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A UNIFIED KRYLOV-BOGOLIUBOV-MITROPOLSKII METHOD FOR SOLVING HYPERBOLIC-TYPE NONLINEAR PARTIAL DIFFERENTIAL SYSTEMS

M. Zahurul Islam
Department of Applied Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh
M. Shamsul Alam and M. Bellal Hossain
Department of Mathematics, Rajshahi University of Engineering and Technology, Rajshahi-6204, Bangladesh, Email address: msalam1964@yahoo.com

Abstract. A general asymptotic solution is presented for investigating the transient response of non-linear systems modeled by hyperbolic-type partial differential equations with small nonlinearities. The method covers all the cases when eigen-values of the corresponding unperturbed systems are real, complex conjugate, or purely imaginary. It is shown that by suitable substitution for the eigen-values in the general result that the solution corresponding to each of the three cases can be obtained. The method is an extension of the unified Krylov-Bogoliubov-Mitropolskii method, which was initially developed for un-damped, under-damped and over-damped cases of the second order ordinary differential equation. The methods also cover a special condition of the over-damped case in which the general solution is useless.
Keywords: Unified KBM method, Oscillation, Non-oscillation

1. INTRODUCTION

Krylov-Bogoliubov-Mitropolskii (KBM) [1, 2] method is one of the widely used techniques to obtain analytical solutions of weakly nonlinear ordinary differential equations. The method was originally developed to find periodic solutions of second-order nonlinear ordinary differential equations. Popov [3] extended the method to damped nonlinear systems. Murty, Deekshatulu and Krisna [4] investigated nonlinear over-damped systems by this method. Murty [5] used their earlier solution [4] as a general solution for un-damped, damped and over-damped cases, which is the basis of the unified theory. Since Murty's technique is a generalization of the KBM method, many authors extended this technique in various oscillatory and non-oscillatory systems. Bojadziev and Edwards [6] investigated nonlinear damped oscillatory and non-oscillatory systems with varying coefficients following Murty's [5] unified method. Recently Shamsul [7, 8] has presented a unified formula to obtain a general solution of an n-th order ordinary differential equation with constant and slowly varying coefficients. The KBM method was extended to partial differential equation with small nonlinearities by Mitropolskii and Mosenkov [9]. Bojadziev and Lardner [10] extended the KBM method to hyperbolic-type partial differential equation

$$\rho(x)u_{tt} = \frac{\partial}{\partial x}(\chi(x)u_x) + \varepsilon f(x, u, u_x, u_t), \tag{1}$$

where the subscript denotes differentiations, ε is a small parameter and $f$is a given nonlinear function.

Bojadziev and Lardner [10] mainly investigated the mono-frequent solution of (1). Bojadziev and Lardner [11] also found mono-frequent type solutions of the partial differential equation with a linear damping effect, $-2\rho(x)ku_t$, of the form

$$\rho(x)(u_{tt}+2ku_t)=\frac{\partial}{\partial x}(\chi(x)u_x)+\varepsilon f(x,u,u_x,u_t). \tag{2}$$

In another recent paper, Shamsul, Akber and Zahurul [12] present a general formula to investigate a class of nonlinear partial differential equations. In this paper a general asymptotic solution of (2) is obtained which covers the over-damped, damped and un-damped cases. Thus, the unified KBM method [5] is independent of whether the unperturbed system has real, complex conjugate, or purely imaginary eigen-values whether described by an ordinary or partial differential equation. Moreover a special over-damped solution is obtain which is essential when the general solution fails to give desired results (See [13] for details).

2. THE METHOD

Let us consider that $u(x,t,\varepsilon)$ satisfies a pair of homogeneous boundary conditions involving u and its derivatives at $x=0$ and $x=l$:

$$B_j(u)\equiv\beta_{j1}u(0,t)+\beta_{j2}u_x(0,t)+\beta_{j3}u(l,t)+\beta_{j4}u_x(l,t),\quad j=1,\ 2, \tag{3}$$

where β_{jr}, $j=1,\ 2$ and $r=1,\ 2,\ 3,\ 4$ are eight constants.

The investigation of mono-frequent damped oscillations of equation (2) is of interest in certain problems occurring in mechanics. For instance, such an equation describes the vibration of certain nonlinear elastic system in the presence of strong viscous damping. We shall examine in detail the longitudinal vibrations of a rod. The material of rod is taken to be predominately *Hooken* but with, in addition, small nonlinear elastic characteristic.

First of all, we consider the unperturbed system (2)

$$\rho(x)(u_{tt}^{(0)}+2ku_t^{(0)})=\frac{\partial}{\partial x}(\chi(x)u_x^{(0)}), \tag{4}$$

with boundary conditions (3).

It is well known that with prescribed boundary conditions (3) satisfying certain self-adjoin ness, (4) has a complete set of separable solutions which can be written in the form

$$\phi_n(x)e^{-kt}a_{n,0}\sinh(\omega_n t+\psi_{n,0}),\quad n=1,\ 2,\cdots \tag{5}$$

where $a_{n,0}$ and $\psi_{n,0}$ are arbitrary constants. The set of functions $\{\phi_n(x)\}$, $n=1,\ 2,\cdots$ satisfy the ordinary differential equation

$$\frac{d}{dx}\left(\chi(x)\frac{d\phi_n(x)}{dx}\right)+(k^2-\omega_n^2)\rho(x)\phi_n(x)=0,\quad B_j(\phi_n(x))=0,\quad j=1,\ 2. \tag{6}$$

In order to solve oscillating processes, Bojadziev and Lardner [11] assumed damping is less than critical, *i.e.*, $\omega_n^2>0$. Here we remove this restriction and consider more general ω_s such that $\omega_s^2>0$ or/and $\omega_s^2<0$. It is to be noted that eigen-values are determined from boundary conditions (3). Let us consider $\{\phi_n(x)\}$ are normalized, so that

$$\int_0^l \rho(x)\ \phi_m(\mathrm{x})\phi_n(\mathrm{x})\mathrm{dx}=\delta_{m,n}. \tag{7}$$

Now we shall find a mono-frequent solution of (2) for which $\varepsilon = 0$ corresponds to frequency ω_1. Following the KBM method, we look for a solution of the form

$$u(x,t,\varepsilon) = \phi_1(x)a(t)\sinh\psi(t) + \varepsilon u_1(x,a,\psi) + \mathrm{O}(\varepsilon^2)\cdots, \tag{8}$$

where a and ψ satisfy the equations

$$\dot{a} = -ka + \varepsilon A_1(a) + \mathrm{O}(\varepsilon^2)\cdots, \quad \dot{\psi} = \omega_1 + \varepsilon B_1(a) + \mathrm{O}(\varepsilon^2)\cdots. \tag{9}$$

Substituting (8) into (2), making use of (9) and comparing the coefficients of ε, we obtain

$$\begin{aligned} &\rho(x)\phi_1(x)\Big[\Big(2\omega_1 A_1 - ka^2\frac{dB_1}{da}\Big)\cosh\psi + \Big(-ka\frac{dA_1}{da} + kA_1 + 2\omega_1 aB_1\Big)\sinh\psi\Big] \\ &+ \rho(x)\Big[\Big(-ka\frac{\partial}{\partial a} + \omega_1\frac{\partial}{\partial\psi}\Big)^2 + 2k\Big(-ka\frac{\partial}{\partial a} + \omega_1\frac{\partial}{\partial\psi}\Big)\Big)u_1 = \frac{\partial}{\partial x}\Big(\chi(x)\frac{\partial u_1}{\partial x}\Big] + f^{(0)}, \end{aligned} \tag{10}$$

where $f^{(0)} = f(x, u_0, u_{0,x}, u_{0,t})$ and $u_0 = \phi_1(x)a(t)\sinh\psi(t)$.

Let us expand u_1 as a Fourier series in x using the basis $\{\phi_n(x)\}$

$$u_1(x,a,\psi) = \sum_{j=1}^{\infty} v_j(a,\psi)\phi_j(x). \tag{11}$$

Substituting (11) into (10), multiplying both sides by $\phi_s(x)$ and integrating with respect to x within limits from 0 to l, and making use of (6) and (7) gives

$$\begin{aligned} &\Big[\Big(2\omega_1 A_1 - ka^2\frac{dB_1}{da}\Big)\cosh\psi + \Big(-ka\frac{dA_1}{da} + kA_1 + 2\omega_1 aB_1\Big)\sinh\psi)\Big]\delta_{1,s} \\ &+ \Big[\Big(-ka\frac{\partial}{\partial a} + \omega_1\frac{\partial}{\partial\psi} + k\Big)^2 - \omega_s^2\Big]v_s = F_s(a,\psi), \end{aligned} \tag{12}$$

where

$$F_s(a,\psi) = \int_0^l f^{(0)}(x,a,\psi)\phi_s(x)dx. \tag{13}$$

It is customary to solve (12) for unknown functions A_1, B_1 and v_s, $s = 1, 2, \cdots$ under the assumption that $v_1$does not contain fundamental terms involving $\sinh\psi$ and $\cosh\psi$ (see [4, 5] for details). It is assumed that F_s is expanded as a series of hyperbolic functions

$$F_s = F_{s,0}(a) + F_{s,1}(a)\cosh\psi + F_{s,2}(a)\cosh 2\psi\cdots + G_{s,1}(a)\sinh\psi + G_{s,2}(a)\sinh 2\psi + \cdots \tag{14}$$

It is noted that series (14) becomes a Fourier series when the motion is un-damped or under-damped, *i.e.*, ω_1 as well as ψ, $G_{s,1}$, $G_{s,2}$ are purely imaginary.

Substituting the values of F_s from (14) into (12) and assuming that v_1 excludes terms involving $\sinh\psi$ and $\cosh\psi$, we obtain 0

$$2\omega_1 A_1 - ka^2\frac{dB_1}{da} = F_{1,1}, \tag{15}$$

$$-ka\frac{dA_1}{da} + kA_1 + 2\omega_1 B_1 = G_{1,1}, \tag{16}$$

$$\Big[\Big(-ka\frac{\partial}{\partial a} + \omega_1\frac{\partial}{\partial\psi} + k\Big)^2 - \omega_1^2\Big]v_1 = F_{1,0} + F_{1,2}\cosh 2\psi + G_{1,2}\sinh 2\psi\cdots \tag{17}$$

and

$$\left[\left(-ka\frac{\partial}{\partial a}+\omega_1\frac{\partial}{\partial \psi}+k\right)^2-\omega_s^2\right]v_s=(F_{s,0}+F_{s,1}\cosh\psi+G_{s,1}\sinh\psi,\ s\geq 2. \tag{18}$$

The particular solutions of (15)-(18) give unknown functions A_1, B_1 and v_s, $s=1,\ 2,\cdots$, which complete the determination of the first order solution of (2). The method can be extended to higher order approximations in a similar way.

3. EXAMPLE

As an example of the above procedure we may consider the longitudinal vibration of a nonlinear elastic rod described by equation

$$\rho u_{tt}+\kappa u_t=\sigma_x, \tag{19}$$

where u is longitudinal displacement, σ axial tension, ρ mass per unit length. The term $2\kappa u_t$ represents viscous damping. The stress-strain relation is assumed as

$$\sigma=Ke+\frac{1}{3}\varepsilon Ee^3, \tag{20}$$

where K is Young's modulus, $e=u_x$ axial strain and the second term containing E represents nonlinear elastic behavior. Eliminating σ from (19) and (20) and substituting $\kappa=2\rho k$, we obtain a partial differential equation in the form

$$\rho(u_{tt}+2ku_t)=Ku_{xx}+\varepsilon Eu_x^2u_{xx}. \tag{21}$$

Let us consider the boundary conditions

$$u(0,t)=0,\ hu_x(l,t)+u(l,t)=0. \tag{22}$$

Applying boundary conditions (22), we obtain the eigen-functions and eigen-values of the unperturbed 21) as:

$$\phi_n(x)=c_n\sin p_nx,\quad \lambda_n^2=\frac{Kp_n^2}{\rho},\quad n=1,\ 2,\cdots \tag{23}$$

where $\{p_n\}$ are the eigen-values of equation

$$\tan pl=-hp, \tag{24}$$

and constants $\{c_n\}$ satisfy

$$\frac{\rho c_n^2}{2}(l+\frac{h}{1+h^2p_n^2})=1,\quad n=1,\ 2,\cdots. \tag{25}$$

In (21), the nonlinear function is $f=Eu_x^2u_{xx}$. Therefore, we have

$$F_s=E_s\sinh^3\psi=\frac{1}{4}E_sa^3(\sinh 3\psi-3\sinh\psi), \tag{26}$$

where $E_s=E\int_0^l(\frac{d\phi_1}{dx})^2\frac{d^2\phi_1}{dx^2}\phi_s dx$. Therefore, only non-zero coefficients of $F_{s,n}$ and $G_{s,n}$, $n=1,\ 2,\cdots$, are $G_{s,1}=-\frac{3}{4}E_sa^3$ and $G_{s,3}=\frac{1}{4}E_sa^3$. Substituting the values of $G_{s,1}$ and $G_{s,3}$ into (15)-(18) and solving them, we obtain

$$A_1=\frac{3E_1ka^3}{8(k^2-\omega_1^2)},\quad B_1=\frac{3E_1\omega_1a^2}{8(k^2-\omega_1^2)}, \tag{27}$$

$$v_1 = \frac{E_1 a^3(3k\omega_1 \cosh 3\psi + (k^2 + 2\omega_1^2)\sinh 3\psi)}{16(k^2 - \omega_1^2)(k^2 - 4\omega_1^2)}, \quad k \neq 2\omega_1 \tag{28}$$

and

$$v_s = \frac{-E_s a^3(4k\omega \cosh\psi + (4k^2 + \omega_s^2 - \omega_1^2)\sinh\psi)}{4(4k^2 - (\omega_s - \omega_1)^2)(4k^2 - (\omega_s + \omega_1)^2)}$$
$$+ \frac{3E_s a^3(12k\omega_1 \cosh 3\psi + (4k^2 + \omega_s^2 - 9\omega_1^2)\sinh 3\psi)}{4(4k^2 - (\omega_s - 3\omega_1)^2)(4k^2 - (\omega_s + 3\omega_1)^2)}, \quad s \geq 2. \tag{29}$$

Substituting the values of A_1 and B_1 from (27) into (9), we integrate them with respect to t, and obtain

$$a = \frac{a_0 e^{-kt}}{\sqrt{1 - \frac{3\varepsilon E_1 a_0^2(1-e^{-2kt})}{8(k^2-\omega_1^2)}}}, \quad \psi = \psi_0 + \omega_1 t - \frac{\omega_1}{2k}\ln\left(1 - \frac{3\varepsilon E_1 a_0^2(1 - e^{-2kt})}{8(k^2 - \omega_1^2)}\right), \tag{30}$$

Thus the first order solution of (19) is

$$u(x, t, \varepsilon) = \phi_1(x) a \sinh\psi + \varepsilon \sum_{s=1}^{\infty} \phi_s(x) v_s(a, \psi), \tag{31}$$

where a, ψ, v_1 and v_s, $s \geq 2$ are given respectively by (30), (28) and (29). In the case of an under-damped system, all ω_s are replaced by $i\omega_s$, a by $-ia$, ψ by $i\psi$, $\cosh i\psi$ by $\cos\psi$ and $\sinh i\psi$ by $i\sin\psi$. These yield

$$u(x, t, \varepsilon) = \phi_1(x) a \sin\psi + \varepsilon \sum_{s=1}^{\infty} \phi_s(x) v_s(a, \psi), \tag{32}$$

where a, ψ, v_1 and v_s, $s \geq 2$ are given by

$$a = a_0 e^{-kt}\left(\sqrt{1 + \frac{3\varepsilon E_1 a_0^2(1 - e^{-2kt})}{8(k^2 + \omega_1^2)}}\right)^{-1},$$
$$\psi = \psi_0 + \omega_1 t - \frac{\omega_1}{2k}\ln\left(1 + \frac{3\varepsilon E_1 a_0^2(1 - e^{-2kt})}{8(k^2 + \omega_1^2)}\right), \tag{33}$$

$$v_1 = -\frac{E_1 a^3(3k\omega_1 \cos 3\psi + (k^2 - 2\omega_1^2)\sin 3\psi)}{16(k^2 + \omega_1^2)(k^2 + 4\omega_1^2)} \tag{34}$$

and

$$v_s = \frac{3E_s a^3(4k\omega \cos\psi + (4k^2 + \omega_s^2 - \omega_1^2)\sin\psi)}{4(4k^2 + (\omega_s - \omega_1)^2)(4k^2 + (\omega_s + \omega_1)^2)}$$
$$- \frac{E_s a^3(12k\omega_1 \cos 3\psi + (4k^2 + \omega_s^2 - 9\omega_1^2)\sin 3\psi)}{4(4k^2 + (\omega_s - 3\omega_1)^2)(4k^2 + (\omega_s + 3\omega_1)^2)}, \quad s \geq 2. \tag{35}$$

It is obvious that when $k > 0$, the solution is similar to Bojadziev and Lardner [11], and identical to that obtained in [9] when $k = 0$.

4. A SPECIAL DAMPING CONDITION

Clearly v_1 (See (28)) is not defined for $k = 2\omega_1$. This situation occurs when the difference of $3\lambda_1$ and λ_2 or $3\lambda_2$ and λ_1 are significant (where λ_1 and λ_2 are the eigen-values of the corresponding unperturbed system (19)). In this case v_1 must contain a secular type term te^{-t} (See [13] for details). In this situation we seek a solution of the form [13],

$$u(x,t,\varepsilon) = \phi_1(x)(ae^{-\lambda t} + be^{-\mu t}) + \varepsilon u_1(x,a,b,t) + \mathrm{O}(\varepsilon^2), \tag{36}$$

where a and b satisfy the equations

$$\dot{a} = \varepsilon A_1(a,b,t) + \mathrm{O}(\varepsilon^2), \quad \dot{b} = \varepsilon B_1(a,b,t) + \mathrm{O}(\varepsilon^2). \tag{37}$$

Substituting (36) into (2), utilizing (37) and comparing the coefficients ofε, we obtain

$$\begin{aligned} &\rho(x)\phi(x)\Big[\Big(\frac{\partial A_1}{\partial t} - \lambda A_1 + \mu A_1\Big)e^{-\lambda t} + \Big(\frac{\partial B_1}{\partial t} - \lambda B_1 + \mu B_1)e^{-\mu t}\Big)\Big] \\ &+ \rho(x)\left(\frac{\partial^2}{\partial t^2} + 2k\frac{\partial}{\partial t}\right)u_1 = \frac{\partial}{\partial x}\left(\chi(x)\frac{\partial u_1}{\partial x}\right) + f^0, \end{aligned} \tag{38}$$

where $f^0 = f(x, u_0, u_{0,x}, u_{0,t})$ and $u_0 = \phi_1(x)(ae^{-\lambda t} + be^{-\mu t})$.

Let us expand u_1 as a Fourier series in x using the basis $\{\phi_n(x)\}$ as

$$u_1(x,a,b,t) = \sum_{j=1}^{\infty} v_j(a,b,t)\phi_j(x). \tag{39}$$

Substituting (39) into (38), multiplying both sides by $\phi_s(x)$ and integrating with respect to x within limits 0 to l, and making use of (6) and (7) gives

$$\begin{aligned} &\Big[\Big(\frac{\partial A_1}{\partial t} - \lambda A_1 + \mu A_1\Big)e^{-\lambda t} + \Big(\frac{\partial B_1}{\partial t} - \lambda B_1 + \mu B_1\Big)e^{-\mu t}\Big]\delta_{1,s} \\ &+ \Big(\frac{\partial}{\partial t} + \lambda\Big)\Big(\frac{\partial}{\partial t} + \mu\Big)v_s = F_s(a,b,t), \end{aligned} \tag{40}$$

where

$$F_s(a,b,t) = \int_0^l f^0(x, u_0, u_{0,x}, u_{0,t})\phi_s(x)dx. \tag{41}$$

In general, $F_s(a,b,t)$ can be expanded as a Taylor's series

$$F_s(a,b,t) = \sum_{j,r=0} F_{j,r}(a,b)e^{-(j\lambda + r\mu)t}. \tag{42}$$

In order to determine over-damped solutions of (1), we assume that u_1 does not contain terms with $e^{-(j\lambda + r\mu)t}$, where $j\lambda + r\mu < k(j+r)$, so that the coefficient of the expansion of u_1 does not become large, and u_1 does not contain the secular type term $te^{-(j\lambda + r\mu)t}$.

The function $F_s(a,\, b,\, t)$ becomes

$$F_s(a,b,t) = E_s(a^3e^{-3\lambda t} + 3a^2be^{-(2\lambda+\mu)t} + 3ab^2e^{-(\lambda+2\mu)t} + b^3e^{-3\mu t}). \tag{43}$$

Substituting the values of $F_s(a, b, t)$ from (43) into (40) we obtain the following equations

$$\left(\frac{\partial A_1}{\partial t} - \lambda A_1 + \mu A_1\right) = E_1 b^3 e^{-(3\mu-\lambda)t}, \tag{44}$$

$$\left(\frac{\partial B_1}{\partial t} - \lambda B_1 + \mu B_1\right) = E_1 a b^2 e^{-(\mu+\lambda)t} \tag{45}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu\right) v_s = E_1(a^3 e^{-3\lambda t} + 3ab^2 e^{-(2\lambda+\mu)t}. \tag{46}$$

Solving (44)-(46), we obtain

$$A_1 = \begin{cases} \dfrac{-E_1 b^3 e^{-(-\lambda+3\mu)t}}{2\mu(-\lambda+3\mu)}, & \lambda \neq 3\mu, \\ \dfrac{-E_1 b^3}{2\mu}, & \lambda = 3\mu, \end{cases} \tag{47}$$

$$B_1 = \frac{-E_1 a b^2 e^{-(\lambda+\mu)t}}{2\mu}, \tag{48}$$

and

$$v_1 = E_1\left(\frac{a^3 e^{-3\lambda t}}{2\lambda(3\lambda-\mu)} + \frac{3ab^2 e^{-(2\lambda+\mu)t}}{(\lambda+\mu)(2\lambda)}\right). \tag{49}$$

Substituting the values of A_1 and B_1 from (48) and (49) into (37), and integrating them with respect to t, we obtain

$$a = a_0 - \frac{E_1 \varepsilon b_0^3}{2\mu(-\lambda+3\mu)}(1 - e^{-(-\lambda+3\mu)t}), \quad \lambda \neq 3\mu,$$

$$= a_0 - \frac{E_1 \varepsilon b_0^3}{2\mu} t, \quad \lambda = 3\mu, \tag{50}$$

$$b = b_0 - \frac{E_1 \varepsilon a_0 b_0^2}{2\mu(\lambda+\mu)}(1 - e^{-(\lambda+\mu)t}). \tag{51}$$

Thus the first order special over-damped solution of (19) is

$$u(x, t, \varepsilon) = \phi_1(x)(a e^{-\lambda t} + b e^{-\mu t} + \varepsilon v_1), \tag{52}$$

where a, b and v_1 are given by (50), (51) and (49) respectively.

5. RESULTS AND DISCUSSION

A unified solution is found for a nonlinear partial differential equation based on the works of Murty *et. al.* [4, 5]. In order to test the accuracy of this unified solution, we compare the solution to the numerical solution (consider to be exact). With regard to such a comparison concerning the presented unified method of this paper, we refer to a recent work by Shamsul [7] and as well as a previous work by Murty, Deekshatulu and Krisna [4]. Moreover, we compare the perturbation solution to the unperturbed solution to denote the response of the nonlinear term.

The solution (31) is well established and useful as an over-damped solution of (19). We are interested in comparing it with numerical solution (generated by finite difference method). Let us consider an over-damped case of (19), in which $\rho = 1$, $k = 1.25$, $l = 2$, $K = 1$. The solutions to (24) are 1.144465, 2.543493, 4.048082 $\cdots$, eigen-values are -1.25 ± 0.502693, $-1.25 \pm 2.215143i$, $-1.25 \pm 3.850256i \cdots$, and one set of eigen-value is real and $\omega_1 = 0.502693$. For $\varepsilon = 0.2$ and initial values $[u(x,0) = 0.90667 \sin(1.144465x),\ u_t(x,0) = 0\,]$, $u(x,t,\varepsilon)$ has been evaluated and the corresponding numerical solution of (19) has been computed. The results for $x = 2$ (*i.e.* for the lower end of the rod) and $x = 1$ (*i.e.* for the middle point of the rod) are presented respectively in Fig. 1(a) and Fig. 1(b). From the figures it is clear that solution Eq. (31) compares well the numerical solution.

The solution (32) is also well established and useful as an un-damped and under damped solution of (19). Let us consider the un-damped case of (19), in which, $\rho = 1$, $k = 0.0$, $l = 2$, $K = 1$. The solution of (24) are 1.144465, 2.543493, 4.048082 $\cdots$ or eigen values are 1.144465, 2.543493, 4.048082 $\cdots$. For $\varepsilon = 0.2$ and for initial values $[u(x,0) = 0.90667 \sin(1.144465x),\ u_t(x,0) = 0\,]$, $u(x,t,\varepsilon)$ has been evaluated and the corresponding numerical solution of (19) computed. The results for $x = 2$ respecyively and $x = 1$ are presented respectively in Fig. 2(a) and Fig. 2 (b). From the figures, it is clear that solution (32) compares well with the numerical solution.

For the under damped case, we consider $\rho = 1$, $k = 0.2$, $l = 2$, $K = 1$. The solutions of (24) are 1.144465, 2.543493, 4.048082 $\cdots$or eigen-values are -0.2 ± 1.126854, -0.2 ± 2.535618, $-0.2 \pm 4.043138 \cdots$. For $\varepsilon = 0.5$ and for initial values $[u(x,0) = 0.90667 \sin(1.144465x), u_t(x,0) = 0]$, $u(x,t,\varepsilon)$ evaluated and the corresponding numerical solution of (19) has been computed. The results for $x = 2$ and $x = 1$ are presented in Fig. 3 (a) and Fig.3 (b) respectively. From the figures, it is clear that solution (32) compares well with the numerical solution.

When $k = 2\omega_1$, then solution Eq. (52) is useful for an over-damped solution of equation (19). We are interested to compare it with numerical solution (generated by finite difference method). Let us consider$\rho = 1$, $k = 1.3215$, $l = 2$, $K = 1$. The solutions of (24) are1.144465, 2.543493, 4.048082 $\cdots$ or eigen-values are -1.3215 ± 0.660728, $-1.3215 \pm 2.173245i$, $-1.3215 \pm 3.826304i \cdots$ and one set of eigen-value is real. For $\varepsilon = 0.5$ and initial values $[u(x,0) = 0.90667 \sin(1.144465x),\ u_t(x,0) = 0]$, $u(x,\ t,\ \varepsilon)$ has been evaluated and the corresponding numerical solution of (19) has been computed. The results for $x = 2$ respectively and $x = 1$ are presented respectively in Fig. 4(a) and Fig. 4(b). From the figures, it is clear that solution Eq. (52) compares well with the numerical solution.

6. CONCLUSION

A general formula is presented for obtaining the transient response of nonlinear systems governed by a hyperbolic-type partial differential equation with small nonlinearities. According to the unified theory [4, 5] there exists a general solution, used in three cases, *i.e.* over-damped, under-damped and un-damped. In previous papers [5, 7] only ordinary differential equations are considered. In the present paper, we observe a similar result for partial differential equations.

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GIẢI HỆ PHƯƠNG TRÌNH ĐẠO HÀM RIÊNG PHI TUYẾN DẠNG HYPERBOLIC BẰNG PHƯƠNG PHÁP KROLOV-BOGOLIUBOV-MITROPOLSKII

Một nghiệm tiệm cận tổng quát được biểu diễn để khảo sát đặc trưng của hệ phi tuyến được mô hình bằng các phương trình đạo hàm riêng dạng hyperbolic với hệ số phi tuyến bé. Phương pháp bao gồm tất cả các trường hợp khi các giá trị riêng của hệ không nhiễu loạn tương ứng là thực, liên hợp phức, thuần ảo. Nó cho thấy bằng sự thay thế phù hợp của các giá trị riêng trong kết quả tổng quát, nghiệm tương ứng với mỗi trường hợp trong ba trường hợp là có thể thu được. Phương pháp này là một sự mở rộng của phương pháp Krolov-Bogoliubov-Mitropolskii, nó là sự phát triển ban đầu cho các trường hợp tắt dần, tắt dần chậm, tắt dần quá của phương trình đạo hàm thường bậc hai. Các phương pháp cũng bao gồm điều kiện đặc biệt của trường hợp tắt dần quá trong đó nghiệm tổng quát là không có nghĩa.