động kiểu con lắc ngược được trình bày. Phương pháp tách cấu trúc được sử dụng để thiết lập phương trình vi phân chuyển động của rôbốt. Phương trình chuyển động sau khi tuyến tính hóa là cơ sở để thiết kế bộ điều khiển phản hồi cho rôbốt. Các kết quả mô phỏng số chỉ ra tính hiệu quả của bộ điều khiển.