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FREE VIBRATION OF PRESTRESS TIMOSHENKO BEAMS RESTING ON ELASTIC FOUNDATION

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Abstract. This paper presents a finite element formulation for investigating the free vibration of uniform Timoshenko beams resting on a Winkler-type elastic foundation and prestressing by axial force. Taking the effect of prestress, foundation support and shear deformation into account, a stiffness matrix for Timoshenko-type beam element is formulated using the energy method. The element consistent mass matrix is obtained from the kinetic energy using simple linear shape functions. Employing the formulated element, the natural frequencies of the beams having various boundary conditions are determined for different values of the axial force and foundation stiffness. The vibration characteristics of the beams partially supported on the foundation are also studied and highlighted. Specially, the effects of shear deformation on the vibration frequencies of prestress beams fully and partially supported on the elastic foundation are investigated in detail.

1. INTRODUCTION

Practical problems like railroad tracks, hight way pavements, continuously pipelines ... can be modelled by means of beams on elastic foundation. Static analysis of beams on various types of foundation has been extensively carried out by many researcher [1, 2]. In the context of dynamic analysis, in [3] Rosa described an analytical approach for investigating the effect of foundation support on the vibration characteristics of Timoshenko beams resting on a Pasternak foundation. Using the so-called Rayleigh-Ritz method, Rao has investigated the large vibration characteristics of simply supported and cantilever Timoshenko beams resting on a two-parameter foundation [4]. By solving the governing equation, Hung [5] derived the stiffness and mass matrices used in dynamic stiffness method in asserting the natural frequencies of shear deformable beams resting on a Winkler foundation and under axial force. The method may result in accurate frequencies but requires complex mathematical forms of the stiffness and mass matrices.

From practical point of view, beam prestressed by axial force is widely use as a structural element in civil engineering, since its superiority in sustaining mechanical forces in particular applications. With the presence of axial force, the prestress beam may have different static and dynamic characteristics, comparing to its conventional counterpart. The effect of axial force may be explained by alternation of the bending stiffness of the structural element, and for the case of beam, the vibration frequency is remarkably reduced by the compressive membrane force [6].

Very recently, the author investigated the vibration frequency of slender beams resting on a Winkler foundation by using the finite element method [7]. In the work, a Bernoulli-type beam element has been developed and employed to compute the frequency of the beams using various types of mass matrices. The main objective of this paper is to extend the work in [7] to the case of Timoshenko beams, so that the effect of shear deformation on the dynamic characteristics of beams can be examined in detail. In addition to the shear deformation effect, the influence of partial support by the foundation on the dynamic characteristics of the beams which has not been studied in the cited references is also investigated.

In the context of the finite element analysis, the main difference between the Timoshenko and Bernoulli beams is the ability of adopting different shape functions to interpolate the displacement fields. For the Timoshenko beam, with the introduction of shear deformation, the rotation becomes an independent variable, and the linear functions can be adopted, while for the Bernoulli beam, the cubic polynomials are the lowest order shape functions [8]. Thus, the Bernoulli element may be better in representing the deformed configuration of beam, but in addition to the simplicity of the finite element formulation, the Timoshenko beam has ability in modelling the shear deformation effect, which may be important for the case of stubby beams.

Following the above introduction, the remainder of this paper is organized as follows: Section 2 formulates the stiffness and consistent mass matrices for the prestress Timoshenko beam element resting on an elastic foundation. Section 3 describes the equations of motion for the case of free vibration of a finite element model. The numerical investigation is presented in Section 4. The main conclusions of the paper are summarized in Section 5.

2. FINITE ELEMENT FORMULATION

2.1. Element stiffness matrix

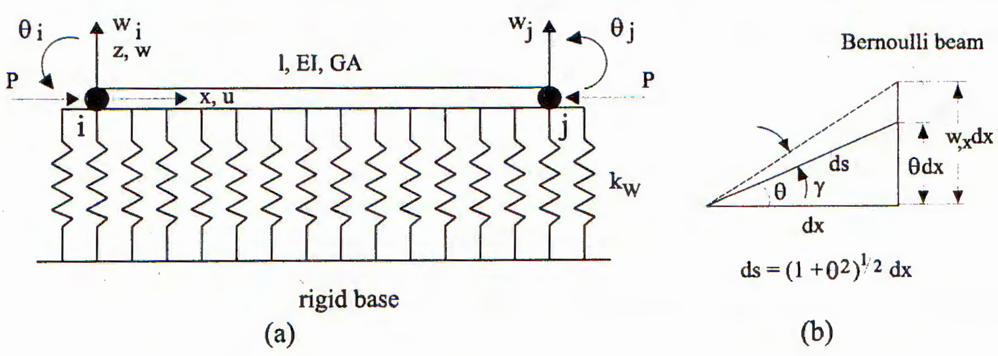

Fig. 1. (*a*) A two-node beam element, (*b*) Geometric relation for a differential element of length dx

Consider a two-node (denoted **i** and **j**) beam element with length l, flexural rigidity EI, shear rigidity GA, prestressed by axial force P as shown in Fig. 1a. The beam is supported on a traditional Winkler elastic foundation, which being modelled by linear springs with stiffness k_W (unit of $force/length^2$). In this Winkler model, the springs are assumed to be independent of each other, and only one parameter k_W is represented for the foundation [1, 9]. The element contains four degree of freedom (d.o.f.), two at each node, namely a transversal displacement and a rotation. Thus, the vector of nodal displacements is given by

$$\mathbf{d} = \{w_i \; \theta_i \; w_j \; \theta_j\}^T, \tag{2.1}$$

where superscript ^{T} denotes the transpose of a vector or a matrix. Assuming linear elastic behavior, the strain energy of the element is obtained as a contribution from strain energy due to the bending and shear deformation of the beam U_B, the energy stemming from deformation of the foundation U_W, and energy of the axial force U_P. The strain energy stored in the beam element is simply given by

$$U_B = \frac{1}{2}\int_0^l EI\chi^2 dx + \frac{1}{2}\int_0^l \psi GA\gamma^2 dx, \tag{2.2}$$

where $\chi = \partial\theta/\partial x$ is the beam curvature; γ is the shear strain, and ψ is the correction factor to allow for cross-sectional warping [10]. The strain energy stemming from the deformation of foundation has a simple form

$$U_F = \frac{1}{2}\int_0^l k_W w^2 dx. \tag{2.3}$$

To derive the energy contributed by the axial force P, we consider herewith a differential element with initial length of dx as shown in Fig. 1b. Let a small lateral displacement $w(x)$ takes place, and denote ds is the new length of the differential element dx. For the case of Bernoulli beam, the rotation is $w_{,x}$, and the new length is computed as $ds = (1 + w_{,x}^2)^{1/2}$. However, taking the shear deformation into account, the total rotation of the element is not $w_{,x}$, but θ, and from Fig. 1b we get

$$ds = (1 + \theta^2)^{1/2} dx \approx (1 + \frac{1}{2}\theta^2)\, dx. \tag{2.4}$$

Thus, the axial membrane strain for the case of Timoshenko beam is given by

$$\epsilon_m = \frac{ds - dx}{dx} \approx \frac{1}{2}\theta^2. \tag{2.5}$$

During a small lateral displacement $w(x)$, the axial force P is still constant (positive in tension). As each element dx lengthens an amount $\epsilon_m dx$, the force P will produce work in amount of $P\epsilon_m dx$. Thus the change in membrane energy is

$$U_P = \frac{1}{2}\int_0^l P\theta^2 dx. \tag{2.6}$$

To this point, we follow the standard approach of the finite element method by introducing an interpolation scheme for the lateral displacement $w(x)$ and rotation $\theta(x)$ as

$$\begin{aligned} w &= N_i w_i + N_j w_j = \frac{l - x}{l} w_i + \frac{x}{l} w_j, \\ \theta &= N_i \theta_i + N_j \theta_j = \frac{l - x}{l} \theta_i + \frac{x}{l} \theta_j, \end{aligned} \tag{2.7}$$

where w_i, w_j, θ_i and θ_j are the values of the lateral displacement and rotation at nodes **i** and **j**. The shear strain $\gamma(x)$ is expressed in terms of the nodal displacements and rotations through

$$\gamma(x) = \frac{\partial w}{\partial x} - \theta. \tag{2.8}$$

Substituting Eqs. (2.7) and (2.8) into Eqs. (2.2)-(2.6), we get

$$\begin{aligned} U_B &= \frac{1}{2l} EI\,(\theta_j - \theta_i)^2 + \frac{1}{2}\psi GAl\left[\frac{w_j - w_i}{l} - \frac{1}{2}(\theta_i + \theta_j)\right]^2, \\ U_W &= \frac{1}{6}(w_i^2 + w_i w_j + w_j^2) l k_W, \\ U_P &= \frac{1}{6}(\theta_i^2 + \theta_i\theta_j + \theta_j^2) lP. \end{aligned} \tag{2.9}$$

In order to avoid the shear locking problem [11], in Eq. (2.9) we have used one-point Gauss quadrature to evaluate the shear strain of the beam as

$$\begin{aligned} \frac{1}{2}\int_0^l \psi GA\gamma^2 dx &= \frac{1}{2}\int_0^l \psi GA\left[\left(\frac{w_j - w_i}{l}\right) - \left(\frac{l-x}{l}\theta_i + \frac{x}{l}\theta_j\right)\right]^2 dx \\ &= \frac{1}{4}\int_{-1}^1 \psi GA\left[\left(\frac{w_j - w_i}{l}\right) - \frac{1}{2}(1-\xi)\theta_i - \frac{1}{2}(1+\xi)\theta_j\right]^2 d\xi. \\ &= \frac{1}{2}\psi GAl\left[\left(\frac{w_j - w_i}{l}\right) - \frac{1}{2}(\theta_i + \theta_j)\right]^2 \end{aligned} \tag{2.10}$$

From Eq. (2.9), the element stiffness matrix is obtained as the summation of stiffness matrices due to bending and shear deformation of the beam, stiffness matrices due to the foundation deformation, and due to the axial force. These matrices are obtained by twice differentiating the corresponding expressions of strain energy with respective to the nodal displacements, and having the form

$$\begin{aligned} \mathbf{k}_B &= \frac{EI}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \frac{\psi GA}{l}\begin{bmatrix} 1 & \frac{1}{2}l & -1 & \frac{1}{2}l \\ \frac{1}{2}l & \frac{1}{4}l^2 & -\frac{1}{2}l & \frac{1}{4}l^2 \\ -1 & -\frac{1}{2}l & 1 & -\frac{1}{2}l \\ \frac{1}{2}l & \frac{1}{4}l^2 & -\frac{1}{2}l & \frac{1}{4}l^2 \end{bmatrix} \\ \mathbf{k}_W &= \frac{1}{6}lk_W\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{k}_P = \frac{1}{6}lP\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{aligned} \tag{2.11}$$

The above stiffness matrices $\mathbf{k}_W$ and $\mathbf{k}_P$ contain many zero coefficients, which are different from the full matrices of Bernoulli beam element, previously derived in [7].

2.2. Element consistent mass matrix

A mass matrix is a discrete representation of a continuous distribution of mass. In the present work, the elastic foundation is considered massless as usually assumed in analysis of beams on foundation [4, 12]. Thus, the element mass matrix is contributed from the mass of the beam only. The kinematic energy of a uniform beam element with inclusion of shear deformation is given by [13]

$$T = \frac{1}{2}\int_0^l \rho A \dot{w}^2 dx + \frac{1}{2}\int_0^l \rho I \dot{\theta}^2 dx, \tag{2.12}$$

where ρ is the mass density; A and I are the area and moment of inertia of cross-section, and $(\dot{...}) = d(...)/dt$ is the velocity of the quantity in the brackets. On the other hand, the kinematic energy can be expressed through the vector of nodal displacements as [13]

$$T = \frac{1}{2}\dot{\mathbf{d}}^T \mathbf{m}\,\dot{\mathbf{d}}, \tag{2.13}$$

where $\dot{\mathbf{d}} = d\mathbf{d}/dt$ with $\mathbf{d}$ defined by Eq. (2.1), and $\mathbf{m}$ is the mass matrix of the element. Substitute (2.7) into Eq. (2.12), we get

$$T = \frac{1}{6}\rho l A(\dot{w}_i^2 + \dot{w}_i\dot{w}_j + \dot{w}_j^2) + \frac{1}{6}\rho l I(\dot{\theta}_i^2 + \dot{\theta}_i\dot{\theta}_j + \dot{\theta}_j^2). \tag{2.14}$$

We can rewrite Eq. (2.14) in a matrix form as

$$T = \frac{1}{12}\rho l \dot{\mathbf{d}}^T \begin{bmatrix} 2A & 0 & A & 0 \\ 0 & 2I & 0 & I \\ A & 0 & 2A & 0 \\ 0 & I & 0 & 2I \end{bmatrix} \dot{\mathbf{d}}. \tag{2.15}$$

From Eqs. (2.13) and (2.15), we obtain the element mass matrix in the form

$$\mathbf{m} = \frac{1}{6}\rho l \begin{bmatrix} 2A & 0 & A & 0 \\ 0 & 2I & 0 & I \\ A & 0 & 2A & 0 \\ 0 & I & 0 & 2I \end{bmatrix}, \tag{2.16}$$

which is a constant positive define mass matrix. The mass matrix derived in this Subsection using the same shape functions as those of displacement field is called the consistent mass matrix [13, 14]. It is noted that the consistent mass matrix of the Timoshenko beam contains some zero coefficients, while that of the Bernoulli beam does not have any zero coefficient.

3. GOVERNING EQUATIONS

The equation of motion for the discretized undamped structure can be written in the forms [13, 15]

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{F}_{\text{ext}}, \tag{3.1}$$

where $\mathbf{D}$ is the vector of structural nodal displacements; $\mathbf{M}$ and $\mathbf{K}$ is the structural mass and stiffness matrices, respectively; $\mathbf{F}_{\text{ext}}$ is the vector of nodal external forces; $\ddot{\mathbf{D}} = \frac{d^2\mathbf{D}}{dt^2}$ is the acceleration of material particles at the structural nodes. The structural mass and stiffness matrices are formed by merging the element mass and stiffness matrices in the standard way of the finite element method

$$\mathbf{M} = \bigcup_{i=1}^{NE}\{\mathbf{m}\}_i; \quad \mathbf{K} = \bigcup_{i=1}^{NE}\{\mathbf{k}\}_i, \tag{3.2}$$

where $\mathbf{m}$ and $\mathbf{k}$ are the element mass and stiffness matrices, formulated in Section 2, and NE is the total element number of structure.

With no external forces, the structure undergoes harmonic motion (caused, perhaps by initial condition), and we can write $\mathbf{D} = \overline{\mathbf{D}} \sin \omega t$, with $\overline{\mathbf{D}}$ is the vibration amplitudes of the nodal displacements $\mathbf{D}$, and ω is the circular frequency (rad/s), so that we can write Eq. (3.1) in the form

$$(\mathbf{K} - \lambda \mathbf{M})\overline{\mathbf{D}} = 0, \tag{3.3}$$

where $\lambda = \omega^2$. Eq. (3.3) is called an eigenvalue problem, which gives nontrivial solution when λ satisfies

$$\det(\mathbf{K} - \lambda \mathbf{M}) = 0. \tag{3.4}$$

Eq. (3.3) can be solved using any standard algorithm to obtain eigenvalues λ and their associated eigenvectors. The frequency corresponding the lowest eigenvalue λ computed from Eq. (3.3) is called the fundamental frequency [15].

4. NUMERICAL INVESTIGATION

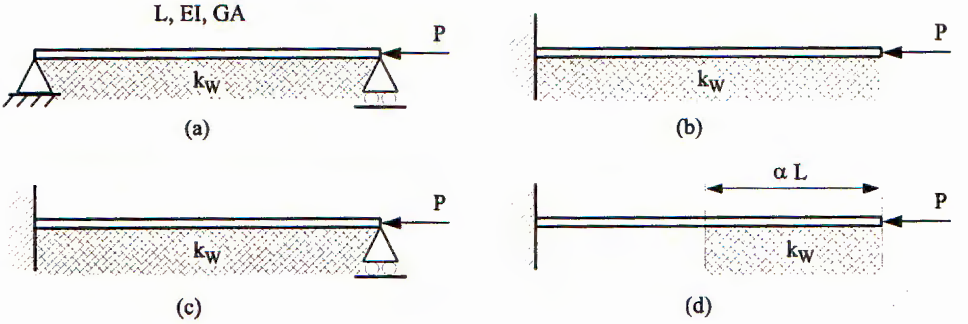

Fig. 2. Beams with with shear deformation for numerical investigation

The eigenvalue problem stated by Eq. (3.3) is formed using the finite element formulations developed in Sec. 2, then solved for the frequencies of prestress beams shown in Fig. 2. Various boundary conditions are considered: clamped at one end and free at other (denoted CF, Fig. 2a), simply supported (SS, Fig. 2b), clamped at one end and simply supported at other (CS, Fig. 2c). The geometry and material data for the beams are the same as those in [7], and listed below:

$L = 5\,m$; $A = 0.01\,m^2$, $I = 1 \times 10^{-5}\,m^4$,
$E = 2.1 \times 10^{11}\,N/m^2$,
$\nu = 0.3$, $\rho = 7\,860\,kg/m^3$,

where L, A, I, E, ν and ρ denote the total length, cross-sectional area, second moment of inertia of cross-section, elastic modulus, Poisson ratio and mass density of the beams, respectively. For the present study, the beam cross-section is rectangular, so that the correction factor ψ in Eq. (2.2) is taken by

$$\psi = \frac{10(1+\nu)}{(12+11\nu)}$$

For the convenience of discussion, we introduce the following dimensionless parameters

$$k_0 = \frac{L^4}{EI} k_W, \tag{4.1}$$

which represents the foundation stiffness, and

$$\mu = \frac{L^2}{EI} P, \tag{4.2}$$

which represents the axial force amplitude. Following the work in [7], we also introduce the so-called frequency parameter, defined as

$$\gamma = \frac{\rho A L^2}{EI} \omega_1^2, \tag{4.3}$$

where ω_1 denotes the fundamental frequency of the beam. Furthermore, in order to study the effect of shear deformation, we introduce herewith a parameter represented the shear deformation, defined as [16]

$$s = \frac{P_E}{GA}, \tag{4.4}$$

where P_E is the Euler buckling load of unsupported SS beam. Thus, for a higher slenderness parameter, the more effect of shear deformation is. In other words, according to Eq. (4.4), the shear deformation becomes more important for the beam having lower shear rigidity.

4.1. Fully supported beams

This Subsection presents the numerical results for the prestress beams fully supported on the elastic foundation. A mesh of 30-equal elements is adopted in the computation. The reason for using the fine mesh comparing the Bernoulli beams (confirm [7]) is the lower order of the shape functions adopted in interpolating the displacement field, Eq. (2.7), so that a fine mesh is needed to ensure the accuracy [7]. Table 1 lists the frequency

Table 1. Frequency parameter of fully supported beams at various values of k_0 and at $\mu = -2.0$

	$k_0 = 0$	$k_0 = 50$	$k_0 = 100$	$k_0 = 150$	$k_0 = 200$
SS beam	77.7954	127.7757	177.7560	227.7363	277.7165
CF beam	2.5009	52.4904	102.4799	152.4694	202.4589
CS beam	214.9376	264.9146	314.8915	364.8685	414.8454

parameter of fully supported SS, CF and CS beams at various values of the foundation stiffness parameter k_0 and at $\mu = -2$. The full pictures describes the dependence of the frequency parameter on the axial load parameter μ and foundation stiffness k_0 are respectively given in Figs. 3-5. It is noted that the effect of the prestress and foundation support on the frequency of the beams obtained in the present study is very much similar to that of Bernoulli beams reported in With the above geometric data, according to Eq. (4.4), $s = 0.0012$, which is very small, and the shear deformation hardly affects the frequency of the beams.

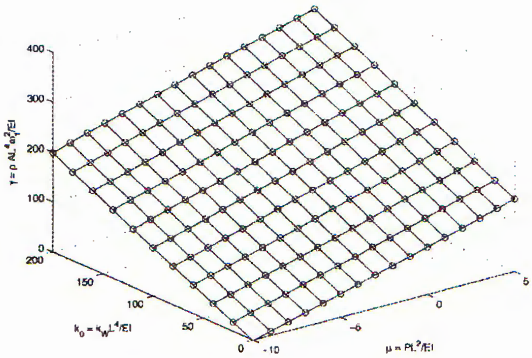

Fig. 3. Influence of axial force and foundation stiffness on frequency parameter of fully supported SS beam

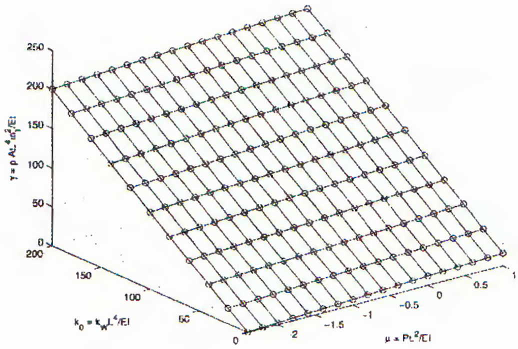

Fig. 4. Influence of axial force and foundation stiffness on frequency parameter of fully supported CF beam

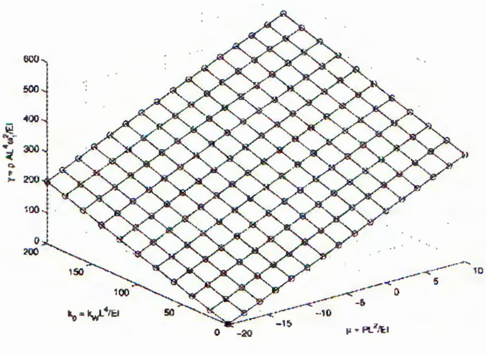

Fig. 5. Influence of axial force and foundation stiffness on frequency parameter of fully supported CS beam

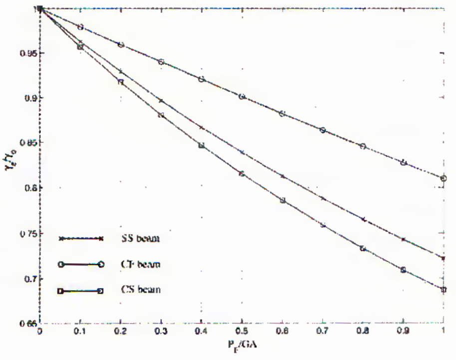

Fig. 6. Effect of shear deformation on frequency parameter of prestress Timoshenko beams fully supported on elastic foundation

In order to study the effect of shear deformation on the dynamic characteristics of the beams, the same approach presented in [?, ?] is followed herewith. In this regard, keeping all the beam data as above, and for $s = 0.1, 0.2, \ldots 1.0$, the computation is performed with different cross-sectional areas $A = 1.0264 \times 10^{-4}, 5.1322 \times 10^{-5}, \ldots 1.0264 \times 10^{-5}$.

Fig. 6 shows the effect of shear deformation on frequency parameter of prestress Timoshenko beams fully supported on the elastic foundation, where γ_0 and γ_s denote the frequency parameter corresponding with slenderness parameter of 0.0012 and s, respectively. With an increase in the slenderness parameter s, a reduction in the frequency parameter is observed, regardless of the boundary condition. In other words, the frequency of the prestress beams is reduced by the shear deformation, and we should take this effect into account for the case of stubby beams. Amongst the three types of boundary conditions considered in the present work, the reduction in the frequency parameter of CS beam is

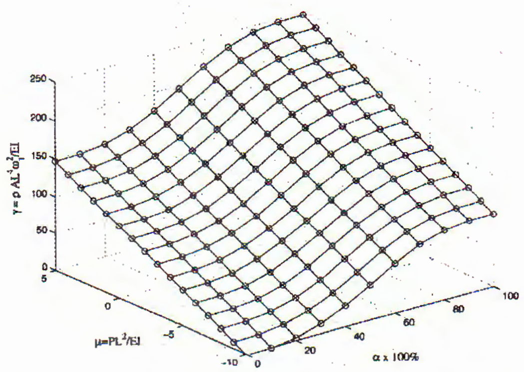

Fig. 7. Frequency parameter of partially supported SS beams at various values of axial force parameter and supporting percentages ($k_0 = 100$)

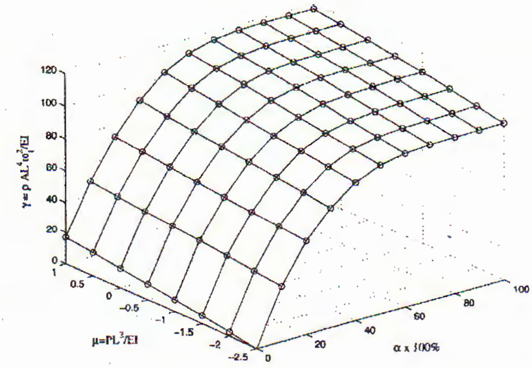

Fig. 8. Frequency parameter of partially supported CF beams at various values of axial force parameter and supporting percentages ($k_0 = 100$)

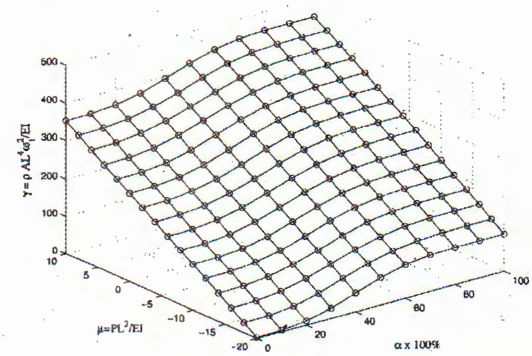

Fig. 9. Frequency parameter of partially supported CS beams at various values of axial force parameter and supporting percentages ($k_0 = 100$)

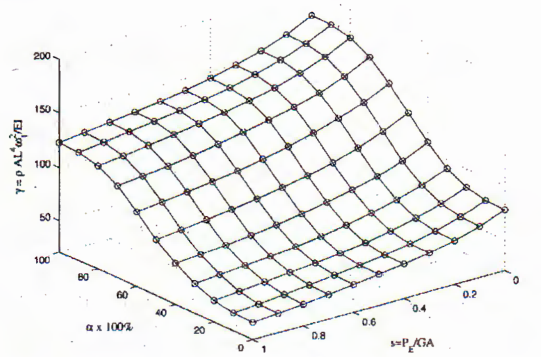

Fig. 10. Frequency parameter of partially supported SS beams at different supporting percentages and various values of slenderness parameter ($\mu = -2$, $k_0 = 100$)

the most pronounced. It is noted that, the effect of shear deformation is independent on the axial load parameter μ and the foundation stiffness parameter k_0.

4.2. Partially supported beams

This Subsection investigates the vibration frequency of the Timoshenko beams partially supported on the elastic foundation. The beams are supposed to be supported in part by the foundation from the left end as typically shown in Fig. 2d for the case of CF beam. The supported part is denoted αL, with $0 \leq \alpha \leq 1$ is called the supporting parameter.

Figs. 7-9 show the frequency parameter of the SS, CF and CS beams as functions of the axial load parameter μ and foundation stiffness parameter k_0, respectively. Similar to the case of Bernoulli beams, a nonlinear relationship between γ and (μ, k_0) is observed, regardless of the boundary conditions. Amongst the three types of boundary conditions, the SS and CS beams show a similar behavior in raising the supporting percentage, while

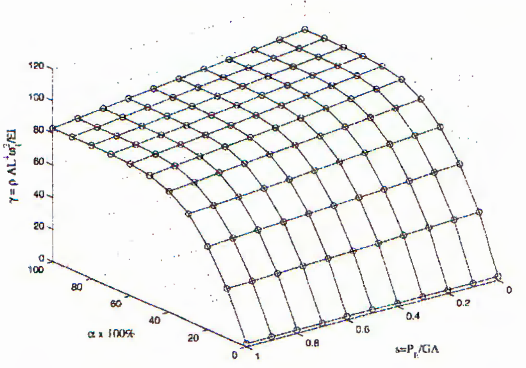

Fig. 11. Frequency parameter of partially supported CF beams at different supporting percentages and various values of slenderness parameter ($\mu = -2$, $k_0 = 100$)

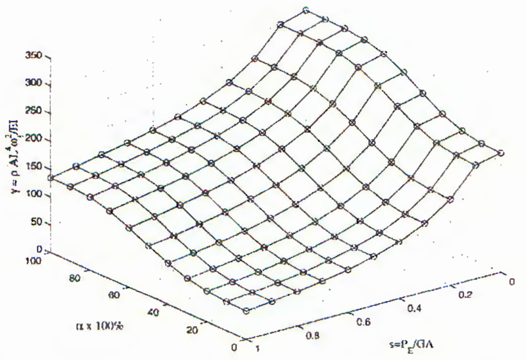

Fig. 12. Frequency parameter of partially supported CS beams at different supporting percentages and various values of slenderness parameter ($\mu = -2$, $k_0 = 100$)

the CF beam is more sensitive to the supporting parameter for the supporting percentage less than 60%.

The effects of shear deformation on the SS, CF and CS beams partially supported on the foundation are given in Figs. 10-12, respectively. The numerical displayed in the figures are computed for the axial load parameter $\mu = -2$ and the foundation parameter $k_0 = 100$. As clearly seen from the figures, the frequency of the beams reduces by the shear deformation, but the reduction depends on the supporting parameter and the type of boundary conditions. While the frequency parameter of SS and CS beams clearly lowers by rasing the shear deformation parameter s, that of CF beam reduces slowly. The reduction in the frequency parameter of all the beams is more clearly at a higher supporting percentage. In other words, the foundation increases the effect of the shear deformation on the vibration frequency of the beams.

5. CONCLUSIONS

The paper has investigated the free vibration of prestress Timoshenko beams resting on a Winkler elastic foundation by the finite element method. A beam element taking the effect of the prestress, foundation support and shear deformation into account has been formulated using the strain energy approach. The consistent mass matrix has been formulated using linear shape functions. The eigenvalue problem has been solved to obtain the natural frequencies of beams with various boundary conditions. The dependence of the frequency parameter on the axial force, foundation stiffness of beams fully and partially supported on the foundation is investigated. The effect of shear deformation on the vibration characteristics of the beam has been examined in detail. In addition to the conclusions which have been made for the Bernoulli beams in [7], the following remarks can be drawn for the prestress Timoshenko beams of the present work:

- The frequency of prestress beams resting on an elastic foundation is affected by the shear deformation, and it is lower for a beam having higher shear deformation parameter.

- The foundation increases the effect of shear deformation on the vibration frequency of prestress beams. For higher foundation supporting percentage, the more shear deformation effect is.
- Amongst three types of the boundary conditions investigated, the CS beam is the most sensitive to the shear deformation effect.

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REFERENCES

1. M. Hetényi, *Beams on Elastic Foundation,* The University of Michigan Press, Ann Arbor, 1946.
2. A. Ghali and A. M. Neville, *Structural Analysis, A Unified classical and matrix approach*, E & FN Spon, London, third edition, 1995.
3. M. A. De Rosa, Free Vibrations of Timoshenko Beams on Two-parameter Elastic Foundation, *Computers & Structures* **57:151** (1995) 156.
4. G. V. Rao, Large-amplitude free vibrations of uniform beams on Pasternak foundation, *Journal of Sound and Vibration* **263** (2003) 954 - 960.
5. Nguyễn Xuân Hùng, *Tính Toán Chính Xác Kết Cấu Trên Máy Tính Chương Trình ADS 2001*, Nhà xuất bản Khoa học và Kỹ thuật, Hà Nội, 2001.
6. J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, Dover Publications Inc., New York, 1985.
7. D. K. Nguyen, Vibration frequency of prestress slender beams resting on Winkler elastic foundation, *Vietnam Journal of Mechanics* **28** (2006) 241-251.
8. M. A. Crisfield, *Finite Elements and Solution Procedures for Structural Analysis, Volume 1: Linear Analysis*, Pineridge Press, Swansea, 1986.
9. S. C. Dutta and R. Roy, A critical review on idealization and modeling for interaction among soil-foundation-structure system, *Computers & Structures* **80** (2002) 1579 - 1594.
10. J. M. Gere and S. P. Timoshenko, *Mechanics of Materials*, Chapman & Hall, London, Third SI edition, 1995.
11. M. A. Crisfield, A faster modified newton-raphson iteration, *Computer Methods in Applied Mechanics and Engineering* **20** (1979) 267 - 278.
12. T. Yokoyama, Vibration analysis of Timoshenko beam-column on two-parameter elastic foundation, *Computers & Structures* **61:995** (1996) 1007.
13. M. Géradin and R. Rixen, *Mechanical vibrations, Theory and Application to Structural Dynamics*, John Wiley and Sons, Chichester, second edition, 1997.
14. R.D. Cook, *Finite Element Modelling for Stress Analysis*, John Wiley & Sons, New York, 1995.
15. L. Meirovithch, *Fundamentals of Vibrations*, McGraw-Hill, Boston, 2001.
16. S.P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, McGraw-Hill, New York, Second edition, 1961.
17. D. K. Nguyen, Post-buckling behavior of beam on two-parameter elastic foundation, *International Journal of Structural Stability and Dynamics* **4** (2004) 21 - 43.
18. D. K. Nguyen, Effects of shear deformation on large deection behavior of elastic frames, *Vietnam Journal of Mechanics* **26** (2004) 167-181.

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DAO ĐỘNG TỰ DO CỦA DẦM TIMOSHENKO DỰ ỨNG LỰC NẰM TRÊN NỀN ĐÀN HỒI

Bài báo trình bày công thức phần tử hữu hạn dùng trong nghiên cứu dao động tự do của dầm Timoshenko có thiết diện đồng nhất, dự ứng lực, nằm trên nền đàn hồi Winkler. Ma trận độ cứng phần tử có tính tới ảnh hưởng của dự ứng lực, nền đàn hồi và biến dạng trượt xây dựng bằng phương pháp năng lượng. Ma trận khối lượng nhất quán nhận được từ biểu thức động năng trên cơ sở các hàm dạng tuyến tính. Sử dụng công thức phần tử hữu hạn phát triển đã xác định tần số dao động riêng của dầm có các điều kiện biên khác nhau, ứng với các giá trị khác nhau của lực dọc trục và độ cứng nền. Các đặc trưng dao động của dầm tựa một phần trên nền đàn hồi cũng được khảo sát. Đặc biệt, ảnh hưởng của biến dạng trượt tới tần số dao động của dầm dự ứng lực nằm hoàn toàn và một phần trên nền đàn hồi được nghiên cứu chi tiết.