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A WEIGHTED DUAL CRITERION FOR STOCHASTIC EQUIVALENT LINEARIZATION METHOD USING PIECEWISE LINEAR FUNCTIONS

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Abstract. A weighted dual mean square criterion for stochastic equivalent linearization method is considered in which the forward and backward replacements are weighted. The normalized weighting coefficient is suggested as a piecewise linear function of the squared correlation coefficient and is defined by the least square method based on the data of Lutes-Sarkani oscillator. The application to two typical nonlinear systems subjected to random excitation shows accurate approximations when the nonlinearity varies from the weak to strong levels.

Keywords: Stochastic equivalent linearization, weighted dual criterion, weighting coefficient.

1. INTRODUCTION

In the study of random vibration, the stochastic equivalent linearization method proposed separately in [1,2] is one of most popular methods for analyzing nonlinear system. The development of its kernel, the classical criterion, leads to several criteria that are summarized in the papers [3, 4] and presented in the books [5–7]. It can be seen that the diverse ideas and new approaches make stochastic equivalent linearization more attractive. Recently, using the dual approach introduced in [8], a dual mean square error criterion of stochastic linearization is proposed in [9] in which dual replacements are used. Its application to investigation of approximate mean-square responses shows good results in cases of Duffing, Van der Pol oscillators but unacceptable in cases of Lutes-Sarkani oscillator with variety of nonlinearities [9]. It is observed that the dual criterion is only effective in a limited range of nonlinearity based on the value of the squared correlation coefficient [9, 10].

We therefore develop a more general form of the dual criterion by considering weighted contributions of forward and backward replacements in which the normalized weighting coefficient depends on the squared correlation coefficient [11, 12]. The simplicity and accuracy of the weighted criterion are checked on several random vibration systems with nonlinear restoring or nonlinear damping.

2. WEIGHTED DUAL CRITERION

2.1. Basic idea of the weighted dual criterion

Anh et al. [9] proposed a dual criterion to the equivalent linearization method by the following expression

$$S_d = \left\langle (A - kB)^2 \right\rangle + \left\langle (kB - \lambda A)^2 \right\rangle \to \min_{k,\lambda} \tag{1}$$

Here $< \cdot >$ denotes the expectation operator, A and B are random nonlinear and linear functions that have zero mean values, k is equivalent linearization coefficient and λ is return coefficient. The first mean square of (1) describes the forward replacement, and the second one is the backward replacement. The replacement process can be illustrated schematically as bellow

$$A \to kB \to \lambda A$$

Applying (1) to several nonlinear systems has shown that the dual criterion can give good results only for a limited level of nonlinearity [9, 10]. A major reason may be that the contributions of the forward and backward replacements in the dual criterion (1) are evaluated with equal influences while in fact they would be different. Thus, we consider a weighted form of (1) as follows [11, 12]

$$S_{wd} = (1 - p) \left\langle (A - kB)^2 \right\rangle + p \left\langle (kB - \lambda A)^2 \right\rangle \to \min_{k,\lambda} \tag{2}$$

where using the normalized weighting coefficient

$$0 \le p \le 1. \tag{3}$$

It is seen from (2) that S_{wd} is the weighted mean of the forward and the backward replacements. In the case $p = 1/2$, (2) has the form of the dual criterion (1) that expresses the equal contributions of replacements. Given a weighting coefficient p and supposing that $\left\langle A^2 \right\rangle > 0$, $\left\langle B^2 \right\rangle > 0$, the equivalent linearization coefficient k and return coefficient λ are determined by the minimum condition from (2)

$$k = (1 - p + p\lambda) \frac{\langle AB \rangle}{\langle B^2 \rangle}, \quad \lambda = k \frac{\langle AB \rangle}{\langle A^2 \rangle}. \tag{4}$$

Solving the system of Eqs. (4) yields

$$k = \frac{1 - p}{1 - \mu p} \frac{\langle AB \rangle}{\langle B^2 \rangle}, \tag{5}$$

$$\lambda = \frac{\mu (1 - p)}{1 - \mu p}, \tag{6}$$

where it is denoted

$$\mu = \frac{\langle AB \rangle^2}{\langle A^2 \rangle \langle B^2 \rangle}, \tag{7}$$

and supposed

$$\mu p \neq 1. \tag{8}$$

For understanding the meaning of μ, we first consider the correlation coefficient of A and B which is defined as (see [13])

$$r = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{\langle AB \rangle}{\left[\langle A^2 \rangle \langle B^2 \rangle\right]^{1/2}}. \tag{9}$$

Following the Schwarz inequality $\langle |AB| \rangle \leq \left[\left\langle A^2 \right\rangle \left\langle B^2 \right\rangle\right]^{1/2}$, one get $|r| \leq 1$. Clearly, the coefficient μ in Eq. (7) is the squared correlation coefficient

$$\mu = r^2, \;\; 0 \leq \mu \leq 1. \tag{10}$$

It is well known that the correlation coefficient r or the squared correlation coefficient r^2 are used as a measure of the linear dependence when the function A is approximated by the linear function kB (see [13]). For this reason, μ is called the linear dependence level between A and B. In statistics, a correlation is an effect size, so guidelines on strength can be suggested (see [13]). Based on the value of μ, it can be seen that:

- When $\mu = 1$ given by $A = \alpha B$, the linear approximation is an exact fit. In this case the linear dependence level is strongest.

- When $\mu = 0$ given by $\langle AB \rangle = 0$, the linear approximation is the worst. In this case A and B are said to be uncorrelated and orthogonal, the linear dependence level is weakest.

Therefore, the important properties of the linear dependence level μ may be used as one of the major possibilities to perform the main features of the weighted dual criterion (2). Moreover, it is noted that μ may be an explicit value in case the considered functions have zero mean. In the next section, we will combine the contributions of replacements with μ to construct the weighting coefficient p.

2.2. Analytical expression of weighting coefficient

In the weighted dual criterion (2), the contribution of the forward and backward replacements is evaluated by the value of the weighting coefficient p. Consider particular cases as follows:

Case (i). This case shows the highest contribution of forward replacement given by $p = 0$. The criterion (2) now reduces to the classical mean square error criterion

$$S_{wd(p=0)} = \left\langle (A - kB)^2 \right\rangle \to \min_k \tag{11}$$

The equivalent linearization coefficient defined from (14) is

$$k = \frac{\langle AB \rangle}{\langle B^2 \rangle}. \tag{12}$$

Substituting (12) into (11) yields the minimum of S_{wd} respect to k

$$\min_k S_{wd(p=0)} = \left\langle A^2 \right\rangle (1 - \mu). \tag{13}$$

We see from (13) that the obtained criterion (11) yields good linear approximation when the value of μ is near to 1, namely in the interval [0.85,1] as shown in [9, 10], especially the best one when the linear dependence is the strongest, $\mu = 1$.

Case (*ii*). This case shows the equal contribution of forward and backward replacements given by $p = 1/2$. The criterion (2) now becomes the dual criterion (1). As shown in [9, 10], the application of the dual criterion for several nonlinear oscillators illustrates how its effective range is related to the values of μ, particularly, when μ belongs to the interval from 1/3 to 2/3.

Case (*iii*). This case shows the highest contribution of backward replacement given by $p = 1$. The criterion (2) now has only

$$S_{wd(p=1)} = \left\langle (kB - \lambda A)^2 \right\rangle \to \min_{k,\lambda} \tag{14}$$

Using the minimum condition in (14), one has

$$k = \lambda \frac{\langle AB \rangle}{\langle B^2 \rangle}, \lambda = k \frac{\langle AB \rangle}{\langle A^2 \rangle}. \tag{15}$$

There are two possible outcome with two equations in (15)

- $\langle AB \rangle^2 / \left[\langle A^2 \rangle \langle B^2 \rangle \right] = 1$ leads to an infinite number of k and λ, but it is contrary to (8) due to $p = 1,\ \ \mu = 1$, so it is rejected.

- leads to trivial solutions $k = \lambda = 0$, it is corresponding to the weakest linear dependence level, $\mu = 0$.

It is observed in above cases (i, ii, iii) that there is a relationship between p, μ and the error of approximate solutions to each other. Since it is difficult to make the exact mathematical expression of p in term of μ, so in this paper we seek for a weighting coefficient in the piecewise linear form of the linear dependence level.

Based on the results of investigation in [9, 11]. It is suggested that $p(\mu)$ can be expressed in the form, $p(\mu) = 1/2$ for $\mu \in [1/3,\ 2/3]$. For simplicity, the linear dependence level is divided into three parts and the corresponding function $p(\mu)$ is a piecewise linear function as below:

- Weak linear dependence level, $0 \le \mu \le 1/3$

$$p(\mu) = \alpha_1 \mu + \beta_1, \tag{16}$$

$$p(0) = 1. \tag{17}$$

- Medium linear dependence level, $1/3 \le \mu \le 2/3$

$$p(\mu) = 1/2. \tag{18}$$

- Strong linear dependence level, $2/3 \le \mu \le 1$

$$p(\mu) = \alpha_2 \mu + \beta_2, \tag{19}$$

$$p(1) = 0, \tag{20}$$

Using those boundary conditions (17), (20) for (16), (19), one has

$$\beta_1 = 1\,,\, p = \alpha_1 \mu + 1 \quad \text{for} \quad 0 \le \mu \le 1/3, \tag{21}$$

$$\beta_2 = -\alpha_2,\, p = \alpha_2 \mu - \alpha_2 \quad \text{for} \quad 2/3 \le \mu \le 1. \tag{22}$$

There are two unknowns α_1 and α_2 in (21), (22). We use the method of least squares for a nonlinear system which has exact solution to find α_1 and α_2. The interpolation steps include:

- Choose systems with different nonlinearities whose exact responses are available so that the exact values of exact equivalent linearization coefficients k_{exact} and exact second moments $< B^2 >_{\text{exact}}$, $< A^2 >_{\text{exact}}$, $< AB >_{\text{exact}}$, can be computed;

- Substituting k_{exact}, $< A^2 >_{\text{exact}}$, $< AB >_{\text{exact}}$, $< B^2 >_{\text{exact}}$ into (7) and (5) to find exact values of squared correlation and weighting coefficients μ_{exact} and p_{exact};

- Based on p_{exact} and μ_{exact} find $\alpha_1,\ \alpha_2$ of linear functions (21), (22) by using the method of least squares.

In order to carry out those interpolation steps we first look for systems with known exact solutions. Although several oscillators that have the exact responses can be found in literature, see for example [1–7], the Lutes-Sarkani oscillator can be chosen due to following reasons:

- It represents a class of nonlinear system and has exact solution.

- It has a continuous linear dependence level.

The equation of Lutes-Sarkani oscillator is governed by

$$\dot{x} + \gamma \left| x \right|^a \operatorname{sgn}(x) = f(t), \tag{23}$$

where a is a real positive number, $f(t)$ is a zero mean, stationary Gaussian white noise with spectral density $S_0 = const$. Indeed, the Eq. (23) may represent a class of Power-law oscillator which has the variable nonlinearity when a varies. Its exact stationary response is given by [6]

$$\sigma^2_{x_{\text{exact}}} = \left(\frac{\pi S_0}{\gamma} \right)^{\frac{2}{a+1}} (a+1)^{\frac{2}{a+1}} \Gamma\left(\frac{3}{a+1} \right) \left[\Gamma\left(\frac{1}{a+1} \right) \right]^{-1}. \tag{24}$$

The equivalent linearization equation to (23) is

$$\dot{x} + kx = f(t), \tag{25}$$

where k is the linearization coefficient. Because $\langle x \rangle = 0$, the variance of approximate solution reduces to the mean square value. Thus, one gets the relationship

$$k_{\text{exact}} = \frac{\pi S_0}{\sigma^2_{x_{\text{exact}}}}. \tag{26}$$

Using the weighted dual criterion with $A = \gamma \left| x \right|^a \operatorname{sgn}(x)\,,\ \ B = \left| x \right| \operatorname{sgn}(x) = x$, first make the following calculations

$$\begin{gathered} \left\langle B^2 \right\rangle = \sigma_x^2, \left\langle A^2 \right\rangle = \tfrac{1}{\sqrt{2\pi}} \gamma^2 2^{\left(a+\frac{1}{2}\right)} \Gamma\left(a + \tfrac{1}{2}\right) \sigma_x^{2a}, \\ \left\langle AB \right\rangle = \tfrac{1}{\sqrt{2\pi}} \gamma 2^{\frac{a}{2}} a \Gamma\left(\tfrac{a}{2}\right) \sigma_x^{a+1}, \end{gathered} \tag{27}$$

where the Gamma function $\Gamma(\upsilon)$ is given by $\Gamma(\upsilon) = \int\limits_0^\infty u^{\upsilon-1} \exp(-u)\, du$. Then using (7), (27) yields the linear dependence level

$$\mu = \frac{a^2}{2\sqrt{\pi}} \left[\Gamma\left(\frac{a}{2}\right) \right]^2 \left[\Gamma\left(a + \frac{1}{2}\right) \right]^{-1}. \tag{28}$$

Substituting (26), (27), (28) into (5) to solve p_{exact} yields

$$p_{\text{exact}} = \frac{\frac{1}{\sqrt{2\pi}}\gamma 2^{\frac{a}{2}} a\Gamma\left(\frac{a}{2}\right)(\sigma_x)_{\text{exact}}^{a+1} - \pi S_0}{\frac{1}{\sqrt{2\pi}}\gamma 2^{\frac{a}{2}} a\Gamma\left(\frac{a}{2}\right)(\sigma_x)_{\text{exact}}^{a+1} - \pi S_0 \frac{a^2}{2\sqrt{\pi}}\left[\Gamma\left(\frac{a}{2}\right)\right]^2\left[\Gamma\left(a+\frac{1}{2}\right)\right]^{-1}}. \tag{29}$$

Table 1. The exact values of $(\mu_{\text{exact}})_j$ and $(p_{\text{exact}})_j$ calculated for the oscillator (25) versus various values of $a$corresponding to weak linear dependence level ($0 \le \mu \le 1/3$)

i	a	$(\mu_{\text{exact}})_i$	$(p_{\text{exact}})_i$
1	4.3415	0.33	0.6392
2	4.5374	0.30	0.6621
3	4.9036	0.25	0.7038
4	5.3395	0.20	0.7505
5	5.8858	0.15	0.8032
6	6.6329	0.10	0.8632
7	7.8662	0.05	0.9313
8	10.6002	0.01	0.9891
9	14.6808	0.001	0.9996

Table 2. The exact values of $(\mu_{\text{exact}})_i$ and $(p_{\text{exact}})_i$ calculated for the oscillator (25) versus various values of a corresponding to strong linear dependence level ($2/3 \le \mu \le 1$)

j	a	$(\mu_{\text{exact}})_j$	$(p_{\text{exact}})_j$
1	0.0390	0.67	0.2616
2	0.0763	0.70	0.2623
3	0.2216	0.80	0.2656
4	0.4244	0.90	0.2722
5	1.0000	1.00	0.0000
6	1.7751	0.90	0.3579
7	2.1999	0.80	0.3967
8	2.5957	0.70	0.4368
9	2.7150	0.67	0.4496

Based on (24) and (28), (29) the values μ_{exact} and p_{exact} are calculated and shown in Tabs. 1, 2 for different powers a corresponding to weak and strong linear dependence levels, respectively. Using those values, the coefficients $\alpha_1,\ \alpha_2$ from (21), (22) are determined by method of least squares with following conditions

$$\sum_{i=1}^{9}\left[\alpha_1(\mu_{\text{exact}})_i + 1 - (p_{\text{exact}})_i\right]^2 \to \min_{\alpha_1} \tag{30}$$

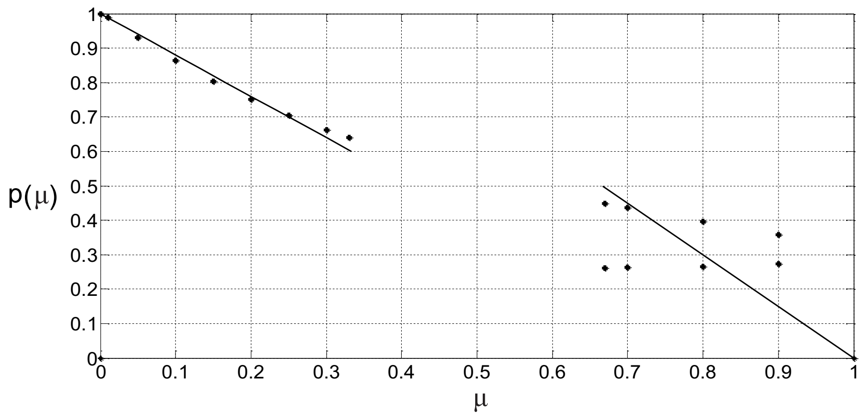

Fig. 1. Interpolation by linear functions

$$\sum_{j=1}^{9}\left[\alpha_2(\mu_{\text{exact}})_j-\alpha_2-(p_{\text{exact}})_j\right]^2\to\min_{\alpha_2} \tag{31}$$

and the results are

$$\alpha_1=-6/5, \tag{32}$$

$$\alpha_2=-3/2. \tag{33}$$

It can be seen in Fig. 1 that for $0\leq\mu\leq1/3$ the exact data denoted by dots and the interpolation linear function are quite close together whereas for $2/3\leq\mu\leq1$ they are quite far apart. Substituting (32), (33) to (21), (22), the weighting coefficient $p(\mu)$ is completely defined. The analytical expression of $p(\mu)$ is

$$p=-6\mu/5+1\quad\text{for}\quad 0\leq\mu\leq1/3, \tag{34}$$

$$p=\ 1/2\quad\text{for}\quad 1/3\leq\mu\leq2/3, \tag{35}$$

$$p=-3\mu/2+3/2\quad\text{for}\quad 2/3\leq\mu\leq1, \tag{36}$$

where the linear dependence level μ is determined by (7). The graph of $p(\mu)$ shown in Fig. 2 where the classical criterion corresponding with $p\left(\mu\right)=0$ is just the lowest line, whereas the dual criterion corresponding with $p\left(\mu\right)=1/2$ is just the middle line. The suggested piecewise linear function $p(\mu)$ makes the weighted dual criterion expresses more diverse behavior according to different linear dependence levels.

It is seen from Fig. 2 that the graph of $p(\mu)$ has a discontinuity point at $\mu=1/3$. From the left, using $p_{\text{left}}=-6\mu/5+1$ and (5) one has

$$p_{\text{left}}=-\frac{6}{5}\cdot\frac{1}{3}+1=\frac{3}{5},\ k_{\text{left}}=\frac{1}{2}\frac{\langle AB\rangle}{\langle B^2\rangle}. \tag{37}$$

From the right, using $p_{\text{right}}=1/2$ and (5) one has

$$k_{\text{right}}=\frac{3}{5}\frac{\langle AB\rangle}{\langle B^2\rangle}. \tag{38}$$

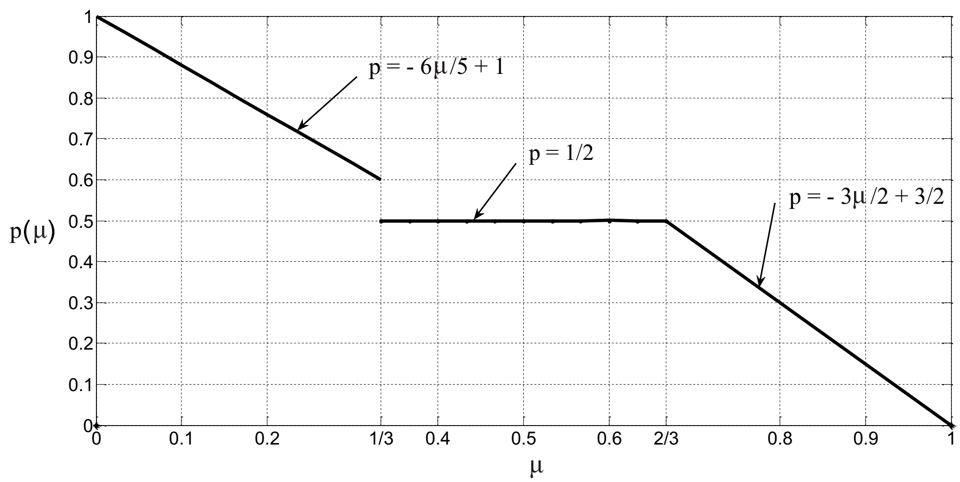

Fig. 2. Graph of the weighting coefficient $p(\mu)$

In order to get a harmonic consideration it is supposed that at $\mu = 1/3$ the equivalent linearization coefficient is to be arithmetic mean as follows

$$k_{1/3} = \frac{1}{2}\left(k_{\text{left}} + k_{\text{right}}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{3}{5}\right)\frac{\langle AB\rangle}{\langle B^2\rangle} = \frac{11}{20}\frac{\langle AB\rangle}{\langle B^2\rangle}. \tag{39}$$

Substituting (39) into (5) yields

$$p\left(1/3\right) = \frac{27}{49}. \tag{40}$$

Summarizing the research above, we can procedure the following steps to calculate the equivalent linearization coefficient k provided by the weighted dual criterion

S1. Calculate the expectations $\left\langle B^2\right\rangle$, $\left\langle A^2\right\rangle$, $\langle AB\rangle$ using the corresponding equivalent linear equation and/or the approximate solution considered;

S2. Calculate the linear dependence level μ from (7) with the supposition that it is explicit;

S3. Use of Tab. 3 to find the corresponding weighting coefficient p and the equivalent linearization coefficient k.

Table 3. Weighting coefficient and linearization coefficient

N	Level	Linear dependence level μ	Weighting coefficient p	Linearization coefficient k
1	Weak	$\mu \in [0,\ 1/3)$	$p = -\frac{6}{5}\mu + 1$	$k = \frac{6\mu}{6\mu^2-5\mu+5}\frac{\langle AB\rangle}{\langle B^2\rangle}$
2	Weak	$\mu = 1/3$	$p = \frac{27}{49}$	$k_{1/3} = \frac{11}{20}\frac{\langle AB\rangle}{\langle B^2\rangle}$
3	Medium	$\mu \in (1/3,\ 2/3]$	$p = \frac{1}{2}$	$k = \frac{1}{2-\mu}\frac{\langle AB\rangle}{\langle B^2\rangle}$
4	Strong	$\mu \in [2/3,\ 1]$	$p = -\frac{3}{2}\mu + \frac{3}{2}$	$k = \frac{3\mu-1}{3\mu^2-3\mu+2}\frac{\langle AB\rangle}{\langle B^2\rangle}$

In the next section the application and accuracy of the weighted dual mean square error criterion (2) will be illustrated and examined for several nonlinear systems.

3. APPLICATION OF WEIGHTED DUAL CRITERION

3.1. Power-law nonlinear restoring oscillator

Consider the following nonlinear system

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + \gamma x^a = \sigma\dot{\xi}(t), \tag{41}$$

where $h, \omega_0, \gamma, \sigma, a$ are positive real constants, γx^a is odd function, $\dot{\xi}(t)$ is Gaussian white noise excitation. The exact mean square solution of this oscillator is [5]

$$\left\langle x^2 \right\rangle_{\text{exact}} = \frac{\int\limits_{-\infty}^{\infty} x^2 \exp\left(-\frac{4h}{\sigma^2}\left(\frac{1}{2}\omega_0^2 x^2 + \frac{1}{a+1}\gamma x^{a+1}\right)\right) dx}{\int\limits_{-\infty}^{\infty} \exp\left(-\frac{4h}{\sigma^2}\left(\frac{1}{2}\omega_0^2 x^2 + \frac{1}{a+1}\gamma x^{a+1}\right)\right) dx}. \tag{42}$$

The equivalent linearization equation to (41) is of the form

$$\ddot{x} + 2h\dot{x} + \left(\omega_0^2 + k\right) x = \sigma\xi(t), \tag{43}$$

where k is the linearization coefficient, and the mean square response of displacement is

$$\left\langle x^2 \right\rangle = \frac{\sigma^2}{4h\left(\omega_0^2 + k\right)}. \tag{44}$$

Using the weighted dual criterion with $A = \gamma x^a$, $kB = kx$, first make the following calculations in step S1

$$\begin{gathered}\left\langle B^2 \right\rangle = \sigma_x^2, \quad \left\langle A^2 \right\rangle = \frac{1}{\sqrt{2\pi}}\gamma^2 2^{\left(a+\frac{1}{2}\right)}\Gamma\left(a+\tfrac{1}{2}\right)\sigma_x^{2a}, \\ \left\langle AB \right\rangle = \frac{1}{\sqrt{2\pi}}\gamma 2^{\frac{a}{2}} a\Gamma\left(\tfrac{a}{2}\right)\sigma_x^{a+1}.\end{gathered} \tag{45}$$

Then in step S2, using (7), (45) yields

$$\mu = \frac{a^2}{2\sqrt{\pi}}\left[\Gamma\left(\frac{a}{2}\right)\right]^2\left[\Gamma\left(a+\frac{1}{2}\right)\right]^{-1}. \tag{46}$$

Table 4. The errors of the approximate mean square responses of Power-law nonlinear restoring oscillator with $a = 1/3$, $h = 0.5$, $\omega_0 = 1$, $\sigma = \sqrt{2}$ and various values of γ

γ	$\left\langle x^2 \right\rangle_{\text{exact}}$	$\left\langle x^2 \right\rangle_c$	error (%)	$\left\langle x^2 \right\rangle_d$	error (%)	$\left\langle x^2 \right\rangle_{wd}$	error (%)	μ	p
0.1	0.9215	0.9213	0.02	0.9305	0.97	0.9240	0.27	0.860	0.209
0.5	0.6821	0.6791	0.45	0.7099	4.08	0.6878	0.84	0.860	0.209
1.0	0.4926	0.4862	1.29	0.5253	6.65	0.4972	0.93	0.860	0.209
5.0	0.1064	0.1007	5.31	0.1188	11.73	0.1055	0.81	0.860	0.209
10	0.0420	0.0393	6.43	0.0472	12.44	0.0414	1.50	0.860	0.209

Table 5. The errors of the approximate mean square responses of Power-law nonlinear restoring oscillator with $a = 5$, $h = 0.5$, $\omega_0 = 1$, $\sigma = \sqrt{2}$ and various values of γ

γ	$\langle x^2\rangle_{\text{exact}}$	$\langle x^2\rangle_c$	error (%)	$\langle x^2\rangle_d$	error (%)	$\langle x^2\rangle_{wd}$	error (%)	μ	p
0.1	0.7026	0.6282	10.59	0.7035	0.14	0.7670	9.17	0.238	0.714
0.5	0.5140	0.4249	17.35	0.4923	4.23	0.5561	8.18	0.238	0.714
1.0	0.4372	0.3511	19.70	0.4106	6.08	0.4686	7.19	0.238	0.714
5.0	0.2865	0.2184	23.76	0.2591	9.55	0.3004	4.83	0.238	0.714
10	0.2352	0.1764	25.00	0.2101	10.67	0.2446	3.97	0.238	0.714
50	0.1453	0.1060	27.02	0.1270	12.55	0.1488	2.44	0.238	0.714
100	0.1172	0.0848	27.61	0.1018	13.11	0.1195	1.96	0.238	0.714

Table 6. The errors of the approximate mean square responses of Power-law nonlinear restoring oscillator with $a = 7$, $h = 0.5$, $\omega_0 = 1$, $\sigma = \sqrt{2}$ and various values of γ

γ	$\langle x^2\rangle_{\text{exact}}$	$\langle x^2\rangle_c$	error (%)	$\langle x^2\rangle_d$	error (%)	$\langle x^2\rangle_{wd}$	error (%)	μ	p
0.1	0.6170	0.4733	23.30	0.5388	12.68	0.7131	15.57	0.082	0.902
0.5	0.4726	0.3354	29.03	0.3869	18.14	0.5374	13.70	0.082	0.902
1.0	0.4156	0.2871	30.92	0.3323	20.03	0.4679	12.61	0.082	0.902
5.0	0.3008	0.1977	34.27	0.2303	23.44	0.3313	10.15	0.082	0.902
10	0.2595	0.1678	35.35	0.1958	24.56	0.2835	9.23	0.082	0.902
50	0.1816	0.1140	37.23	0.1334	26.53	0.1952	7.49	0.082	0.902
100	0.1549	0.0963	37.82	0.1128	27.16	0.1656	6.90	0.082	0.902

Next in step S3, the corresponding equivalent linearization coefficient is calculated by using Tab. 3, then substituting it into (44) to find the mean square response.

Consider the Power-law nonlinear restoring oscillator for various values of a and γ. For evaluating the effectiveness of present criterion, the exact solution and the ones provided by classical and dual criteria available in [5,9] are used. The result by percentage error are shown in Tabs. 4, 5 and 6. It is seen among considered criteria that, the present criterion gives the best approximation while the dual criterion gives better result than the classical linearization does when $a > 1$ but becomes worse when $a < 1$.

3.2. Oscillator with nonlinear damping by displacement and velocity

Consider the nonlinear stochastic oscillator governed by

$$\ddot{x} + \zeta\left(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2\right)^a \dot{x} + \omega_0^2 x = f\left(t\right), \tag{47}$$

where ζ is the damping constant, ω_0 is the natural frequency, a is positive constant, $f(t)$ is a Gaussian white noise process with spectral density $S_0 = const$. The exact mean response of the above oscillator is (see [5])

$$\left\langle x^2 \right\rangle_{\text{exact}} = \frac{1}{\omega_0^2}\left(\frac{\pi S_0}{\zeta}\right)^{1/(a+1)} (a+1)^{1/(a+1)}\,\Gamma\left(\frac{2}{a+1}\right)\left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{48}$$

The equivalent linearization equation is given by

$$\ddot{x} + b\dot{x} + \omega_0^2 x = \sigma\dot{\xi}(t), \tag{49}$$

where b is the linearization coefficient. The mean square response of (49) is

$$\left\langle x^2 \right\rangle = \frac{\pi S_0}{b\omega_0^2}. \tag{50}$$

Applying the weighted dual criterion with $A = \zeta\left(\dot{x}^2/2 + x^2/2\right)\dot{x}$, $B = \dot{x}$, first make the following calculations in step S1

$$\begin{aligned}
&\left\langle B^2 \right\rangle = \left\langle \dot{x}^2 \right\rangle,\\
&\left\langle A^2 \right\rangle = \left\langle \left[\zeta\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{\omega_0^2}{2}x^2\right)^a \dot{x}\right]^2 \right\rangle = \zeta^2\Gamma(2a+2)\left\langle \dot{x}^2 \right\rangle^{2a+1},\\
&\left\langle AB \right\rangle = \left\langle \zeta\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{\omega_0^2}{2}x^2\right)^a \dot{x}^2 \right\rangle = \zeta\Gamma(a+2)\left\langle \dot{x}^2 \right\rangle^{a+1},
\end{aligned} \tag{51}$$

with notice of $\left\langle \dot{x}^2 \right\rangle = \omega_0^2\left\langle x^2 \right\rangle$. Then in step S2 using (7), (51) yields the squared correlation coefficient

$$\mu = [\Gamma(a+2)]^2\,[\Gamma(2a+2)]^{-1} \tag{52}$$

Next in step S3, the corresponding equivalent linearization coefficient is calculated by using Tab. 3, then substituting it into (50) to find the approximate response.

Consider the oscillator with nonlinear damping by displacement and velocity when a varies. For evaluating the effectiveness of present criterion, the exact solution and the ones provided by classical and dual criteria are used. The result by percentage error is shown in Tab. 7. For $a = 0.0399$ corresponding with strong linear dependence level, μ closes to 1 and $p(\mu)$ closes to 0. For $a = 7.1107$ corresponding with weak linear dependence level, μ closes to 0 and $p(\mu)$ closes to 1. The classical linearization only gives good results in case μ belongs to the range from 0.85 to 1 while the dual one shows its effectiveness with the value of μ in the range from 0.4 to 1. The present criterion provides the best linear approximation with maximum error is less than 7.5% in comparison with 57.84% and 54.8% provided by the classical and dual ones.

4. CONCLUSIONS

The development of the dual mean square criterion of the equivalent linearization method leads to a weighted dual mean square criterion in which the normalized weighting coefficient is used for evaluating different contributions of the replacements. The linear dependence level derived from the squared correlation coefficient allows to outline main features of the proposed criterion. Treating weighting coefficient as a function depending on the squared correlation coefficient, its main restrictions is introduced. Using the least squares method for a typical Power-law oscillator, a approximation of weighting coefficient

Table 7. Errors of approximate mean square responses of considered oscillator with various values of a

a	$\langle x^2\rangle_{\text{exact}}$	$\langle x^2\rangle_c$	error (%)	$\langle x^2\rangle_d$	error (%)	$\langle x^2\rangle_{wd}$	error (%)	μ	p_{wd}
0.0399	0.9838	0.9834	0.04	0.9844	0.05	0.9834	0.04	0.999	0.002
0.1298	0.9521	0.9482	0.42	0.9565	0.46	0.9483	0.41	0.990	0.015
0.2106	0.9280	0.9186	1.01	0.9376	1.03	0.9194	0.93	0.975	0.038
0.3074	0.9033	0.8857	1.95	0.9194	1.78	0.8885	1.64	0.950	0.075
0.3864	0.8859	0.8607	2.85	0.9067	2.36	0.8665	2.18	0.925	0.113
0.4568	0.8720	0.8396	3.72	0.8963	2.79	0.8497	2.56	0.900	0.150
0.5836	0.8502	0.8042	5.41	0.8784	3.32	0.8261	2.83	0.850	0.225
0.7008	0.8329	0.7741	7.06	0.8617	3.46	0.8125	2.45	0.800	0.300
0.8138	0.8184	0.7473	8.68	0.8451	3.27	0.8072	1.37	0.750	0.375
0.9253	0.8056	0.7227	10.30	0.8282	2.80	0.8099	0.53	0.700	0.450
1.0000	0.7979	0.7071	11.38	0.8165	2.33	0.8165	2.33	0.667	0.500
1.0376	0.7942	0.6995	11.92	0.8105	2.06	0.8105	2.06	0.650	0.500
1.1523	0.7836	0.6774	13.55	0.7921	1.07	0.7921	1.07	0.600	0.500
1.2710	0.7738	0.6561	15.22	0.7727	0.14	0.7727	0.14	0.550	0.500
1.3955	0.7644	0.6351	16.92	0.7522	1.60	0.7522	1.60	0.500	0.500
1.5277	0.7554	0.6143	18.68	0.7306	3.28	0.7306	3.28	0.450	0.500
1.6701	0.7466	0.5934	20.51	0.7077	5.21	0.7077	5.21	0.400	0.500
1.8260	0.7378	0.5722	22.45	0.6832	7.41	0.6832	7.41	0.350	0.500
1.8820	0.7349	0.5650	23.12	0.6746	8.21	0.6952	5.40	0.333	0.551
2.0000	0.7290	0.5503	24.51	0.6568	9.91	0.7205	1.16	0.300	0.640
2.1992	0.7199	0.5273	26.76	0.6281	12.76	0.7234	0.48	0.250	0.700
2.4353	0.7104	0.5024	29.27	0.5962	16.07	0.7255	2.14	0.200	0.760
2.7298	0.6999	0.4747	32.18	0.5598	20.02	0.7257	3.69	0.150	0.820
3.1302	0.6876	0.4416	35.78	0.5158	24.98	0.7215	4.94	0.100	0.880
5.2277	0.6453	0.3245	49.72	0.3624	43.85	0.6590	2.11	0.010	0.988
7.1107	0.6230	0.2627	57.84	0.2861	54.08	0.5956	4.40	0.001	0.999

is found in the form of piecewise linear function. It should be stressed that the performance of the squared correlation coefficient is implemented for zero mean stationary process. Two typical random vibrations with nonlinear damping and restoring, respectively, are examined. The results show good accurate approximations when the nonlinearity varies from the weak to strong levels. Further investigation, however, is needed to be extended to other nonlinear systems.

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