ASYMPTOTIC EXPANSION OF THE DISPERSION EQUATION OF LAMB WAVES IN PERIODICALLY LAYERED ELASTIC MEDIA

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Abstract. The present paper deals with the problem on Lamb waves propagation in periodically layered, compressible elastic media with initial deformations, in the case of long wavelength approximation (i.e. $\varepsilon = k.h << 1$, where k is the wave number, h is the thickness of one periodicity cell). With the assumption that $\varepsilon << 1$, the dispersion equation is written as:

$$\frac{\omega^2}{k^2} = \Omega_1 + \varepsilon\Omega_2 + \varepsilon^2\Omega_3 + \cdots = \sum_{m=0}^{\infty} \varepsilon^m \Omega_{m+1}.$$

The main aim of this paper is to find formulae for determining the coefficients $\Omega_i, (i \geq 1)$. In particular, we prove that $\Omega_{2n} = 0$ $(n \geq 1)$, derive formulae for Ω_1, Ω_3, and construct recurrent formulae for Ω_{2n+1} $(n \geq 2)$. Based on these formulae, the solution with any order of accuracy can be obtained. This research is an extension of the investigation by Norris and Santosa [Norris A. and Santosa F., Wave Motion 16 (1992), 33-55] from SH waves (one-component waves) to Lamb waves (two-component waves).

Keyword. Lamb waves, Wave propagation, Periodically layered media, Asymptotic expansion.

1. INTRODUCTION

Wave propagation in elastic media has received an intensive consideration because of its wide range of applications in various fields in science such as seismology, acoustics, material sciences, construction,... (see for example, [1], [2]). Since thin structures are often encountered in engineering and technology, the wave propagation in thin layers, or long-wave propagation, has been the subject of many investigations, see for instance [3]-[17] and references therein.

In the wave propagation studies, it is important to derive dispersion equations being of the form: $\omega = \omega(k, p_1, p_2, ...)$, where ω is the wave frequency, k is the wave number, and p_i (i=1, 2, ...) are the characteristic parameters of the material. The dispersion equations can be used to solve the direct (forward) problems which study the effects of material parameters on the wave velocity $c = \omega/k$ or to solve the inverse problems which determine material parameters from the measured values of the wave speed c.

When the layers are thin, or the considered wave is a long wave (low-frequency wave), we have:

$$0 < \varepsilon = k.h << 1 \tag{1}$$

where h is a typical length (it is the thickness of one periodicity cell for the periodically layered structures, for instance). Taking assumption (1) into account, the dispersion equation can be expanded into a power series of the small parameter ϵ as follows:

$$c^2 = \frac{\omega^2}{k^2} = \Omega_1 + \varepsilon\Omega_2 + \varepsilon^2\Omega_3 + \ldots = \sum_{m=0}^{\infty} \Omega_{m+1}\varepsilon^m \tag{2}$$

here the coefficients Ω_m need to be determined.

Norris and Santosa [6] studied the propagation of SH-wave (one-component wave) in a periodically layered infinite elastic medium without initial deformations. They derived the formulae of Ω_1, Ω_3 and proved that $\Omega_2 = 0$.

In this paper, the results obtained by Norris and Santosa [6] are extended to Lamb wave (two-component wave). In particular, we derive the expressions for Ω_1, Ω_3 and demonstrate $\Omega_2 = 0$. Moreover, we establish recurrent formulae for Ω_{2n+1} ($n \geq 2$) and prove that $\Omega_{2n} = 0, \forall n \geq 1$. The material is assumed with initial deformations.

The layout of the paper is as follows. The problem setting and mathematical formulations are introduced in Section 2. The asymptotic analysis, the expressions for Ω_1, Ω_3 and the vanishing of Ω_2 are discussed in Section 3. Section 4 presents recurrent formulae for calculating the higher-order coefficients Ω_m in the expansion (2).

2. PROBLEM SETTING AND MATHEMATICAL FORMULATIONS

Let us consider a periodically layered, infinite elastic medium. Each periodicity cell consists of N different layers ($N \geq 2$). Suppose that the material layers are compressible isotropic and subject to homogeneous initial deformations (see [18], [19]). At the initial state, we introduce a system of orthogonal Cartesian coordinates $Oz_1z_2z_3$ in which the coordinate plane Oz_1z_2 coincides with the bottom plane of the first material layer of the periodicity cell (see Fig. 1). By h_i and h we denote, respectively, the thickness of the i-th layer ($i = 1, ..., N$) and the thickness of one periodicity cell at the initial state, then we have:

$$h = h_1 + h_2 + \cdots + h_N. \tag{3}$$

Let us consider a Lamb wave propagating along the layers, in the Oz_1-direction, and its displacement vector lies on the plane Oz_1z_3. The components of the wave vector have the form $\vec{k}(k_1, k_2, k_3) = (k, 0, 0)$, and non-zero components of the displacement vector of the Lamb wave are: $u_1 = u_1(z_1, z_3, \tau)$, $u_3 = u_3(z_1, z_3, \tau)$, where τ is the time.

The following system of equations is used to study the problem (see [18], [19]):

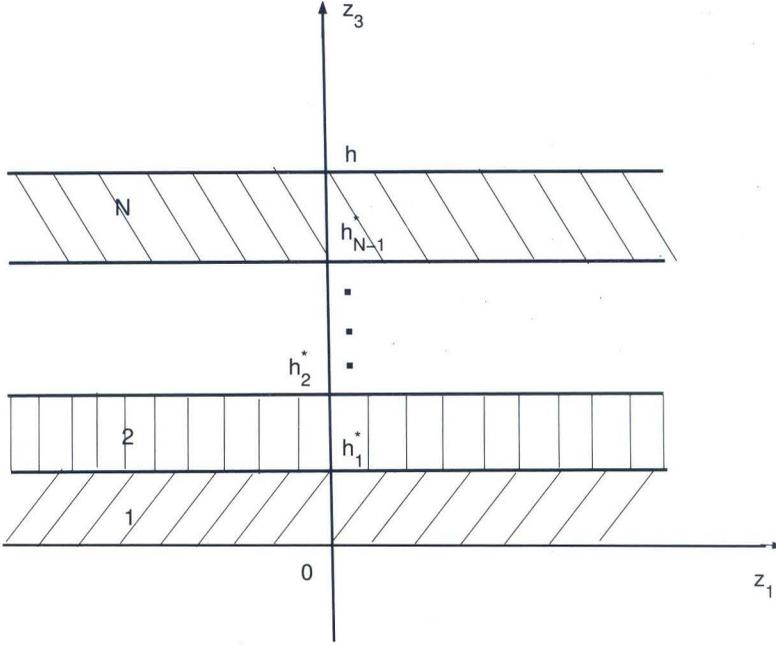$$L_{m\alpha}u_\alpha = 0 \qquad m, \alpha = 1, 3, \tag{4}$$

where:

Fig. 1. The periodicity cell

$$L_{m\alpha} = \frac{\partial}{\partial z_i}\left[\tilde{\omega}_{im\alpha\beta}(z_3)\frac{\partial}{\partial z_\beta}\right] - \tilde{\rho}(z_3)\delta_{m\alpha}\frac{\partial^2}{\partial \tau^2} \quad , \qquad i, \beta = 1, 3, \tag{5}$$

$\tilde{\rho}(z_3)$ is the mass density of the medium at the initial state. Note that $\tilde{\omega}_{im\alpha\beta}(z_3)$, $\tilde{\rho}(z_3)$ are periodic functions of period h. They are constant on each interval (h_{i-1}^*, h_i^*), $i = 1, \ldots, N$ (see [19]), where:

$$h_i^* = h_1 + h_2 + \ldots + h_i, \ i = 1, 2, \ldots, N, \ h_0^* = 0.$$

The surface tractions per unit area $\tilde{P}_m (m = 1, 3)$ at the initial state at the plane $z_3 = $ const are calculated as follows (see [19]):

$$\tilde{P}_m = \tilde{\omega}_{3m\alpha\beta}(z_3).\frac{\partial u_\alpha}{\partial z_\beta}, \qquad m = 1, 3. \tag{6}$$

Equations (4) are written in the matrix form as

$$L\mathbf{u} = \tilde{\rho}\frac{\partial^2 \mathbf{u}}{\partial \tau^2}, \tag{7}$$

where:

$$L\mathbf{u} = \frac{\partial}{\partial z_h}\left(A_{hk}\frac{\partial \mathbf{u}}{\partial z_k}\right), \tag{8}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_3 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} \tilde{\omega}_{1111} & 0 \\ 0 & \tilde{\omega}_{1331} \end{bmatrix}, \qquad A_{33} = \begin{bmatrix} \tilde{\omega}_{3113} & 0 \\ 0 & \tilde{\omega}_{3333} \end{bmatrix}, \tag{9}$$

$$A_{13} = \begin{bmatrix} 0 & \tilde{\omega}_{3311} \\ \tilde{\omega}_{3131} & 0 \end{bmatrix}, \qquad A_{31} = \begin{bmatrix} 0 & \tilde{\omega}_{3131} \\ \tilde{\omega}_{3311} & 0 \end{bmatrix}.$$

It should be noted that when the medium is at the natural state, we have

$$\tilde{\omega}_{1111} = \tilde{\omega}_{3333} = \lambda + 2\mu, \quad \tilde{\omega}_{1331} = \tilde{\omega}_{3113} = \tilde{\omega}_{3131} = \mu, \quad \tilde{\omega}_{3311} = \lambda.$$

In the matrix form, formula (6) becomes

$$\tilde{P} = A_{31}\frac{\partial \mathbf{u}}{\partial z_1} + A_{33}\frac{\partial \mathbf{u}}{\partial z_3}, \tag{10}$$

where:

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_3 \end{bmatrix}. \tag{11}$$

We find the solution of equation (7) in the form

$$\mathbf{u} = \mathbf{U}(z_3)e^{i(kz_1 - \omega\tau)}. \tag{12}$$

Substituting (12) into (7) yields:

$$\frac{d}{dz_3}\left(A_{33}\frac{d\mathbf{U}}{dz_3} + ikA_{31}\mathbf{U}\right) + ikA_{13}\frac{d\mathbf{U}}{dz_3} + (\tilde{\rho}\omega^2 E - k^2 A_{11})\mathbf{U} = 0, \tag{13}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U}(\mathbf{z_3}) = \begin{bmatrix} U_1(z_3) \\ U_3(z_3) \end{bmatrix}.$$

Substitution of equation (12) into equation (10) leads to

$$\tilde{P} = P(z_3)e^{i(kz_1 - \omega\tau)}, \qquad P = A_{31}\frac{d\mathbf{U}}{dz_1} + A_{33}\frac{d\mathbf{U}}{dz_3},$$

where

$$P = \begin{bmatrix} P_1 \\ P_3 \end{bmatrix}. \tag{14}$$

Equation (13) can be rewritten as

$$\frac{d}{kdz_3}\left(A_{33}\frac{d\mathbf{U}}{kdz_3} + iA_{31}\mathbf{U}\right) + iA_{13}\frac{d\mathbf{U}}{kdz_3} + (\tilde{\rho}\frac{\omega^2}{k^2}E - A_{11})\mathbf{U} = 0. \tag{15}$$

Denote $Y_1 = \mathbf{U}$, $Y_2 = A_{33}\frac{d\mathbf{U}}{kdz_3} + iA_{31}\mathbf{U}$, then equation (15) yields

$$\frac{d\mathbf{Y}}{kdz_3} = B \cdot \mathbf{Y}, \tag{16}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \qquad B = \begin{bmatrix} -iA_{33}^{-1}A_{31} & A_{33}^{-1} \\ -(A_{13}A_{33}^{-1}A_{31} + \tilde{\rho}\frac{\omega^2}{k^2} - A_{11}) & -iA_{13}A_{33}^{-1} \end{bmatrix}. \tag{17}$$

Introducing the new variable $y_3 = z_3/h$, then equation (16) is written as:

$$\frac{d\boldsymbol{Y}}{dy_3} = \varepsilon B \cdot \boldsymbol{Y}. \tag{18}$$

where

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 \\ b_{21} & 0 & 0 & b_{24} \\ b'_{31} & 0 & 0 & b_{34} \\ 0 & b'_{42} & b_{43} & 0 \end{bmatrix}, \tag{19}$$

in which

$$b_{12} = b_{43} = -i\frac{\tilde{\omega}_{3131}}{\tilde{\omega}_1},$$

$$b_{13} = \frac{1}{\tilde{\omega}_1},$$

$$b_{21} = b_{34} = -i\frac{\tilde{\omega}_{3311}}{\tilde{\omega}_3},$$

$$b_{24} = \frac{1}{\tilde{\omega}_3}, \tag{20}$$

$$b'_{31} = -\left(\frac{\tilde{\omega}_{3311}^2}{\tilde{\omega}_3} + \tilde{\rho}\frac{\omega^2}{k^2} - \tilde{\omega}_{1111}\right),$$

$$b'_{42} = -\left(\frac{\tilde{\omega}_{3131}^2}{\tilde{\omega}_3} + \tilde{\rho}\frac{\omega^2}{k^2} - \tilde{\omega}_{3131}\right),$$

$$\tilde{\omega}_1 = \tilde{\omega}_{3113}, \quad \tilde{\omega}_3 = \tilde{\omega}_{3333}.$$

Note that $B(y_3)$ is a periodic function of period 1. In summary, we need to solve equation (18) on the interval $[0, 1]$, with assumption $\varepsilon = k.h \ll 1$. The periodicity of the medium leads to condition

$$\boldsymbol{Y}(1) = \boldsymbol{Y}(0). \tag{21}$$

3. ASYMPTOTIC SOLUTION, EXPRESSIONS FOR Ω_1, Ω_3

Suppose that the layers are all thin, or the Lamb wave is a long-wave (low-frequency wave) which leads to $0 < \varepsilon \ll 1$. As mentioned above, when $\varepsilon \ll 1$, the dispersion equation of the Lamb wave can be written as

$$\frac{\omega^2}{k^2} = \Omega_1 + \varepsilon\Omega_2 + \varepsilon^2\Omega_3 + \cdots + \varepsilon^{j-1}\Omega_j + \cdots = \sum_{m=0}^{\infty} \varepsilon^m \Omega_{m+1}. \tag{22}$$

Our task now is to determine the coefficients $\Omega_1, \Omega_2, \Omega_3 \ldots$. Substituting equation (22) into formulas (19), (20), matrix B can be expressed as follows

$$B = B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots + \varepsilon^j B_j + \cdots, \tag{23}$$

where

$$B_j = \tilde{\rho}\Omega_{j+1}L, \quad \forall j = 1, 2, 3\ldots, \tag{24}$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 \\ b_{21} & 0 & 0 & b_{24} \\ b_{31} & 0 & 0 & b_{34} \\ 0 & b_{42} & b_{43} & 0 \end{bmatrix},$$

and

$$b_{31} = -\left(\frac{\tilde{\omega}_{3311}^2}{\tilde{\omega}_3} + \tilde{\rho}\Omega_1 - \tilde{\omega}_{1111}\right), \qquad b_{42} = -\left(\frac{\tilde{\omega}_{3131}^2}{\tilde{\omega}_3} + \tilde{\rho}\Omega_1 - \tilde{\omega}_{3131}\right). \tag{25}$$

In order to solve the equation (18), we seek Y in the form

$$\boldsymbol{Y} = Y_0 + \varepsilon^1 Y_1 + \varepsilon^2 Y_2 + \varepsilon^3 Y_3 + \cdots = \sum_{n=0}^{\infty} \varepsilon^n Y_n. \tag{26}$$

We introduce new matrices $R_n(y_3) = [r_{ij}^n(y_3)]_{4\times4}$, $T_n(y_3) = [t_{ij}^n(y_3)]_{4\times4}$ and $S_n = [s_{ij}^n]_{4\times4}$ defined as follows

$$R_n(y_3) = \sum_{m=0}^{n} \int^{y_3} B_{n-m}R_{m-1}; \quad S_n = R_n(1); \quad \forall n = 0, 1, 2, \ldots; \tag{27}$$

$$T_n(y_3) = \sum_{m=1}^{n} \int^{y_3} B_{n-m}R_{m-1}, \quad \forall n = 1, 2, \ldots; \qquad T_0 = \int^{y_3} B_0, \tag{28}$$

where

$$\int^{y_3} \Gamma = \int_0^{y_3} \Gamma(x)dx; \qquad R_{-1} = diag(1,1,1,1) = I. \tag{29}$$

From the above definitions, $R_n(y_3)$ relates to $T_n(y_3)$ by the relation

$$R_n(y_3) = T_n(y_3) + \int^{y_3} B_n, \qquad \forall n \geq 1. \tag{30}$$

Lemma 1:

$$Y_j(y_3) = R_{j-1}(y_3)Y(0), \qquad \forall j = 1, 2, \ldots. \tag{31}$$

Proof: Taking into account (23) and (26), we have from (18)

$$\sum_{j=0}^{\infty} \varepsilon^j \frac{dY_j}{dy_3} = \sum_{i=0}^{\infty} \varepsilon^{i+1} B_i \sum_{k=0}^{\infty} \varepsilon^k Y_k. \tag{32}$$

The coefficients of the same order of ε on both sides in (32) must be the same. Therefore, we have the following differential equations

$$\frac{dY_0}{dy_3} = 0, \tag{33}$$

$$\frac{dY_1}{dy_3} = B_0 Y_0, \tag{34}$$

$$\frac{dY_2}{dy_3} = B_1 Y_0 + B_0 Y_1, \tag{35}$$

$$\frac{dY_3}{dy_3} = B_2 Y_0 + B_1 Y_1 + B_0 Y_2, \tag{36}$$

$$\dots$$

$$\frac{dY_j}{dy_3} = \sum_{s=0}^{j-1} B_{j-s-1} Y_s. \tag{37}$$

Taking integration of equations (33)-(36) leads to

$$Y_0 const = Y(0);$$

$$Y_1(y_3) = \left(\int^{y_3} B_0 \right) Y(0) = R_0 Y(0);$$

$$Y_2(y_3) = \left(\int^{y_3} B_1 + \int^{y_3} B_0 S_0 \right) Y(0) = R_1 Y(0);$$

$$Y_3(y_3) = \left(\int^{y_3} B_2 + \int^{y_3} B_1 S_1 + \int^{y_3} B_0 S_2 \right) Y(0) = R_2 Y(0). \tag{38}$$

Thus, the statement (31) holds true for $j = 1, 2, 3$. Assume that the statement (31) holds true for any $j = 1, 2, 3 \ldots n-1$. We will prove that it also holds true for $j = n$. From (37), we have:

$$\frac{dY_n}{dy_3} = \sum_{s=0}^{n-1} B_{n-s-1} Y_s. \tag{39}$$

From the inductive assumption: $Y_s = R_{s-1} Y(0)$ and the above equation, we have

$$\frac{dY_n}{dy_3} = \sum_{s=0}^{n-1} B_{n-s-1} R_{s-1} Y(0). \tag{40}$$

Integration of this equation gives:

$$Y_n = \left(\sum_{s=0}^{n-1} \int^{y_3} B_{n-s-1} R_{s-1}\right) Y(0) \qquad \text{or} \qquad Y_n = R_{n-1} Y(0) \tag{41}$$

Thus, statement (31) holds true for $j = n$ and it is valid for $\forall j = 1, 2, \ldots$.

Substitution of (31) into (26) leads to

$$\boldsymbol{Y}(y_3) = \left[I + \varepsilon^1 R_0(y_3) + \varepsilon^2 R_1(y_3) + \varepsilon^3 R_2(y_3) + \cdots \right] Y(0)$$

$$= \left[\sum_{n=0}^{\infty} \varepsilon^n R_{n-1}(y_3) \right] Y(0). \tag{42}$$

Taking the periodicity condition (21) into account and noting that $S_n = R_n(1)$, from (42) we have

$$\boldsymbol{Y}(1) = \left[I + \varepsilon^1 S_0 + \varepsilon^2 S_1 + \varepsilon^3 S_2 + \cdots \right] Y(0) = \left[\sum_{n=0}^{\infty} \varepsilon^n S_{n-1} \right] Y(0)$$

or

$$\boldsymbol{Y}(0) = \left[\sum_{n=0}^{\infty} \varepsilon^n S_{n-1} \right] Y(0). \tag{43}$$

Equation (43) leads to

$$\det \left[\sum_{n=0}^{\infty} \varepsilon^n S_n \right] = \det \left[S_0 + \varepsilon^1 S_1 + \varepsilon^2 S_2 + \cdots + \varepsilon^j S_j + \cdots \right] = 0. \tag{44}$$

Equation (44) can be written as

$$\det \left[\sum_{i=0}^{\infty} \varepsilon^i A^i, \sum_{j=0}^{\infty} \varepsilon^j B^j, \sum_{k=0}^{\infty} \varepsilon^k C^k, \sum_{h=0}^{\infty} \varepsilon^h D^h \right] = 0. \tag{45}$$

where A^j, B^j, C^j, D^j is the 1st, 2nd, 3th and 4th column of matrix $S_j, j = 0, 1, 2, \ldots$

Expansion of the first, second, third and forth column of the above determinant yields

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \varepsilon^i \varepsilon^j \varepsilon^k \varepsilon^h \det[A^i B^j C^k D^h] = 0. \tag{46}$$

hence

$$\sum_{\substack{0 \le i,j,k,h \le m}}^{i+j+k+h=m} \det[A^i B^j C^k D^h] = 0, \qquad (m = 0, 1, 2, \ldots). \tag{47}$$

Calculating Ω_1

Corresponding to $m = 0$, equation (47) gives

$$\det S_0 = 0, \qquad \Leftrightarrow \begin{vmatrix} s_{21}^0 & s_{24}^0 \\ s_{31}^0 & s_{21}^0 \end{vmatrix} \begin{vmatrix} s_{12}^0 & s_{13}^0 \\ s_{42}^0 & s_{12}^0 \end{vmatrix} = 0. \tag{48}$$

From (27) it follows $S_0 = \int^1 B_0$, thus we have

$$s_{ij}^0 = \langle b_{ij} \rangle, \qquad i, j = 1, 2, 3, 4. \tag{49}$$

where $S_0 = [s_{ij}^0]$ and $\langle f \rangle = \int^1 f = \int_0^1 f(y_3) dy_3$. From (48) it implies

case 1: $(s_{21}^0)^2 = s_{31}^0 s_{24}^0 \implies$

$$\Omega_1 = \left[\langle \tilde{\omega}_{1111} \rangle - \left\langle \frac{\tilde{\omega}_{3311}^2}{\tilde{\omega}_3} \right\rangle + \frac{\left\langle \frac{\tilde{\omega}_{3311}}{\tilde{\omega}_3} \right\rangle^2}{\left\langle \frac{1}{\tilde{\omega}_3} \right\rangle} \right] \langle \tilde{\rho} \rangle^{-1}. \tag{50}$$

case 2: $(s_{12}^0)^2 = s_{13}^0 s_{42}^0 \implies$

$$\Omega_1 = \left[\langle \tilde{\omega}_{1313} \rangle - \left\langle \frac{\tilde{\omega}_{3131}^2}{\tilde{\omega}_1} \right\rangle + \frac{\left\langle \frac{\tilde{\omega}_{3131}}{\tilde{\omega}_1} \right\rangle^2}{\left\langle \frac{1}{\tilde{\omega}_1} \right\rangle} \right] \langle \tilde{\rho} \rangle^{-1}. \tag{51}$$

Proving $\Omega_2 = 0$.

From the definition of S_i, we have

$$S_1 = \int^1 B_0 \int B_0 + \int^1 B_1 = \begin{bmatrix} s_{11}^1 & 0 & 0 & s_{14}^1 \\ 0 & s_{22}^1 & s_{23}^1 & 0 \\ s_{31}^1 & s_{32}^1 & s_{33}^1 & 0 \\ s_{41}^1 & s_{42}^1 & 0 & s_{44}^1 \end{bmatrix}, \tag{52}$$

where

$$\int^{y_3} \Gamma_1 \int \Gamma_2 = \int_0^{y_3} \Gamma_1(x_1) dx_1 \int_0^{x_1} \Gamma_2(x_2) dx_2,$$
$$\int^{y_3} \Gamma_1 \int \Gamma_2 \int \Gamma_3 = \int_0^{y_3} \Gamma_1(x_1) dx_1 \int_0^{x_1} \Gamma_2(x_2) dx_2 \int_0^{x_2} \Gamma_2(x_3) dx_3, ..., y_3 \in [0, 1].$$

Corresponding to $m = 1$, equation (47) gives

$$\det[A^1 B^0 C^0 D^0] + det[A^0 B^1 C^0 D^0] + \det[A^0 B^0 C^1 D^0] + \det[A^0 B^0 C^0 D^1] = 0. \tag{53}$$

Taking into account (52) and $s_{31}^1 = s_{42}^1 = \Omega_2 \langle \tilde{\rho} \rangle$, equation (53) provides

$$\Omega_2 \langle \tilde{\rho} \rangle \left\{ s_{24}^0 [(s_{12}^0)^2 - s_{13}^0 s_{42}^0] + s_{13}^0 [(s_{21}^0)^2 - s_{31}^0 s_{24}^0] \right\} = 0. \tag{54}$$

From (50) and (51): if $(s_{12}^0)^2 - s_{13}^0 s_{42}^0 = 0$ then $(s_{21}^0)^2 - s_{31}^0 s_{24}^0 \neq 0$ or, if $(s_{12}^0)^2 - s_{13}^0 s_{42}^0 \neq 0$

then $(s_{21}^0)^2 - s_{31}^0 s_{24}^0 = 0$. Hence, we get from (54)

$$\boxed{\Omega_2 = 0}. \tag{55}$$

Calculating Ω_3

From the definitions of matrices S_i, T_i, we have

$$S_2 = T_2(1) + \int^1 B_2 = \begin{bmatrix} 0 & s_{12}^2 & s_{13}^2 & 0 \\ s_{21}^2 & 0 & 0 & s_{24}^2 \\ s_{31}^2 & 0 & 0 & s_{34}^2 \\ 0 & s_{42}^2 & s_{43}^2 & 0 \end{bmatrix}, \tag{56}$$

where

$$T_2 = \int^{y_3} B_0 \int B_0 \int B_0 = \begin{bmatrix} 0 & t_{12}^2 & t_{13}^2 & 0 \\ t_{21}^2 & 0 & 0 & t_{24}^2 \\ t_{31}^2 & 0 & 0 & t_{34}^2 \\ 0 & t_{42}^2 & t_{43}^2 & 0 \end{bmatrix}. \tag{57}$$

Corresponding to $m = 2$, equation (47) gives

$$\begin{aligned} &\det[A^2 B^0 C^0 D^0] + \det[A^0 B^2 C^0 D^0] + \det[A^0 B^0 C^2 D^0] + \det[A^0 B^0 C^0 D^2] \\ &+ \det[A^1 B^1 C^0 D^0] + \det[A^1 B^0 C^1 D^0] + \det[A^1 B^0 C^0 D^1] \\ &+ \det[A^0 B^1 C^1 D^0] + \det[A^0 B^1 C^0 D^1] + \det[A^0 B^0 C^1 D^1] = 0. \end{aligned} \tag{58}$$

We introduce the quantities M, a, b defined as

$$M = \det[A^1 B^1 C^0 D^0] + \det[A^1 B^0 C^1 D^0] + \det[A^0 B^1 C^0 D^1] + \det[A^0 B^0 C^1 D^1],$$

and

$$a = \begin{vmatrix} s_{12}^0 & s_{13}^0 \\ s_{42}^0 & s_{43}^0 \end{vmatrix}, \qquad b = \begin{vmatrix} s_{21}^0 & s_{24}^0 \\ s_{31}^0 & s_{34}^0 \end{vmatrix}. \tag{59}$$

Taking into account $\Omega_2 = 0$, from (52), (56) and (58) we have

Case 1: $(s_{21}^0)^2 = s_{31}^0 s_{24}^0$, then $b = 0 \Rightarrow \det[A_0 B_2 C_0 D_0] = \det[A_0 B_0 C_2 D_0] = 0$, henceforth:

$$\Omega_3 = (\langle \tilde{\rho} \rangle \, s_{24}^0)^{-1} \left[s_{21}^2 s_{34}^0 - t_{31}^2(1) s_{24}^0 + s_{34}^2 s_{21}^0 - s_{24}^2 s_{31}^0 + M a^{-1} \right]. \tag{60}$$

Case 2: $(s_{12}^0)^2 = s_{13}^0 s_{42}^0$, then $a = 0 \Rightarrow \det[A^2 B^0 C^0 D^0] = \det[A^0 B^0 C^0 D^2] = 0$, hence:

$$\Omega_3 = (\langle \tilde{\rho} \rangle \, s_{42}^0)^{-1} \left[s_{12}^2 s_{43}^0 - s_{13}^2 s_{42}^0 + s_{43}^2 s_{12}^0 - t_{42}^2(1) s_{13}^0 + M b^{-1} \right]. \tag{61}$$

4. CALCULATING COEFFICIENTS OF HIGHER ORDER

In this section, we will prove that $\Omega_{2n} = 0$ and construct recurrent formulae for calculating Ω_{2n+1}. Before doing that, some definitions and lemmas are introduced.

Definition 1: $MT1$, $MT2$ are sets of matrices which are of the form:

$$mt1 = \begin{bmatrix} \times & 0 & 0 & \times \\ 0 & \times & \times & 0 \\ 0 & \times & \times & 0 \\ \times & 0 & 0 & \times \end{bmatrix}, \qquad mt2 = \begin{bmatrix} 0 & \times & \times & 0 \\ \times & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ 0 & \times & \times & 0 \end{bmatrix}, \tag{62}$$

respectively.

Formulas (62) indicate entries being definitely zero of the matrices $mt1$ and $mt2$.

From the definition of $MT1$ and $MT2$, we immediately have the following lemmas:

Lemma 2: Summation of matrices in $MT1$ ($MT2$) belongs to $MT1$ ($MT2$).

Lemma 3: Let $mt1 \in MT1, mt2 \in MT2$, then: $B_j mt2 \in MT1$; $B_j mt1 \in MT2$ for all $j \geq 0, j \in \mathbb{N}$.

Lemma 4: The determinant of a matrix which is formed by replacing any column of a $mt1$-matrix ($mt2$-matrix) with the corresponding column of a $mt2$-matrix ($mt1$-matrix) is equal to zero.

We now prove the following statement:

$$\Omega_{2n} = 0, \qquad \forall n = 1, 2, 3.... \tag{63}$$

Proof: It has been proved in the previous section that $\Omega_2 = 0$. Thus, the above statement holds true for $n = 1$. Assume that statement (63) holds true for any $j = 1, 2, \ldots, n - 1$, that means: $\Omega_2 = \Omega_4 = \ldots = \Omega_{2n-2} = 0$. We will prove (63) is true for $j = n$.

From the definition of matrix B_k, it is seen that $B_k \in MT2, \forall k \geq 0$. The inductive assumption gives

$$B_{2j-1} = \tilde{\rho}\Omega_{2j} = 0 \quad \forall j = 1, 2, \ldots n - 1. \tag{64}$$

We will show that

$$R_{2j-1} \in MT1, R_{2j} \in MT2, \qquad \forall j = 0, 1, \ldots, n - 1. \tag{65}$$

Indeed, $B_0 \in MT2 \Rightarrow R_0 = \int^{y_3} B_0 \in MT2$, and $R_{-1} = I \in MT1$. Hence (65) is true for $j = 0$. Assume that (65) is true for every $j = 0, 1, 2, ..., k - 1$ with $k \leq n - 1$, that means:

$$\begin{cases} R_{-1}, R_1, R_3, \ldots R_{2k-3} \in MT1, \\ R_0, R_2, \ldots R_{2k-2} \in MT2, \qquad k \leq n - 1. \end{cases} \tag{66}$$

We will prove that (65) is true for $j = k$, that means:

$$R_{2k-1} \in MT1, R_{2k} \in MT2, \quad k \leq n-1. \tag{67}$$

Indeed, from (27) and (64) we have (noting that $k \leq n-1$)

$$R_{2k-1} = \int^{y_3} B_{2k-2}R_0 + \int^{y_3} B_{2k-4}R_2 + ... + \int^{y_3} B_0 R_{2k-2}$$

$$R_{2k} = \int^{y_3} B_{2k} + \int^{y_3} B_{2k-2}R_1 + ... + \int^{y_3} B_2 R_{2k-3} + \int^{y_3} B_0 R_{2k-1}. \tag{68}$$

From Lemmas 2, 3, (66), (68) and noting that $B_k \in MT2, \forall k \geq 0$ it deduces (67), i. e. (65) is proved. From (28) and (64) we have

$$T_{2n-1} = \int^{y_3} B_{2n-2}R_0 + \int^{y_3} B_{2n-4}R_2 + \cdots + \int^{y_3} B_0 R_{2n-2}. \tag{69}$$

Therefore, $T_{2n-1} \in MT1$ according to Lemmas 2, 3, (65) and the fact that $B_k \in MT2, \forall k \geq 0$. From (30), we have

$$R_{2n-1} = \int^{y_3} B_{2n-1} + T_{2n-1}, \tag{70}$$

This yields

$$R_{2n-1} = \begin{bmatrix} t_{11}^{2n-1} & 0 & 0 & t_{14}^{2n-1} \\ 0 & t_{22}^{2n-1} & t_{23}^{2n-1} & 0 \\ \Omega_{2n} \int^{y_3} \tilde{\rho} & t_{32}^{2n-1} & t_{33}^{2n-1} & 0 \\ t_{41}^{2n-1} & \Omega_{2n} \int^{y_3} \tilde{\rho} & 0 & t_{44}^{2n-1} \end{bmatrix}. \tag{71}$$

Since $S_{2n-1} = R_{2n-1}(1)$, it follows

$$S_{2n-1} = \begin{bmatrix} t_{11}^{2n-1}(1) & 0 & 0 & t_{14}^{2n-1}(1) \\ 0 & t_{22}^{2n-1}(1) & t_{23}^{2n-1}(1) & 0 \\ \langle \tilde{\rho} \rangle \Omega_{2n} & t_{32}^{2n-1}(1) & t_{33}^{2n-1}(1) & 0 \\ t_{41}^{2n-1}(1) & \langle \tilde{\rho} \rangle \Omega_{2n} & 0 & t_{44}^{2n-1}(1) \end{bmatrix}. \tag{72}$$

On view of (65) and $S_{n-1} = R_{n-1}(1)$ for every $n \geq 0$, it is clear that

$$\begin{cases} S_{-1}, S_1, S_3, ..., S_{2n-3} \in MT1 \\ S_0, S_2, ..., S_{2n-2} \in MT2. \end{cases} \tag{73}$$

With $m = (2n-1)$, equation (47) is

$$\det[A^{2n-1}B^0C^0D^0] + \det[A^0B^{2n-1}C^0D^0] + \det[A^0B^0C^{2n-1}D^0]+$$

$$+ \det[A^0B^0C^0D^{2n-1}] + \sum_{\substack{u+v+s+t=2n-1 \\ 0 \leq u,v,s,t < 2n-1}} \det[A^uB^vC^sD^t] = 0. \tag{74}$$

Since $u + v + s + t = 2n - 1 \implies u + v + s + t$ is an odd number. Therefore, either one of u, v, s, t is odd and the others are even or three of them are odd and the remainder is even. In both cases

$$\det[A^u B^v C^s D^t] = 0, \quad u + v + s + t = 2n - 1, \quad 0 \le u, v, s, t < 2n - 1$$

according to Lemma 4 and taking into account (65). Thus we have

$$\sum_{\substack{u+v+s+t=2n-1 \\ 0 \le u,v,s,t < 2n-1}} \det[A^u B^v C^s D^t] = 0. \tag{75}$$

From (72), it follows

$$\begin{aligned}
\det[A^{2n-1} B^0 C^0 D^0] &= -s_{24}^0 s_{31}^{2n-1}[(s_{12}^0)^2 - s_{13}^0 s_{42}^0], \\
\det[A^0 B^{2n-1} C^0 D^0] &= -s_{42}^{2n-1} s_{13}^0[(s_{21}^0)^2 - s_{31}^0 s_{24}^0], \\
\det[A^0 B^0 C^{2n-1} D^0] &= 0, \quad \det[A^0 B^0 C^0 D^{2n-1}] = 0.
\end{aligned} \tag{76}$$

Substituting (75) and (76) into (74) yields

$$\Omega_{2n} \langle \tilde{\rho} \rangle \left\{ s_{24}^0[(s_{12}^0)^2 - s_{13}^0 s_{42}^0] + s_{13}^0[(s_{21}^0)^2 - s_{31}^0 s_{24}^0] \right\}. \tag{77}$$

Noting that if $(s_{12}^0)^2 - s_{13}^0 s_{42}^0 = 0$ then $(s_{21}^0)^2 - s_{31}^0 s_{24}^0 \ne 0$, or if $(s_{12}^0)^2 - s_{13}^0 s_{42}^0 \ne 0$ then $(s_{21}^0)^2 - s_{31}^0 s_{24}^0 = 0$, from (77) it deduces

$$\boxed{\Omega_{2n} = 0}. \tag{78}$$

The proof of the statement (63) is finished.

Recurrent formulas for Ω_{2n+1}.
On view of (63) it follows that $B_{2n-1} = 0 \; \forall \; n \ge 1$. Therefore, from (28) we have

$$T_{2n} = \int^{y_3} B_{2n-2} R_1 + \int^{y_3} B_{2n-4} R_3 + \cdots + \int^{y_3} B_0 R_{2n-1}, \; n \ge 0. \tag{79}$$

Since $B_{2n-1} = 0 \; \forall \; n \ge 1$, according to the proof of the statement (65), it is clear that

$R_{2n-1} \in MT1$, $R_{2n} \in MT2$, $\forall \; n \ge 0$. Thus, from (79) it deduces $T_{2n} \in MT2$ ($\forall \; n \ge 0$) according to Lemmas 2, 3 and $B_k \in MT2, \forall k \ge 0$. From (30), we have

$$R_{2n} = \int^{y_3} B_{2n} + T_{2n}.$$

This and $T_{2n} \in MT2$ lead to

$$R_{2n} = \begin{bmatrix} 0 & t_{12}^{2n} & t_{13}^{2n} & 0 \\ t_{21}^{2n} & 0 & 0 & t_{24}^{2n} \\ t_{31}^{2n} + \Omega_{2n+1} \int^{y_3} \tilde{\rho} & 0 & 0 & t_{34}^{2n} \\ 0 & t_{42}^{2n} + \Omega_{2n+1} \int^{y_3} \tilde{\rho} & t_{43}^{2n} & 0 \end{bmatrix}. \tag{80}$$

Since $S_{2n} = R_{2n}(1)$, we have:

$$S_{2n} = \begin{bmatrix} 0 & t_{12}^{2n}(1) & t_{13}^{2n}(1) & 0 \\ t_{21}^{2n}(1) & 0 & 0 & t_{24}^{2n}(1) \\ t_{31}^{2n}(1) + \Omega_{2n+1}\langle\tilde{\rho}\rangle & 0 & 0 & t_{34}^{2n}(1) \\ 0 & t_{42}^{2n}(1) + \Omega_{2n+1}\langle\tilde{\rho}\rangle & t_{43}^{2n}(1) & 0 \end{bmatrix}. \tag{81}$$

Corresponding to $m = 2n$, equation (47) is

$$\det[A^{2n}B^0C^0D^0] + \det[A^0B^{2n}C^0D^0]$$
$$+ \det[A^0B^0C^{2n}D^0] + \det[A^0B^0C^0D^{2n}]$$
$$+ \sum_{\substack{u+v+s+t=2n \\ 0 \le u,v,s,t < 2n}} \det[A^uB^vC^sD^t] = 0. \tag{82}$$

From (81), we have

$$\det[A^{2n}B^0C^0D^0] = a(s_{21}^{2n}s_{34}^0 - s_{31}^{2n}s_{24}^0),$$

$$\det[A^0B^{2n}C^0D^0] = b(s_{12}^{2n}s_{43}^0 - s_{42}^{2n}s_{13}^0),$$

$$\det[A^0B^0C^{2n}D^0] = b(s_{12}^0s_{43}^{2n} - s_{42}^0s_{13}^{2n}),$$

$$\det[A^0B^0C^0D^{2n}] = a(s_{21}^0s_{34}^{2n} - s_{31}^0s_{24}^{2n}).$$

It follows from (82) that
Case 1: $(s_{21}^0)^2 = s_{31}^0s_{24}^0 \Rightarrow b = 0 \Rightarrow \det[A^0B^{2n}C^0D^0] = \det[A^0B^0C^{2n}D^0] = 0$, then

$$s_{21}^{2n}s_{34}^0 - s_{31}^{2n}s_{24}^0 + s_{34}^{2n}s_{21}^0 - s_{24}^{2n}s_{31}^0 + \sum_{\substack{u+v+s+t=2n \\ 0 \le u,v,s,t < 2n}} \det[A^uB^vC^sD^t]a^{-1} = 0.$$

This provides

$$\boxed{\Omega_{2n+1} = (\langle\tilde{\rho}\rangle\, s_{24}^0)^{-1}\begin{bmatrix} s_{21}^{2n}s_{34}^0 - t_{31}^{2n}(1)s_{24}^0 + s_{34}^{2n}s_{21}^0 - s_{24}^{2n}s_{31}^0 + \\ +\sum_{\substack{u+v+s+t=2n \\ 0 \le u,v,s,t < 2n}} \det[A^uB^vC^sD^t]a^{-1} \end{bmatrix},} \tag{83}$$

for every $n \ge 1$.

Case 2: $(s_{12}^0)^2 = s_{13}^0s_{42}^0 \Rightarrow a = 0 \Rightarrow \det[A^{2n}B^0C^0D^0] = \det[A^0B^0C^0D^{2n}] = 0$, then

$$s_{12}^{2n}s_{43}^0 - s_{13}^{2n}s_{42}^0 + s_{43}^{2n}s_{12}^0 - s_{42}^{2n}s_{13}^0 + \sum_{\substack{u+v+s+t=2n \\ 0 \le u,v,s,t < 2n}} \det[A^uB^vC^sD^t]b^{-1} = 0.$$

This yields

$$\Omega_{2n+1} = (\langle \tilde{\rho} \rangle \, s_{42}^0)^{-1} \begin{bmatrix} s_{12}^{2n} s_{43}^0 - s_{13}^{2n} s_{42}^0 + s_{43}^{2n} s_{12}^0 - t_{42}^2(1) s_{13}^0 + \\ + \sum_{\substack{u+v+s+t=2n \\ 0 \le u,v,s,t < 2n}} \det[A^u B^v C^s D^t] b^{-1} \end{bmatrix}, \tag{84}$$

for every $n \ge 1$. The formulas (83) and (84) are the recurrent formulas for determining $\Omega_{2n+1}, n \ge 1$.

5. CONCLUSIONS

In this paper, we consider the long-Lamb wave propagation in a periodically layered, compressible elastic medium subject to a homogeneous initial deformation. Taking the assumption $\varepsilon = k.h \ll 1$, the dispersion of the wave is expressed as a power series of the small parameter ε whose coefficients are Ω_m, $m \ge 1$. We have derived the expressions for Ω_1, Ω_3, and proved that $\Omega_2 = 0$. We have also established the recurrent formulae for calculating the non-zero higher-order coefficients Ω_{2k+1}, $k \ge 2$, and proved that $\Omega_{2k} = 0 \ \forall k \ge 1$. Based on these formulae, the solution with any order of accuracy can be obtained.

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KHAI TRIỂN TIỆM CẬN CỦA PHƯƠNG TRÌNH TÁN SẮC CỦA SÓNG LAMB TRONG CÁC MÔI TRƯỜNG ĐÀN HỒI PHÂN LỚP TUẦN HOÀN

Bài báo nghiên cứu sự truyền của sóng Lamb trong các môi trường đàn hồi nén được, phân lớp tuần hoàn, có biến dạng trước, trong trường hợp xấp xỉ sóng dài (tức là khi $0 < \varepsilon = k.h \ll 1$, trong đó k là số sóng, h là độ dầy của một chu kỳ). Với giả thiết $\varepsilon \ll 1$ phương trình tán sắc của sóng được viết dưới dạng:

$$\frac{\varpi^2}{k^2} = \Omega_1 + \varepsilon\Omega_2 + \varepsilon^2\Omega_3 + ... = \sum_{m=0}^{\infty} \varepsilon^m \Omega_{m+1}$$

Mục đích chính của bài báo là tìm các công thức xác định các hệ số Ω_i $(i \geq 1)$ ... Các tác giả đã chứng minh được rằng $\Omega_{2n} = 0$ với mọi $n \geq 1$, tìm ra các công thức xác định Ω_1, Ω_3, và xây dựng được các công thức truy hồi để tính Ω_{2n+1}, $n \geq 2$. Sử dụng các công thức này, ta có thể tìm được nghiệm với độ chính xác tùy ý. Bài báo là sự mở rộng của nghiên cứu thực hiện bởi Noris và Santosa [Norris A. and Santosa F., Wave Motion 16 (1992), 33-55] từ sóng SH (sóng một thành phần) sang sóng Lamb (sóng hai thành phần).