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A CAUCHY LIKE PROBLEM IN PLANE ELASTICITY*

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Abstract. Let Ω be a bounded domain in the plane, representing an elastic body. Let Γ_0 be a portion of the boundary Γ of Ω, Γ_0 being assumed to be paralled to the x – axis. It is proposed to determine the stress field in Ω from the displacements and surface stresses given on Γ_0. Under the assumption of plane stress, it is shown that $\sigma_x + \sigma_y$ is a harmonic function. An Airy stress function is introduced, from which the stress field is computed.

Consider an elastic body represented by a bounded domain Ω in the plane. Let Γ_0 be a portion of the boundary Γ of Ω assumed to be paralled to the x – axis (cf. Fig. 1). We propose to determine the stress field in Ω from the displacements and surface stresses given on Γ_0.

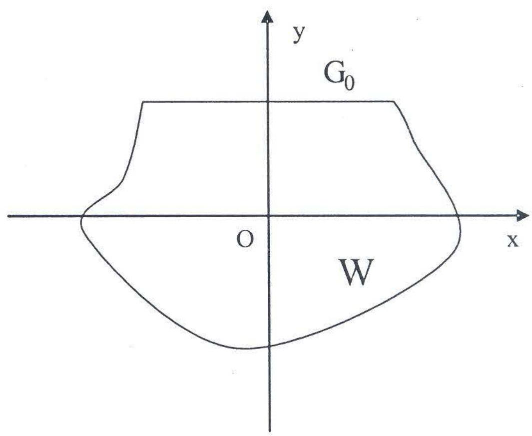

Fig. 1

Cauchy like problems in plane elasticity are treated in [1], [4] and others (cf. References). For a derivation of basic relations on stresses and displacements, we follow {TG}. Assume plane stress. We denote the displacements in the x - and y -directions respectively

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by u and v and the stress components by σ_x, σ_y and τ_{xy}. Now, we have

$$\varepsilon_x = \frac{\partial u}{\partial x}; \ \varepsilon_y = \frac{\partial v}{\partial y}; \ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{1}$$

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y), \ \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x), \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} = \dfrac{2(1+\nu)}{E}\tau_{xy} \end{cases} \tag{2}$$

In the absence of body forces, we have

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \tag{3}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{4}$$

These are the equilibrium equations for our problem.

We now derive the compatibility equations. We have

$$\varepsilon_x = \frac{\partial u}{\partial x}, \ \varepsilon_y = \frac{\partial v}{\partial y}, \ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5}$$

from which we get upon differentiating with recpect to y, then with respect to x

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_x}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{6}$$

This differential relation, called the condition of compatibility, must be satisfied by the strain components. By using Hooke's law, the condition (6) can be transformed into a relation between the components of stress.

We have

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y), \ \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x), \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} = \dfrac{2(1+\nu)}{E}\tau_{xy}. \end{cases} \tag{7}$$

Substituting into (6), we find

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}. \tag{8}$$

Differentiating equation (3) with respect to x, equation (4) with respect to y and adding together, we find

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}. \tag{9}$$

Substituting into (8), the compatibility equation in terms of stress components becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{10}$$

Before going further, we make the assumption that there exists an open set Ω_0 with $\Gamma_0 \subset \Omega_0 \cap \Omega$ such that the stress field is analytically continued to Γ_0. Let

$$\begin{matrix} l\sigma_x + m\tau_{xy} = \overline{X} \\ m\sigma_y + l\tau_{xy} = \overline{Y} \end{matrix} \quad \text{on} \quad \Gamma_0, \tag{11}$$

where l, m are the x and y - components of the exterior normal to Γ_0 and furthermore

$$\begin{matrix} u = \overline{u}(x) \\ v = \overline{v}(x) \end{matrix} \quad \text{on} \quad \Gamma_0. \tag{12}$$

It can be shown that

$$\sigma_x \quad \text{and} \quad \frac{\partial \sigma_x}{\partial_y} \quad \text{are known on } \Gamma_0, \tag{13}$$

$$\sigma_y \quad \text{and} \quad \frac{\partial \sigma_y}{\partial_y} \quad \text{are known on } \Gamma_0. \tag{14}$$

Thus $\sigma_x + \sigma_y$ is seen as solution of a Cauchy problem on Ω. As is well-known, the problem admits at most one solution. It is also known that the problem is ill-posed. Since by (10) $\sigma_x + \sigma_y$ is harmonic on Ω, it is analytic on Γ_0. Thus if $z_n = (x_n, k)$ $n = 1, 2, ...$ is a sequence of points of Γ_0 with $x_i \neq x_j$ for $i \neq j$ and accumulating at a point interior to Γ_0, then $\sigma_x + \sigma_y$ is uniquely determined by it values on (z_n). Hence the Cauchy problem for the Laplace equation can be formulated as a moment problem, it has been regularized by various methods (cf. e.g. [2] chapter 6).

Now we introduce the Airy stress function as follows. Let $f = \sigma_x + \sigma_y$, we define

$$\varphi(x, y) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\xi, \eta) \ln[(x - \xi)^2 + (y - \eta)^2] d\xi d\eta, \tag{15}$$

where f is set equal to 0 in the complement of Ω. We let

$$\overline{\sigma}_x = \frac{\partial^2 \varphi}{\partial y^2}, \quad \overline{\sigma}_y = \frac{\partial^2 \varphi}{\partial x^2}, \quad \overline{\tau}_{xy} = \frac{\partial^2 \varphi}{\partial x \partial y}. \tag{16}$$

It can be checked that $\overline{\sigma}_x$, $\overline{\sigma}_y$ and $\overline{\tau}_{xy}$ satisfy the equalibrium equations and the compatibility relation. Hence

$$\overline{\sigma}_x = \sigma_x, \qquad \overline{\sigma}_y = \sigma_y, \qquad \overline{\tau}_{xy} = \tau_{xy}. \tag{17}$$

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BÀI TOÁN TỰA CAUCHY TRONG ĐÀN HỒI PHẲNG

Xét một vật thể đàn hồi biểu diễn bởi một miền Ω bị chặn trong mặt phẳng. Gọi Γ_0 là một phần của biên Γ của miền Ω , Γ_0 được giả thiết là song song với trục tọa độ x. Vấn đề đặt ra là xác định trường ứng suất trong Ω từ các chuyển vị và ứng suất mặt cho trước trên Γ_0. Dưới các giả thiết của ứng suất phẳng, bài báo đã chỉ ra $\sigma_x + \sigma_y$ là một hàm điều hòa. Một hàm ứng suất Airy được giới thiệu để từ đó có thể tính được trường ứng suất .